UFFE HAAGERUPS THIRD TALK AT MASTERCLASS ON VON NEUMANN ALGEBRAS AND GROUP ACTIONS

1. Crossed products

If $\mathcal{A} \subseteq B(H)$ is a von Neumann algebra and G is a discrete group and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a action of G on $\mathcal{A}$, let

$$(\pi(a)\xi)(g) = \alpha_g^{-1}(a)\xi(g), \qquad \forall \xi \in l^2(G,H), a \in \mathcal{A} \tag{1.1}$$

and

$$(\lambda(g)\xi)(h) = \xi(h^{-1}g), \qquad \forall \xi \in l^2(G,H), g \in G \tag{1.2}$$

Define

$$\mathcal{M} := \mathcal{A} \rtimes G := (\pi(\mathcal{A}) \cup \lambda(G))'' \tag{1.3}$$

then

$$\lambda(g)\pi(a)\lambda(g)^* = \pi(\alpha_g(a)) \tag{1.4}$$

and if we identify a with $\pi(a) \in \mathcal{M}$, we get

$$\mathcal{M} := (\mathcal{A} \cup \lambda(G))'' \tag{1.5}$$

and

$$\lambda(g)a\lambda(g)* = \alpha_g(a). \tag{1.6}$$

2. Group measure space construction

As a special case take $\mathcal{A} = L^\infty(X,\mu)$ where X is a standard Borel space and μ is a σ-finite measure, and let $\alpha_g(f) = f(\sigma_g^{-1}x)$ for an action $\sigma : g \to \mathrm{Aut}(X,[\mu])$, the Borel transformations of X preserving the measure class.

Definition 1. *σ is an ergodic action iff for every G-invariant Borelset $B \subseteq X$ either $\mu(B) = 0$ or $\mu(X\backslash B) = 0$.*

Definition 2. *σ is free if for μ-almost all $x \in X$ $g \to gx$ is a 1-to-1-map from G to X.*

Theorem 3 (Murray + von Neumann, $\approx$ 1940). *If σ is free and ergodic then $\mathcal{M} = L^\infty(X,\mu) \rtimes_\alpha G$ is a factor and $\mathcal{A} = L^\infty(X,\mu)$ is a MASA (maximal abelian selfadjoint subalgebra) in $\mathcal{M}$.*

Theorem 4 (Murray + von Neumann, $\approx$ 1940). *Assume a free and ergodic action*

- *$\mathcal{M}$ is a I_∞-factor iff Ω is infinite but countable*
- *$\mathcal{M}$ is a II_1-factor iff Ω is uncountable and there exist a G-invariant finite measure $\nu \in [\mu]$*
- *$\mathcal{M}$ is a II_∞-factor iff Ω is uncountable and there exists a G-invariant σ-finite, but not finite measure $\nu \in [\mu]$*

Date: 27/01/2010.

- *$\mathcal{M}$ is a III-factor iff Ω is uncountable and there does not exist a G-invariant σ-finite measure $\nu \in [\mu]$*

From now on we look at the II_1-factor case only.

Let (X, μ) be a uncountable standard Borel space with a probability measure, and let $\sigma : G \to \mathrm{Aut}(X, \mu)$ be a Borel transformation of X which leaves μ invariant. Now define $\mathcal{M} = L^\infty(X, \mu) \rtimes_\sigma G$. Then $\mathcal{A} = L^\infty(X, \mu)$ is a Cartan MASA in $\mathcal{M}$, where Cartan means that for $N(\mathcal{A}) = \{u \in U(\mathcal{M}) | u\mathcal{A}a^* = \mathcal{A}\}$ holds $N(\mathcal{A})'' = \mathcal{M}$.

Theorem 5 (Voiculescu $\approx$ 1995)**.** *For $2 \leq n < \infty$, $L(\mathbb{F}_n)$ has no Cartan MASA. Hence $L(\mathbb{F}_n)$ cannot be obtained by the group measure space construction.*

3. Equivalence relation

Assume $(\mathcal{A}_1 \subseteq \mathcal{M}_1)$ and $(\mathcal{A}_2 \subseteq \mathcal{M}_2)$ both obtained be group meausure space construction from $(X_i, \mu_i, \Gamma_i, \sigma_i)$ where the actions are free and ergodic.

We write $(\mathcal{A}_1 \subseteq \mathcal{M}_1) \cong (\mathcal{A}_2 \subseteq \mathcal{M}_2)$ iff there exists a von Neumann algebra isomorphism $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ with $\theta\mathcal{A}_1 = \mathcal{A}_2$.

Definition 6. (Γ_1, σ_1) *is* orbit equivalent *(OE) to* (Γ_2, σ_2) *if there exist nullsets $N_1 \subseteq X_i$ and a Borel isomorphism $\chi : X_1 \backslash N_1 \to A_2 \backslash N_2$ mapping σ_1-orbits onto σ_2-orbits.*

Theorem 7 (Singer 1955)**.** $(\mathcal{A}_1 \subseteq \mathcal{M}_1) \cong (\mathcal{A}_2 \subseteq \mathcal{M}_2)$ *iff* (Γ_1, σ_1) *is OE to* (Γ_2, σ_2).

Definition 8. *Let $\alpha : \Gamma \to Aut(X, \mu)$ and $x, y \in X$ then $x \sim_\alpha y$ iff x and y are in the same Γ-orbit, that is $y = \alpha_\gamma(x)$ for some $\gamma \in \Gamma$.*

Theorem 9 (Dye, 1959)**.** *Any two ergodic actions of $\mathbb{Z}$ on (X, μ) are OE.*

Theorem 10 (Ornstein + Weiss)**.** *If Γ_1, Γ_2 are amendable groups acting ergodicly on (X_i, μ_i) then the actions are OE.*

Theorem 11 (Connes + Weiss)**.** *If Γ has property T then it has at least two non-OE ergodic actions on (X_i, μ_i).*

Theorem 12 (Furmann, 1988)**.** *If Γ has an action on (X, μ) which is OE to the standard action of $SL(2, \mathbb{Z})$ on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ for $n \geq 3$ then $\Gamma \cong SL(n, \mathbb{Z})$.*

Definition 13. *Let $Y \subseteq X$ be Borel spaces and α a free and ergodic action of G on these. We define the equivalence relation R_α by $x \sim_\alpha y$ if x, y are in the same G-orbit, and $R_\alpha \subseteq X \times X$ by $R_\alpha = \{(x, y) | x \sim_a lphay\}$ and $R_{\alpha|Y} = R_\alpha \cap Y \times Y$.*

Observation 14. *If G is free and ergodic acting on (X, μ), and $Y, Z \subseteq X$ are Borel, with $\mu(Y) = \mu(Z) > 0$ then $R_{\alpha|Y} \sim_{OE} R_{\alpha|Z}$.*

Corollary 15. *Let $\mathcal{A} \subseteq \mathcal{M}$ come from (X, μ, G, α) with α free and ergodic. Let $p, q \in P(\mathcal{A})$ such that $\tau(p) = \tau(q)$ then $(p\mathcal{A} \subseteq p\mathcal{M}p) \cong (q\mathcal{A} \subseteq q\mathcal{M}q)$.*

4. Fundamental group op the equivalence relation

Definition 16. *Let $\mathcal{A} \subseteq \mathcal{M}$ and let $t \in]0, 1]$, choose $pt \in P(\mathcal{A})$ such that $\tau(p) = t$. Define $(\mathcal{A}_t \subseteq \mathcal{M}_t)$ by the isomorphism class of $(p_t\mathcal{A} \subseteq p_t\mathcal{M}p_t)$. For $t > 0$ arbitrary define $(\mathcal{A}_t \subseteq \mathcal{M}_t)$ as the isomorphism class of $(A \otimes D_n(\mathbb{C}))_{t/n} \subseteq (\mathcal{M} \otimes M_n(\mathbb{C}))_{t/n}$ where $D_n \subseteq M_n$ are the diagonal matrices and $n \geq t$.*

It has to be proven that the isomorphism class is independent of n.

Definition 17. *Put*

$$\begin{aligned}
\mathcal{F}(R_\alpha) &= \mathcal{F}(\mathcal{A} \subseteq \mathcal{M}) &\quad(4.1)\\
&= \{t > 0 | (\mathcal{A}_t \subseteq \mathcal{M}_t) \cong (\mathcal{A} \subseteq \mathcal{M})\} &\quad(4.2)\\
&= grp\{\tau(p) | p \in P(\mathcal{A}), (p\mathcal{A} \subseteq p\mathcal{M}p) \cong (\mathcal{A} \subseteq \mathcal{M})\} &\quad(4.3)\\
&= \{\frac{\tau(p)}{\tau(q)} | p, q \in P(A), (p\mathcal{A} \subseteq p\mathcal{M}p) \cong (q\mathcal{A} \subseteq q\mathcal{M}q)\} &\quad(4.4)
\end{aligned}$$

Theorem 18 (Gabriau, 2000 + 2002)**.** *Assume that G acts free and ergodis on (X, μ). If either*

- *one of the L^2-Betti numbers $\beta_k^{(2)}$ for $k \in \mathbb{N}$ is non-zero og*
- *the cost $C(G)$ of G is greater than 1*

then $\mathcal{F}(R_\alpha) = \{1\}$.

5. Cost of an equivalence relation

Let R_α be an equivalnece relation conming frim a free and ergodic action of G or such an equivalence relation cut down to a Borel set.

Definition 19. *A* graphing *of R_α is a countable family $\Phi = (\phi_i)_{i \in I}$ of partial Borel isomorphisms $\phi_i : A_i \to B_i$ where $A_i, B_i \subseteq X$ are Borel sets, satisfying that for $\phi(x) \sim_\alpha x$ for all $x \in A_j$ and that R_α is generated by $\{\phi_i(x) \sim_\alpha x | i \in I, x \in A_i\}$.*

Remark 20. *Under these conditions do ϕ_i preserve measures in particular $\mu(A_i) = \mu(B_i)$ for all $i \in I$.*

Definition 21. *The* cost *of a graphing Φ of R_α is defined as*

$$C(\Phi) = \sum_{i \in I} \mu(A_i) \tag{5.1}$$

and the cost of the equivalence relation R_α is defined as

$$C(R_\alpha) = \inf_{\Phi} C(\Phi) \tag{5.2}$$

where the infimum is taken over all graphings of R_α. We define by

$$C(\Gamma) = \inf_{\alpha} C(R_\alpha) \tag{5.3}$$

the cost of a group, where the infimum is taken over all free actions of Γ and Γ is said to have fixed price *if the cost is equal for all free actions α.*

Theorem 22 (Gabriau)**.** *The cost of a group is obtained by a relation (hence it is a minimum).*

Example 23 (Gabriau, 2000)**.**

- *For $2 \leq n \leq \infty$ we have $C(\mathbb{F}_n) = n$ and $\mathbb{F}_n$ has fixed prize.*
- *$C(SL(2, \mathbb{Z})) = \frac{13}{12}$ and it has fixed prize.*
- *$C(PSL(2, \mathbb{Z})) = \frac{7}{6}$ and it has fixed prize.*
- *If $\Lambda \subseteq \Gamma$ is a subgroup of finite index $[\Gamma : \Lambda] = |\Gamma/\Lambda| < \infty$ then $C(\Lambda) = 1 + [\Gamma : \Lambda](C(\Gamma) - 1)$. If Γ has fixed prize so has Λ.*
- *If $\Gamma = \Gamma_1 \star \Gamma_2$ (free product) and Γ_1, Γ_2 have fixed prize then $C(\Gamma) = C(\Gamma_1) + C(\Gamma_2)$ and fixed prize.*
- *If $|\Gamma| < \infty$ then $C(\Gamma) = 1 - \frac{1}{\Gamma}$*

- *If* $|\Gamma| = \infty$ *then* $C(\Gamma) \geq 1$
- *If* Γ *is infinite amendable then* $C(\Gamma) = 1$

Remark 24. *The cost of the free groups above can be obtained from them beeing subrgoups of finite index in* $\mathbb{F}_2$*, which again is a subgroup of finite index in* $SL(2,\mathbb{Z})$ *respective* $PSL(2,\mathbb{Z})$*.*

Theorem 25 (Gabriau, 2000)**.** *For* $Y \subseteq X$ *Borel,* $\mu(Y) > 0$ *there holds* $C(R_{\alpha|Y}) - 1 = \frac{1}{\mu(Y)}(C(R_\alpha) - 1)$*.*

Theorem 26 (Gabriau)**.** *If* Γ *acts freely and ergodicly on* (X,μ) *and* $C(\Gamma) > 1$ *then* $\mathcal{F}(R_\alpha) = \mathcal{F}(\mathcal{A} \subseteq \mathcal{M}) = \{1\}$*.*

Proof. Let $\alpha : \Gamma \to \mathrm{Aut}(X,\mu)$ be a free ergodic action. Then $1 < C(\Gamma) \leq C(R_\alpha)$. Now let $t \in]0,1[$ and choose $Y \subseteq X$ Borel set such that $\mu(Y) = t$ hence

$$C(R_{\alpha|Y}) = 1 + \frac{1}{\mu(Y)}(C(R_\alpha) - 1) > C(R_\alpha). \tag{5.4}$$

But OE actions have the same cost hence $R_{\alpha|Y} \not\sim R_\alpha$ and a result of [Singer, 1955] gives us $(p\mathcal{A} \subseteq p\mathcal{M}p) \ncong (\mathcal{A} \subseteq \mathcal{M})$ when $p \in P(\mathcal{A})$ with $\tau(p) = t$.

As this holds for all $t \in]0,1[$ we have $\mathcal{F}(R_\alpha) \cap]0,1[= \emptyset$ and hence (as $\mathcal{F}$ is a group) we have $\mathcal{F}(R_\alpha) = \{1\}$. □