UFFE HAAGERUPS SECOND TALK AT MASTERCLASS ON VON NEUMANN ALGEBRAS AND GROUP ACTIONS

Let G be a discrete group and $\lambda : G \to B(l^2(G))$ its left regular representation. Then

$$L(G) = \lambda(G)'' = \overline{\text{span}\{\lambda(g)|g \in G\}}^{SO} \tag{0.1}$$

and

$$C_r^*(G) = \overline{\text{span}\{\lambda(g)|g \in G\}}^{\|\cdot\|} \tag{0.2}$$

and $C^*(G)$ is the full / universal C*-algebra of G.

$(l^1(G), \star)$ and $\pi : G \to B(H)$ unitary and $\tilde{\pi} : l^1(G) \to B(H)$, where

$$\tilde{\pi}(f) = \sum_{g \in G} f(g)\pi(g). \tag{0.3}$$

Then $C^*(G)$ is the completion of $l^1(G)$ in the norm $\|f\|_u = \sup\{\|\tilde{\pi}(f)\|, \pi \text{ unitary representation of } G\}$, and there exists a canonical surjection from $C^*(G)$ onto $C_r^*(G)$, i.e. $C_r^*(G)$ is a quotient of $C^*(G)$.

Theorem 1. *If G is a discrete group TFAE*

- *G amendable*
- *$L(G)$ injective*
- *$C_r^*(G)$ nuclear C^*-algebra*
- *$C^*(G)$ nuclear C^*-algebra*
- *The canonical surjection $C^*(G) \to C_r^*(G)$ is an isomorphism*
- *There exists a net (ϕ_α) of positiv definite functions from G to $\mathbb{C}$ such that supp(ϕ_α) is finite and $\lim_\alpha \phi_\alpha(g) = 1$ for alle $g \in G$.*

Definition 2. *A function $\phi : G \to \mathbb{C}$ is* positive definite *if $\forall n \in \mathbb{N} \forall g_1, \dots, g_n \in G : (\phi(g_i^{-1} g_j)_{i,j}) \in M_n(\mathbb{C})^+$ that is $\forall c_1, \dots c_n \in \mathbb{C} : \sum_{i,j=1}^n c_i \bar{c}_j \phi(g_i^{-1} g_j) \geq 0$.*

Observation 3. *There is a 1-1-corespondence between states on $C^*(G)$ and positive definite functions ϕ with $\phi(e) = 1$.*

Theorem 4 (Haagerup, 1979)**.** *If $2 \leq n < \infty$ and $x \to |x|$ is the wordlength of $x \in \mathbb{F}_n$, then $x \to \exp(-\lambda|x|)$ is a positive definite functions for all $\lambda > 0$.*

Corollary 5. *There exists positive definite functions $\phi_m : \mathbb{F}_m \to \mathbb{C}$ such that $\phi_n(e) = 1$, $\phi_n \in C_0(\mathbb{F}_m)$ and $\phi_n(g) \to 1$ for all $g \in G$.*

Bevis. Use $\phi_n(x) = \exp(-\frac{|x|}{n})$. □

Date: 26/01/2010.

1. Property H and property T

Definition 6 (Connes). *G has property H (Haagerup property) iff there exists a sequence (ϕ_n), $\phi_n : G \to \mathbb{C}$, positive definite, $\phi_n(e) = 1$, $\phi_n \in C_0(G)$ and $\phi_n(g) \to 1$ for all $g \in G$.*

Amenability implies property H, but the converse is false and the free groups are counterexamples.

Definition 7. *G has Kazhdan's property T if the trivial representation $\tau : G \to \{1\} \in M_1(\mathbb{C})$ is an isolated point in the spectrum of $C^*(G)$. Let $\tilde{\tau} : C^*(G) \to M_1(\mathbb{C}) = \mathbb{C}$ be a multiplicative functional, then kernel($\tilde{\tau}$) is a maximal ideal in $C^*(G)$, and kernel($\tilde{\tau}$) is an isolated point in the hull/kernel topology on the primitive ideals of $C^*(G)$.*

Proposition 8. *G has property T iff for all nets (ϕ_α) of positive definite functions on G such that $\phi_\alpha(e) = 1$ and $\phi_\alpha(g) \to 1$ for all $g \in G$ you have that $\|\phi_\alpha - 1\|_\infty \to 0$.*

Observation 9. *G has both property H and T iff G is finite.*

2. Exact groups

Definition 10 (Kirchberg). *A C^*-algebra A is* exact *if for all pairs of C^*-algebras $J \lhd B$, where J is a closed two-sided ideal of B,*

$$0 \to A \otimes_{\min} J \to A \otimes_{\min} B \to A \otimes_{\min} B/J \to 0 \tag{2.1}$$

is an exat sequence.

Definition 11 (Kirchberg + S. Wasserman). *G is called* exact *if $C_r^*(G)$ is an exact C^*-algebra.*

Remark 12. *$\mathbb{F}_n$ is exact (i.e. $C_r^*(\mathbb{F}_n)$ is exact) for $n \geq 2$, but $C^*(\mathbb{F}_n)$ is not exact.*

Definition 13. *A function $\chi : X \times X \to \mathbb{C}$ is of* positive type *if $(\chi(x_i, x_j)_{i,j}) \in M_n(\mathbb{C})^+$ for all $n \in \mathbb{N}, x_1, \dots x_n \in X$.*

Theorem 14 (Ozawa, 2000). *G is exact iff there exists a net χ_α of functions $\chi_\alpha : G \times G \to \mathbb{C}$ of positive type such that $\chi_\alpha(x, x) = 1$ for all $x \in G$ and*

$$F_\alpha = \{y^{-1}x | (x, y) \in supp(\chi_\alpha)\} \tag{2.2}$$

is finite for all α and $\chi_\alpha(x, y) \to 1$ for all $x, y \in G$.

Amenability implies exactness.

Group	Amenable	Property H	Property T	Exact
finite	yes	yes	yes	yes
inf. amendable	yes	yes	no	yes
$\mathbb{F}_n \quad n \geq 2$	no	yes	no	yes
$SL(2, \mathbb{Z})$	no	yes	no	yes
$SL(n, \mathbb{Z}) \quad n \geq 3$	no	no	yes	yes
$SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$	no	no	no	yes

Unknown if property H implies exact, but the converse is false.

Theorem 15 (Connes + Kirchberg, 1991). *Every discrete subgroup of a connected locally compact group is exact.*

$$SL(n,\mathbb{Z}) \hookrightarrow SL(n,\mathbb{R}) \tag{2.3}$$

and $SL(n,\mathbb{R})$ is a connected Lie group.

$$\mathbb{F}_2 \hookrightarrow SL(2,\mathbb{Z}) \hookrightarrow SL(n,\mathbb{Z}) \tag{2.4}$$

for instance is $\mathbb{F}_2$ generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \tag{2.5}$$

in $SL(2,\mathbb{Z})$.

3. Gromov's uniform embedability in a Hilbert space

If G is a finite group and $S \subseteq G$ is a set of generators of G, such that $S = S^{-1}$ and we for all $g \in G$ define

$$l(g) = \min\{k | g = s_1 S_2 \dots s_k, s_i \in S\} \tag{3.1}$$

and $d(x,y) = l(y^{-1}x)$ is an invariant metric on G.

Definition 16. *G is* uniformly embedable *in a Hilbert space iff there exists $\phi : G \to H$, where H is a Hilbert space and functions $f, g : [0,\infty[\to [0,\infty[$ such that $f(t) \to_{t\to\infty} \infty$ and $g(t) \to_{t\to\infty} \infty$ and $f(d(k,h)) \leq \|\phi(g) - \phi(h)\| \leq g(d(k,h))$ for all $k, h \in G$.*

Theorem 17 (Ozawa)**.** *G exact implies that G is embedable in a Hilbert space.*

Theorem 18 (Gomov)**.** *There exists a finitly generated group G which is not embedable in a Hilbert space, and hence not exact.*

Observation 19. *Amenability, property H, property G and exactness of a group G can all be characterized in terms of $L(G)$.*

- *G amenable iff $L(G)$ injective*
- *G property H iff $L(G)$ has property H*
- *G property T iff $L(G)$ has propert T*

Definition 20 (Connes + Choda ≈ 1982/1983)**.** *If $\mathcal{M}$ is a II_1-factor then $\mathcal{M}$ has* property H *iff there exists a net (T_α) of completely positive maps $T_\alpha : \mathcal{M} \to \mathcal{M}$ such that $T_\alpha(1) \leq 1$, $\tau \circ T_\alpha \leq \tau$, and the extension $\tilde{T}_\alpha$ of T_α to $L^2(\mathcal{M},\tau)$ is compact ($T_\alpha \in K(L^2(\mathcal{M},\tau))$) and $\|T_\alpha(x) - x\|_2 \to 0$ for all $x \in \mathcal{M}$ ($\|x\|_2 = \tau(x^*x)^{1/2}$).*

Definition 21 (Connes + Jones, 1985)**.** *If $\mathcal{M}$ is a II_1-factor and K is a $\mathcal{M}$-$\mathcal{M}$-bimodule and (S_n) a sequence of unit vectors in K then (S_n) is almost central iff $\|xS_n - S_n x\| \to 0$ for all $x \in \mathcal{M}$, and (S_n) is almost tracial iff $\| < \cdot S_n, S_n > -\tau\| \to 0$ and $\| < S_n \cdot, S_n > -\tau\| \to 0$.*

$\mathcal{M}$ has property T iff every normal Hilbert $\mathcal{M}$-$\mathcal{M}$-bimodule K which admits an almost central and almost tracial sequence (S_n) of unitvectors contains a central unit vector S (i.e. $xS = Sx$ for all $x \in \mathcal{M}$).

Ozawa introduced in 2008 *weak exactnes* of von Neumann algebras and proved G exact iff $L(G)$ weak exact.

4. The Thompson groups F and T

Definition 22. *The group F is the group of order preserving homeomorphisms of $[0,1]$ which are piecewise linear and slopes are in $2^{\mathbb{Z}}$ and breakpoints are in $\mathbb{Z}[\frac{1}{2}] = \{\frac{k}{2^n} | k \in \mathbb{Z}, n \in \mathbb{N}\}$ the dyadic numbers.*

The group T is the same on $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, where $g \in T$ iff $g = \tau_a \circ h$ for $h \in F$ and τ_a is the translation by $a \in \mathbb{Z}[\frac{1}{2}]$.

Theorem 23.

$$F = < A, B | [AB^{-1}, A^{-1}BA] = [AB^{-1}, A^2BA^{-2}] = 1 > \tag{4.1}$$

and T has a similar representation with three generators and six relations.

F is not simple, as for the commutator subroup $F' = [F, F]$, we have $F/F' \cong \mathbb{Z}^2$ and F' is simple and not finitely generated.

T is simple, and was the first example of an infinitely presented simple group.

Problem 24. *Is F amenable?*

Recent attempts of A. Akhmedov from february 2009 concluding NO with a gap in the proof and E.T. Shavgulidez from march 2009 concluding YES with a gap in the proof.

5. The classes of groups EA, AG and NF

Definition 25. *The class EA of elementary amenable groups is the smalles class of groups*

- *containing all finite groups*
- *containing all abelian groups*
- *closed under taking subgroups, quotients and extensions*
- *closed with respect to inductive limits*

Definition 26. *The class AG is the class of all amenable groups.*

Definition 27. *The class NF is the class of all groups not containing a copy of $\mathbb{F}_2$.*

$$EA \subsetneq AG \subsetneq NF \tag{5.1}$$

Where strictness of the first inequality is due to Grigorshuk in 1988 and strictness of the second inequality is due to Olshanski and Sapir in 2002.

It is known that $F \notin EA$ but $F \in NF$, so $F \in NF \backslash EA$ so either $F \in AG \backslash EA$ or $F \in NF \backslash AG$ but which one is not known.

Property	F	T
amenable	?	no (as $\mathbb{F}_2 \hookrightarrow T$)
exact	?	?
property H	yes	yes [Farley, 2003]
property T	no	no [Farley, 2003]

G has property H iff G has Gromov's aT-amenability, at this implies that it satisfies the Baum-Connes conjecture with coefficients.

Definition 28. *A II_1-factor $\mathcal{M}$ is a* McDuff-factor *iff $\mathcal{M} \cong \mathcal{M} \otimes \mathcal{R}$, where $\mathcal{R}$ is the hyperfinite II_1-factor.*

$\mathcal{R}$ is a McDuff-factor.

Definition 29. *A sequence* $(x_n) \in \mathcal{M}$ *is a central sequence if* $\sup \|x_n\| < \infty$ *and* $\|ax_n - x_n a\|_2 \to 0$ *for all* $a \in \mathcal{M}$ *and it is nontrivial if* $\|x_n - \tau(x_n)1\|_2 \nrightarrow 0$.

$L(\mathbb{F}_n)$ are not McDuff-factors, as they have no nontrivial central sequences.

Theorem 30 (Jolissaint, 1998). *$L(F)$ and $L(F')$ are McDuff-factors.*

Theorem 31 (Jolissaint, 2006). *$L(F')$ is asymptotically abelian, i.e. there exists a sequence $(\alpha_n) \subseteq Aut(L(F'))$ such that $\|\alpha_n(x)y - y\alpha_n(x)\|_2 \to 0$ for all $x, y \in L(F')$.*

Theorem 32 (Haagerup + Picioroga, to appear J.O.T.). *For $n \in \mathbb{N}, n \geq 4$*

(5.2)

$$F =< g_0, \dots, g_n | g_{i-1}g_i g_{i+1} = g_i g_{i+1} g_{i-1} g_i, 1 \leq i \leq n-1, g_i g_j = g_j g_i, |i-j| \geq 2 >$$

Theorem 33 (Haagerup + Picioroga). *$C^*(F)$ and $C_r^*(F)$ are not residually finite, i.e. they do not embed in $\prod_{k=1}^{\infty} M_{n_k}(\mathbb{C}) = (\bigoplus M_{n_k}(\mathbb{C}))_{l^\infty}$.*

Remark 34. *$C^*(\mathbb{F}_2)$ is residually finite [Choi] and $C_r^*(\mathbb{F}_2)$ is not residually finite, because it is simple.*