# UFFE HAAGERUPS FIRST TALK AT MASTERCLASS ON VON NEUMANN ALGEBRAS AND GROUP ACTIONS

## 1. Von Neumann Algebras

**Definition 1.** *Let $H$ be a Hilbert space (usually finite dimensional and separable), and let $B(H)$ be the *-algebra of bounded operators on $H$. $\mathcal{M} \subseteq B(H)$ is a* von Neumann Algebra *iff*

- *$\mathcal{M}$ is a *-subalgebra of $B(H)$*
- *$1 \in \mathcal{M}$*
- *$\mathcal{M}$ is SO-closed*

**Theorem 2** (von Neumann, 1929)**.** *$\mathcal{M} \subseteq B(H)$ is a von Neumann algebra iff $\mathcal{M} = \mathcal{M}^*$ and $\mathcal{M} = \mathcal{M}''$.*

**Definition 3.** *The* center *of $\mathcal{M}$ is denoted by $Z(\mathcal{M})$ and defined as $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$.*

*$\mathcal{M}$ is a* factor *iff $Z(\mathcal{M}) = \mathbb{C} \cdot 1$.*

| Type of factor $\mathcal{M}$ | Model / characterization |
|---|---|
| $I_n, \quad n \in \mathbb{N}$ | $M_n(\mathbb{C})$ |
| $I_\infty$ | $B(H), \quad \dim(H) = \infty$ |
| $II_1$ | $\dim(\mathcal{M}) = \infty, \quad \exists$ tracial state $\tau : \mathcal{M} \to \mathbb{C} \forall a, b \in \mathcal{M} : \tau(ab) = \tau(ba)$ |
| $II_\infty$ | $\mathcal{M} \cong \mathcal{N} \bar{\otimes} B(H), \quad \dim(H) = \infty, \quad \mathcal{N} \quad II_1 -$ factor |
| $III$ | Everything else |

If $H$ is a separable Hilbert space and $\mathcal{M} \subseteq B(H)$ a von Neumann algebra, then

$$\mathcal{M} = \int_\Omega^\oplus \mathcal{M}(\omega) d\omega \tag{1.1}$$

where each $\mathcal{M}(\omega)$ is a factor. This is the desintegration of $\mathcal{M}$ in factors.

## 2. Projections in a von Neumann algebra

**Definition 4.** *If $\mathcal{M}$ is a von Neumann algebra, we set*

$$P(\mathcal{M}) = \{p \in \mathcal{M} | p = p^* = p^2\} \tag{2.1}$$

*the set of* projections *in $\mathcal{M}$.*

**Theorem 5.**

$$\mathcal{M} = \overline{span(P(\mathcal{M}))}^{\|\cdot\|} \tag{2.2}$$

**Definition 6.** *For $p, q \in P(\mathcal{M})$ we say $p \sim q$ iff $\exists u \in \mathcal{M} : p = u^*u, q = uu^*$.*

**Definition 7.** *$\mathcal{M}$ is* finite *iff $(p \in P(\mathcal{M}) : p \sim 1 \Rightarrow p = 1)$.*

*$\mathcal{M}$ is* properly infinite *iff $\exists p, q \in P(\mathcal{M}) : p \perp q, p \sim q \sim 1$ In a von Neumann algebra (but not in a general C*-algebra), this is the same as $\exists p, q \in P(\mathcal{M}) : p \perp q, p + q = 1, p \sim q \sim 1$.*

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Type $I_n$ and type $II_1$ factors are finite, type $I_\infty$, type $II_\infty$ and type $III$ factors are properly infinite.

## 3. Traces

A $II_1$-factor $\mathcal{M}$ has a unique trace state $\tau$:

- $\tau$ is normal
- $\tau$ is faithful $((a \geq 0, \tau(a) = 0) \Rightarrow a = 0)$
- $\forall p, q \in P(\mathcal{M}) : p \sim q \Leftrightarrow \tau(p) = \tau(q)$
- $\tau(P(\mathcal{M})) = [0, 1]$

A $I_\infty$-factor or $II_\infty$ factor $\mathcal{M}$has a normal faithful semifinite trace $\tau$ defined on $\mathcal{M}^+$ (Semifinite means that the set $\{a \in \mathcal{M} | \tau(a * a) < \infty\}$ is SO-dense in $\mathcal{M}$.)

- $\tau$ is unique up to multiplication by $\lambda \in ]0, \infty[$
- $\forall a \in \mathcal{M} : \tau(a^*a) = \tau(aa^*)$
- If $\mathcal{M}$ is a type $II_\infty$-factor, $\tau(P(\mathcal{M})) = [0, \infty]$

## 4. group von Neumann Algebras

**Definition 8.** *If $G$ is a discrete group (usually countable).Then we define $L(G)$ as the von Neumann algebra generated by the left regular representation of $G$. For this define $\lambda : G \to B(l^2(G))$ as $\forall f \in l^2(G) \forall x, y \in G : (\lambda(x)f)(y) = f(x^{-1}y)$. Set*

$$L(G) = \overline{span\{\lambda(x) | x \in G\}}^{SO} \tag{4.1}$$

*and*

$$C_r^*(G) = \overline{span\{\lambda(x) | x \in G\}}^{\|\cdot\|}. \tag{4.2}$$

*Observe $C_r^*(G) \subseteq L(G)$.*

$L(G)$ is a finite von Neumann algebra.

**Definition 9.** *A group $G$ is an* ICC-group *iff $\forall g \in G, g \neq e : \{hgh^{-1} | h \in H\}$ is infinite.*

**Theorem 10.** *Let $G \neq \{e\}$ then TFAE*

- *$G$ is an ICC-group*
- *$L(G)$ is a factor*
- *$L(G)$ is a $II_1$-factor*

## 5. Injectivity and hyperfiniteness of von Neumann algebras

**Definition 11.** *$\mathcal{M} \subseteq B(H)$ is* injective *iff $\exists p \in P(B(H)), \|p\| = 1$ of $B(H)$ onto $\mathcal{M}$.*

**Definition 12.** *$\mathcal{M}$ is* hyperfinite *(or AFD) on a separable Hilbert space iff $\mathcal{M} = \cup_{n=1}^{\infty} \mathcal{M}_n$ where $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \dots$ are finite dimensional von Neumann algebras.*

**Theorem 13** (Connes, 1976)**.** *If $H$ separable, and $\mathcal{M} \subseteq B(H)$. Then $\mathcal{M}$ is injective iff $\mathcal{M}$ is hyperfinite.*

**Theorem 14** (Murray + von Neumann, 1940)**.** *There is up to isomorphism only one hyperfinite factor of type $II_1$ on a separable Hilbert space.*

## 6. Amendable groups

**Definition 15.** *A discrete group $G$ is* amendable *if $\exists \mu : \mathcal{P}(G) \to [0,1]$ finite additive measure with $\mu(G) = 1$ such that $\forall A \subseteq G \forall x \in G : \mu(xA) = \mu(A)$.*

**Observation 16.** *$G$ is amendable iff $\exists$ state $m : l^\infty(G) \to \mathbb{C}$ such that $\forall f \in l^\infty(G) \forall x \in G : m(xf) = m(f)$ where $\forall x, y \in G : (xf)(y) = f(x^{-1}y)$.*

**Theorem 17** (Sakai, Connes)**.** *If $G$ is a discrete countable group TFAE*

- *$G$ amendable*
- *$L(G)$ is injective*
- *$L(G)$ is hyperfinite*

*where the equivalence of the first two statements is old, and does not require countability.*

**Remark 18.** *$G$ amendable iff $C_r^*(G)$ nuclear.*

**Corollary 19.** *If $G$ is a amendable, countable ICC-group then $L(G) \cong \mathcal{R}$ the unique hyperfinite $II_1$-factor.*

**Definition 20.** *A group $G$ is* solvable *iff $\exists G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$ such that $G_{k+1}/G_k$ is abelian.*

**Definition 21.**

$$\mathbb{F}_n = \text{ group generated by } \{g_1, \ldots, g_n | \text{ no relations } \} \tag{6.1}$$

$$SL(n, \mathbb{Z}) = \{g \in M_n(\mathbb{Z}) | \det(g) = 1\} \tag{6.2}$$

$$PSL(n, \mathbb{Z}) = \begin{cases} SL(n, \mathbb{Z}) & n \text{ odd} \\ SL(n, \mathbb{Z})/\{\pm 1\} & n \text{ even} \end{cases} \tag{6.3}$$

$$S_n = \{ \text{ permutations of } \{1, \ldots, n\}\} \tag{6.4}$$

$$S_\infty = \bigcup_{n=1}^{\infty} S_n \tag{6.5}$$

| Group | Amendable |
|---|---|
| finite groups | yes |
| abelian groups | yes |
| solvable groups | yes |
| inductive limits of amendable groups | yes |
| $S_\infty$ | yes |
| $\mathbb{F}_n, \quad 2 \leq n \leq \infty$ | no |
| $SL(n, \mathbb{Z}), \quad 2 \leq n$ | no |
| $PSL(n, \mathbb{Z}), \quad 2 \leq n$ | no |

**Remark 22.** *$PSL(n, \mathbb{Z})$ is an ICC-group.*

**Remark 23.** *$L(\mathbb{F}_n) \not\cong L(S_\infty) \cong \mathcal{R}$ for $2 \leq n \leq \infty$ as $L(\mathbb{F}_n)$ is not injective, but $\mathbb{F}_n, S_\infty$ are ICC-groups, so $L(\mathbb{F}_n)$ is a $II_1$-factor not hyperfinite.*

## 7. The fundamental group of $II_1$-factors

**Recall 24.** *Let $\mathcal{M}$ be a $II_1$-factor, then*

- *$\mathcal{M}$ has unique trace state $\tau$*
- $p, q \in P(\mathcal{M}) : p \sim q \Leftrightarrow \tau(p) = \tau(q)$
- $\forall t \in [0,1] \exists p_t \in P(\mathcal{M}) : t = \tau(p_t)$

**Observation 25.** *$p, q \in P(\mathcal{M}) : p \sim q \Rightarrow p\mathcal{M}p \cong q\mathcal{M}q$ as von Neumann algebras on $pH$ and $qH$ and these are $II_1$-factors too.*

**Definition 26.** *If $t \in ]0,1]$ put $\mathcal{M}_t = p_t \mathcal{M} p_t$ which is welldefined up to isomorphism.*

*For general $t \in ]0,\infty[$ define $\mathcal{M}_t = M_n(\mathcal{M})_{t/n}$ where $n \in \mathbb{N}, n \geq 2$. (Have to check that $\mathcal{M}_t$ doesn't depend on $n$ up to isomorphism.*

**Proposition 27.**

$$(\mathcal{M}_s)_t = \mathcal{M}_{st} \qquad \forall s,t > 0 \tag{7.1}$$

*and*

$$\mathcal{M}_n = M_n(\mathcal{M}) = \mathcal{M} \otimes M_n(\mathbb{C}) \qquad \forall n \in \mathbb{N} \tag{7.2}$$

**Definition 28.** *The* fundamental group *of $\mathcal{M}$ is the set*

$$\mathcal{F}(\mathcal{M}) = \{t \in ]0,\infty[ | \mathcal{M}_t \cong \mathcal{M}\} \tag{7.3}$$

**Remark 29.** *$\mathcal{F}(\mathcal{M})$ is a multiplicative subgroup of $(],\infty[,\cdot)$ and*

$$\mathcal{F}(\mathcal{M}) \cap [0,1] = \{\tau(p) | p \in P(\mathcal{M}) : p\mathcal{M}p \cong \mathcal{M}\} \tag{7.4}$$

*and $\mathcal{F}(\mathcal{M})$ is the multiplicative group generated by $\{\tau(p) | p \in P(\mathcal{M}) : p\mathcal{M}p \cong \mathcal{M}\}$ and*

$$\mathcal{F}(\mathcal{M}) = \{\frac{\tau(p)}{\tau(q)} | p, q \in P(\mathcal{M}) : p\mathcal{M}p \cong q\mathcal{M}q\} \tag{7.5}$$

**Observation 30.** *$M_n(\mathcal{M}) \cong \mathcal{M}$ iff $n \in \mathcal{F}(\mathcal{M})$ and $(\forall n \in \mathbb{N} : M_n(\mathcal{M}) \cong \mathcal{M})$ iff $\mathbb{Q}^+ \subseteq \mathcal{F}(\mathcal{M})$.*

**Example 31** (Murray + von Neumann $\approx$ 1940)**.** *If $\mathcal{R}$ is the hyperfinite $II_1$-factor, $\mathcal{F}(\mathcal{R}) = ]0,\infty[$.*

**Theorem 32** (Connes $\approx$ 1975)**.** *If $G$ has Kazhdan's property T (e.g. $G \cong PSL(n,\mathbb{Z}), n \geq 3$) then $\mathcal{F}(L(G))$ is countable, in particular $\mathcal{F}(L(G)) \subsetneq ]0,\infty[$.*

**Theorem 33** (Radulescu + Dykema $\approx$ 1994)**.** $\mathcal{F}(L(\mathbb{F}_\infty)) = ]0,\infty[$

**Problem 34.** *Compute $\mathcal{F}(L(\mathbb{F}_n))$ for $2 \leq n < \infty$.*

Using free probability one can prove [Radulescu + Dykema] that $L(\mathbb{F}_n) \cong L(\mathbb{F}_2)_{t_n}$ where $t_n = \frac{1}{\sqrt{n-1}}$. From this follows that $\mathcal{F}(L(\mathbb{F}_n)) = \mathcal{F}(L(\mathbb{F}_2))$ for $2 \leq n < \infty$.

**Theorem 35** (Radulescu + Dykema)**.** *Either*

- *$\mathcal{F}(L(\mathbb{F}_2)) = \{1\}$ in which case $L(\mathbb{F}_2), L(\mathbb{F}_3), \ldots, L(\mathbb{F}_\infty)$ are all nonisomorphic $II_1$-factors or*
- *$\mathcal{F}(L(\mathbb{F}_2)) = ]0,\infty[$ in which case $L(\mathbb{F}_2), L(\mathbb{F}_3), \ldots, L(\mathbb{F}_\infty)$ are all isomorphic $II_1$-factors or*

## 8. Interpolated free group factors

**Definition 36.** *For $t \in ]1, \infty[$ define $L(\mho_t) = L([F_2)_{\frac{1}{\sqrt{t-1}}}$.*

**Remark 37.** *$L(\mathbb{F}_{\frac{7}{6}} \cong L(PSL(2,\mathbb{Z})) \cong L(\mathbb{Z}_2 \star \mathbb{Z}_3)$ hence $\mathcal{F}(L(PSL(2,\mathbb{Z})) = \mathcal{F}(L(\mathbb{F}_2))$.*

**Theorem 38** (Sorin + Popa, 2004-2006)**.** *There exists a $II_1$-factor $\mathcal{M}$ on a separable Hilbert space such that $\mathcal{F}(\mathcal{M}) = \{1\}$. (e.g. $\mathcal{M} = L(\mathbb{Z}_2 \rtimes SL(2,\mathbb{Z}))$)*

**Theorem 39** (Popa, Popa + Adrian + Petersen)**.** *For all countable subgroups $\Gamma \subseteq ]0, \infty[$ there exists a $II_1$-factor $\mathcal{M}$ such that $\{(\mathcal{M}) = \Gamma$.*

**Theorem 40** (Popa + Vaes, 2008)**.** *There exists a $II_1$-factor $\mathcal{M}$ such that $\mathcal{F}(\mathcal{M}) \subsetneq ]0, \infty[$ but $\mathcal{F}(\mathcal{M})$ is not countable.*

## 9. The "Group measure space"construction

**Definition 41.** *Let $\mathcal{A} \subseteq B(H)$ be a von Neumann algebra and $G$ be a discrete group and $\alpha : G \to Aut(\mathcal{A})$ be an action of $G$ on $\mathcal{A}$ (a group homomorphism).*

*Then the crossed product $\mathcal{M} = \mathcal{A} \rtimes_\alpha G$ acts on $l^2(G, H)$ and $M = \{\pi(\mathcal{A}) \cup \lambda(G)\}''$ where $\pi : \mathcal{A} \to B(l^2(G, H))$ is given by*

$$(\pi(a)\xi)(g) = \alpha_g^{-1}(a)\xi(g) \qquad \xi \in l^2(G, H), g \in G \tag{9.1}$$

*where*

$$l^2(G, H) \cong \bigoplus_{g \in G} H \tag{9.2}$$

*and $\lambda : G \to B(l^2(G, H))$ is given by*

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h) \qquad \xi \in l^2(G, H), g, h \in G. \tag{9.3}$$

$(\pi, \lambda)$ is a ??? representation of $(\mathcal{A}, G, \alpha)$:

$$\lambda(g)\pi(a)\lambda(g)^{-1} = \pi(\alpha_g(a)) \qquad a \in \mathcal{A}, g \in G. \tag{9.4}$$

and $\mathcal{A} \cong \pi(\mathcal{A}) \subseteq \mathcal{A} \rtimes_\alpha G$ and $\mathcal{A} \rtimes -\alpha G$ is generated by $\mathcal{A}$ and $\lambda(g)$ for $g \in G$ and

$$\lambda(g)a\lambda(g)^{-1} = \alpha_g(a). \tag{9.5}$$

## 10. Special case

If $H$ is a seperable Hilbert space and $\mathcal{A}$ is abelian then $\mathcal{A} \cong L^\infty(\Omega, \mu)$ for some standard Borel space $\Omega$ and $\sigma$-finite measure $\mu$.

Note that $L^\infty(\Omega, \mu) \cong L^\infty(\Omega, \nu)$ when $\nu \in [\mu]$ that is when $\mu$ and $\nu$ have the same null sets ($\nu \ll \mu$ and $\mu \ll \nu$).

Given a group homomorphism $\sigma : G \to \mathrm{Iso}(\Omega, [\mu])$, that is a $[\mu]$-preserving Borel isomorphism of $\Omega$), one can associate an action $\alpha : G \to \mathrm{Aut}(L^\infty(\Omega, \mu))$ by

$$\alpha(f)(\omega) = f(\sigma^{-1}\omega) \qquad f \in L^\infty(\Omega, \mu). \tag{10.1}$$

Then $\mathcal{M} = L^\infty(\Omega) \rtimes_\alpha G$ is called the von Neumann algebra obtained from the "Group measure space-construction.