A Noncommutative Gauss Map

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- We write $\alpha = [a_1, a_2, ...]$ in it's **continued fraction decomposition**.
- This provides a nice, algorithmic approximation of irrational numbers by rational numbers.

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$$e_n(x) := \left| m_n(x) - \frac{\ln(1+x)}{\ln 2} \right|$$

for large n.

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- (Wirsing 74) There is an optimal constant $q \sim .303$ such that $e_n(x) \leq q^n$.
- Why would Gauss say a good estimate would be "very desirable"?

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  $= m_n(\frac{1}{2}) - m_n(\frac{1}{3}) \sim \frac{\ln(9/8)}{\ln 2} + (.303..)^n.$

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- We recall candidates for "noncommutative irrational numbers"
- And for the "noncommutative unit interval."

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- Let's think of $\mathcal{C}_\theta$ as a "noncommutative irrational number."

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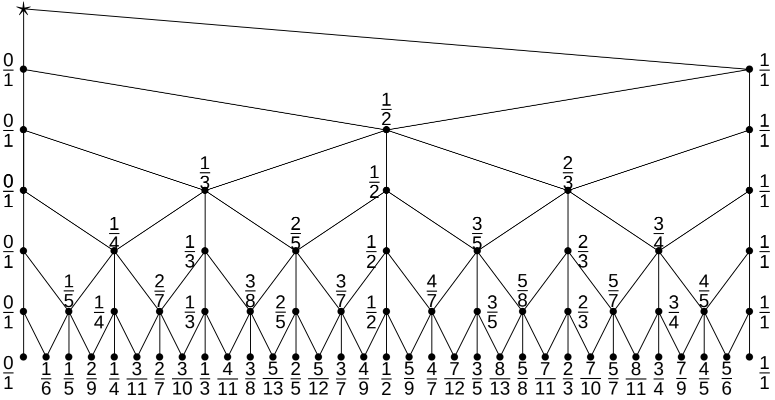

Figure: Bratelli diagram of $\mathfrak{A}$

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- So if we (incorrectly) view $\mathfrak{A}$ as operator valued continuous functions on its maximal ideal space..then each function evaluated at θ takes values in the Effros Shen algebra $\mathcal{C}_\theta$.
- For these reasons, we feel it is reasonable to attach the moniker "noncommutative unit interval" to $\mathfrak{A}$.

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- These are the properties we want our noncommutative extension to inherit.

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We let τ_μ be the unique tracial extension of μ to $\mathfrak{A}$.

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- Let's outline this extension for $s = 1$.

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Extension of Composition of g_1

The map $g_1 : [0,1] \to [\frac{1}{2}, 1]$ shrinks $[0,1]$ in half and then flips it. We want to imitate this on the Bratelli diagram of $\mathfrak{A}$:

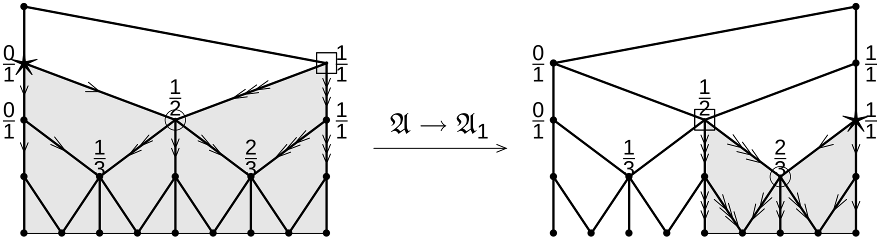

Figure: Bratelli Diagram of $\mathfrak{A}$.....Bratelli Diagram of Quotient of $\mathfrak{A}$

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- For this reason, we define a CP map that preserves as much trace as possible with induced map an L^2-isometry..

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$$\widetilde{\mathbb{G}} = \sum_{s=1}^{\infty} \widetilde{G}_s \frac{1+\theta}{(\theta+s)(\theta+s+1)}.$$

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5. $\phi(x) = \tau_\mu(\widetilde{\mathbb{G}}(x))$ *for* $x \in \mathfrak{A}$.

Mange tak!