Continuous spectrum of automorphism groups

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Motivation.

- $U(t) = e^{iHt}$ a group of unitaries on a Hilbert space $\mathcal{H}$.
  $\mathcal{H} = \mathcal{H}_{\mathrm{pp}} \oplus \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}$.

- $\alpha_t = e^{iDt}$ a group of isometries on a Banach space $\mathfrak{A}$.
  $\mathfrak{A}_{\mathrm{pp}}$ - eigenvectors. Concepts of $\mathfrak{A}_{\mathrm{c}}$, $\mathfrak{A}_{\mathrm{ac}}$, $\mathfrak{A}_{\mathrm{sc}}$ missing to date.

- Continuous spectrum of $\{\alpha_t\}_{t\in\mathbb{R}}$ is essential for the understanding of particle aspects in QFT: [D. Buchholz 94]
  Let $A \in \mathfrak{A}$, $\omega \in \mathfrak{A}^*$

$$\begin{aligned} \sigma_\omega^+(A) &:= \lim_{t\to\infty} \int d^s x \, \omega(\alpha_{(t,\vec{x})} A) \\ &= \sum_\lambda \int d^s p \, \rho_{\lambda,\omega}(\vec{p}) \, \langle \vec{p}, \lambda | A | \vec{p}, \lambda \rangle. \end{aligned}$$

Outline

1. Spectral theory of groups of isometries.
   - Arveson theory.
   - Pure-point and continuous spectrum.
   - Absolutely continuous and singular-continuous spectrum.
2. Spectral theory of the transposed action.
3. Unitarily implemented groups of automorphisms.
4. Example: Spacetime translation automorphisms in QFT.
5. Conclusions.

1(a). Spectral theory of groups of isometries. Arveson theory.

- Framework:

(a) $(\alpha, \mathfrak{A})$, $\mathfrak{A}$ - Banach space,

(b) $\mathfrak{A}_* \subset \mathfrak{A}^*$ closed, α^* - invariant and s.t. $\sup_{\varphi \in \mathfrak{A}_{*,1}} |\varphi(A)| = \|A\|$,

(c) $\hat{\mathfrak{A}}_* \subset \mathfrak{A}_*$ norm-dense.

- General concepts:

(a) $\operatorname{Sp} \alpha := \overline{\bigcup_{\substack{A \in \mathfrak{A} \\ \varphi \in \mathfrak{A}_*}} \operatorname{supp} \varphi(\widetilde{A}(\,\cdot\,))}$, $\quad \varphi(\widetilde{A}(p)) := \int d^d x \, e^{-ipx} \varphi(\alpha_x(A))$,

(b) $\widetilde{\mathfrak{A}}(\Delta) := \{ A \in \mathfrak{A} \mid \forall_{\varphi \in \mathfrak{A}_*} \operatorname{supp} \varphi(\widetilde{A}(\,\cdot\,)) \subset \Delta \}$.

1(b). Pure-point and continuous spectrum.

- Pure-point spectrum:

(a) $\widetilde{\mathfrak{A}}(\{q\}) = \{\, A \in \mathfrak{A} \mid \alpha_x(A) = e^{iqx} A \ \text{ for all } \ x \in \mathbb{R}^d \,\}$,

(b) $\mathfrak{A}_{\rm pp} := \mathrm{Span}\{\, \mathfrak{A}(\{q\}) \mid q \in \mathrm{Sp}\,\alpha \,\}^{\text{n-cl}}$,

(c) $\mathrm{Sp}_{\rm pp}\alpha := \{\, q \in \mathrm{Sp}\,\alpha \mid \mathfrak{A}(\{q\}) \neq \{0\} \,\}$.

- Continuous spectrum:

(a) $\mathfrak{A}_{\rm c} := \{\, A \in \mathfrak{A} \mid \underset{\substack{q \in \mathbb{R}^d \\ \varphi \in \mathfrak{A}_*}}{\forall} \ \lim\limits_{K \nearrow \mathbb{R}^d} \frac{1}{|K|} \int_K e^{-iqx} \varphi(\alpha_x(A))\, d^d x = 0 \,\}$,

(b) $\mathrm{Sp}_{\rm c}\alpha := \mathrm{Sp}\,\alpha|_{\mathfrak{A}_{\rm c}}$.

1(c). Absolutely continuous and singular-continuous spectrum.

- Absolutely continuous spectrum:

(a) $\mathfrak{A}_{\rm ac} := \{\, A \in \mathfrak{A}_{\rm c} \mid \forall_{\varphi \in \hat{\mathfrak{A}}_*} \; \varphi(\widetilde{A}(\,\cdot\,)) \in L^1(\mathbb{R}^d, d^dp) \,\}^{\text{n-cl}},$

(c) $\mathrm{Sp}_{\rm ac}\alpha := \mathrm{Sp}\,\alpha|_{\mathfrak{A}_{\rm ac}}.$

- Singular-continuous spectrum:

(a) $\mathfrak{A}_{\rm sc} := \mathfrak{A}_{\rm c}/\mathfrak{A}_{\rm ac},$

(b) $\mathrm{Sp}_{\rm sc}\alpha := \mathrm{Sp}\,\underline{\alpha},$ where $\underline{\alpha}_x[A] = [\alpha_x(A)].$

2. Spectral theory of the transposed action.

- Input: (a) $(\alpha^*, \mathfrak{A}_*)$, equipped with $\mathfrak{A}$.
  (b) $\hat{\mathfrak{A}} \subset \mathfrak{A}$ norm-dense.

- Output: $\widetilde{\mathfrak{A}}_*(\Delta)$, $\mathfrak{A}_{*,\mathrm{pp}}$, $\mathfrak{A}_{*,\mathrm{c}}$, $\mathfrak{A}_{*,\mathrm{ac}}$, $\mathfrak{A}_{*,\mathrm{sc}} \ldots$

- Properties:

(a) $\langle \widetilde{\mathfrak{A}}_*(\Delta), \widetilde{\mathfrak{A}}(\Delta') \rangle = 0$ for $\Delta \cap \Delta' = \emptyset$,

(b) $\langle \mathfrak{A}_{*,\mathrm{pp}}, \mathfrak{A}_{\mathrm{c}} \rangle = \langle \mathfrak{A}_{*,\mathrm{c}}, \mathfrak{A}_{\mathrm{pp}} \rangle = 0$.

- Proposition. Suppose that $\mathfrak{A} = \mathfrak{A}_{\mathrm{pp}} \oplus \mathfrak{A}_{\mathrm{c}}$. Then

(a) $\dim \widetilde{\mathfrak{A}}(\{0\}) \geq \dim \widetilde{\mathfrak{A}}_*(\{0\})$,

(b) $\mathfrak{A}_{\mathrm{pp}}^{\perp} = \mathfrak{A}_{*,\mathrm{c}}$.

3(a). Automorphism groups of C^*-algebras.

- Framework:

(a) $(\alpha, \mathfrak{A}, \omega_0)$, $\mathfrak{A}$ - unital C^*-algebra, ω_0 - pure, invariant state.

(b) $(\pi, \mathcal{H}, \Omega)$ - the GNS triple.

(c) $\mathfrak{A}_*$ - predual of $\pi(\mathfrak{A})''$.

(d) $\hat{\mathfrak{A}}_* \subset \mathfrak{A}_*$ s.a., $(\Psi|\pi(\,\cdot\,)\Omega) \in \hat{\mathfrak{A}}_*$, for Ψ from some dense set.

- Key assumption: $\ker\omega_0 \subset \mathfrak{A}_{\mathrm{c}}$.

- Proposition.

(a) $\mathfrak{A} = \mathfrak{A}_{\mathrm{pp}} \oplus \mathfrak{A}_{\mathrm{c}}$, $\quad \mathfrak{A}_{\mathrm{pp}} = \mathrm{Span}\{I\}$, $\quad \mathfrak{A}_{\mathrm{c}} = \ker\omega_0$,

(b) $\mathfrak{A}_* = \mathfrak{A}_{*,\mathrm{pp}} \oplus \mathfrak{A}_{*,\mathrm{c}}$, $\quad \mathfrak{A}_{*,\mathrm{pp}} = \mathrm{Span}\{\omega_0\}$, $\quad \mathfrak{A}_{*,\mathrm{c}} = \ker I$.

Moreover, $\mathrm{Sp}_{\mathrm{c}}\alpha = \mathrm{Sp}_{\mathrm{c}}\alpha^*$.

3(b). Implementing group of unitaries.

- Recall: There exists $\mathbb{R}^d \ni x \to U(x)$ on $\mathcal{H}$ s.t.

(a) $\pi(\alpha_x(A)) = U(x)\pi(A)U(x)^{-1}$,

(b) $U(x)\Omega = \Omega$,

(c) If $A \in \widetilde{\mathfrak{A}}(\Delta_1)$ then $\pi(A)\widetilde{\mathcal{H}}(\Delta_2) \subset \widetilde{\mathcal{H}}(\overline{\Delta_2 + \Delta_1})$.

- Proposition. Let $\mathcal{H}_{\mathrm{pp}}$, $\mathcal{H}_{\mathrm{c}}$, $\mathcal{H}_{\mathrm{ac}}$ be defined w.r.t. U. Then

(a) $\mathcal{H}_{\mathrm{pp}} = \mathrm{Span}\{\Omega\}$,

(b) $\mathcal{H}_{\mathrm{c}} = \{\,\pi(\mathfrak{A}_{\mathrm{c}})\Omega\,\}^{\text{n-cl}}$,

(c) $\mathcal{H}_{\mathrm{ac}} \supset \pi(\mathfrak{A}_{\mathrm{ac}})\Omega$.

3(c). Relations between $\mathrm{Sp}\,\alpha$ and $\mathrm{Sp}\,U$.

• Theorem. There holds:

(a) $\mathrm{Sp}_{\mathrm{pp}} U = \mathrm{Sp}_{\mathrm{pp}} \alpha^{(*)} = \{0\}$,

(b) $\mathrm{Sp}_{\mathrm{c}} U - \mathrm{Sp}_{\mathrm{c}} U \subset \mathrm{Sp}_{\mathrm{c}} \alpha$,

(c) $\pm \mathrm{Sp}_{\mathrm{ac}} U \subset \mathrm{Sp}_{\mathrm{ac}} \alpha^{*}$,

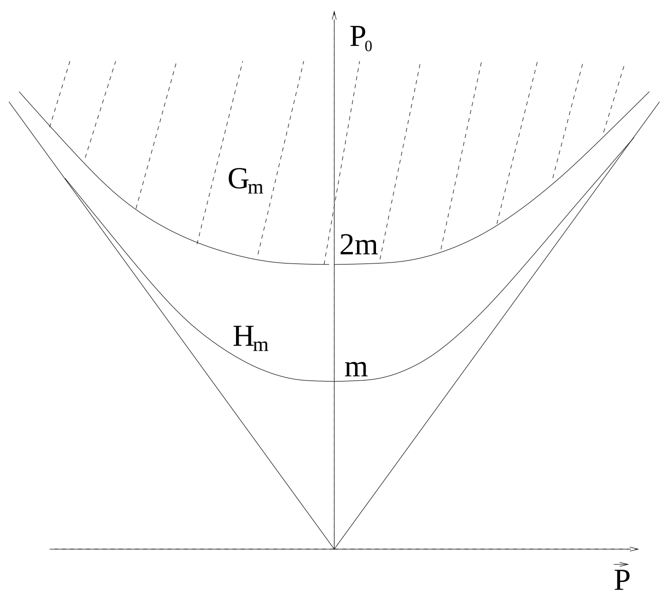
(d) $\pm \mathrm{Sp}_{\mathrm{sc}} U \subset \mathrm{Sp}_{\mathrm{sc}} \alpha$.

4(a). Spacetime translations in QFT.

• Theorem. Let $\mathbb{R}^{s+1} \ni x \to \alpha_x$ be spacetime translations in QFT with a pure vacuum state ω_0 and s.t. $\mathrm{Sp}\, U = \{0\} \cup H_m \cup G_m$.

(a) $\mathrm{Sp}_{\mathrm{pp}}\alpha^{(*)} = \{0\}$, (b) $\mathrm{Sp}_{\mathrm{c}}\alpha^{(*)} = \mathbb{R}^{s+1}$,

(c) $\mathrm{Sp}_{\mathrm{ac}}\alpha^{*} \supset \pm G_m$, (d) $\mathrm{Sp}_{\mathrm{sc}}\alpha \supset \pm H_m$.



5(b). Space translations in QFT.

- Theorem. Let $\mathbb{R}^s \ni \vec{x} \to \beta_{\vec{x}}$ be space translation automorphisms in QFT with a pure vacuum state ω_0. Then:

(a) $\mathrm{Sp}_{\mathrm{pp}}\beta^{(*)} = \{0\}$, (b) $\mathrm{Sp}_{\mathrm{c}}\beta^{(*)} = \mathbb{R}^s$,

(c) $\mathrm{Sp}_{\mathrm{ac}}\beta^{(*)} = \mathbb{R}^s$, (d) $\mathrm{Sp}_{\mathrm{sc}}\beta^{(*)} \subset \{0\}$.

- Proof of (c), (d) relies on the following estimate [D. Buchholz 90]

$$\int d^s p\, |\vec{p}|^{s+1+\varepsilon} |\omega(\widetilde{A}(\vec{p}))|^2 < \infty, \quad A \in \hat{\mathfrak{A}},\ \omega \in \hat{\mathfrak{A}}_*$$

5. Conclusions.

(a) For a group of isometries α acting on a Banach space $\mathfrak{A}$, we defined $\mathfrak{A}_{\mathrm{c}}$, $\mathfrak{A}_{\mathrm{ac}}$, $\mathfrak{A}_{\mathrm{sc}}$ and the corresponding spectra.

(b) We found necessary and sufficient conditions for $\mathfrak{A} = \mathfrak{A}_{\mathrm{pp}} \oplus \mathfrak{A}_{\mathrm{c}}$ in terms of the transposed action $(\alpha^*, \mathfrak{A}_*)$.

(c) In a C^*-algebraic setting we found relations between $\mathrm{Sp}\,\alpha$ and $\mathrm{Sp}\,U$, where $\pi\,\alpha_x(\,\cdot\,) = U(x)\pi(\,\cdot\,)U(x)^{-1}$.

(d) We studied the spectrum of spacetime translations in QFT. Its analysis is essential for the understanding of particle aspects.

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