Affine category over a Planar algebra

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P - planar algebra

AP - "affine category over P"

Objects / "Colors": $\{+,-\} \times \mathbb{N}_0$

$$\mathrm{Mor}(\varepsilon k, \eta l) = AP_{\varepsilon k, \eta l}$$

$$AP_{0,0} := \begin{pmatrix} AP_{+0,+0} & AP_{-0,+0} \\ AP_{+0,-0} & AP_{-0,-0} \end{pmatrix} \text{ ring}$$

$N \subset M$ - extremal, $[M:N] < \infty$

$$P = P^{N \subset M}$$

Fusion ring of bimodules $(\oplus, \otimes_{N/M})$

$$\begin{pmatrix} [N-N] & [N-M] \\ [M-N] & [M-M] \end{pmatrix} \underset{\text{Goal}}{\cong} AP_{0,0}^{N \subset M}$$

(suggested by Bisch, Jones)

Affine representation:

Functor: $AP \longrightarrow$ Hil. sp.

Importance of Affine representations

- Construction of subfactors with given principal graph

  E_6, E_8 - Jones

  Haagerup graph - Peters

- Group case - Affrep(P) equivalent to the category of representations of the quantum double of the group

  Ghosh

- TQFT

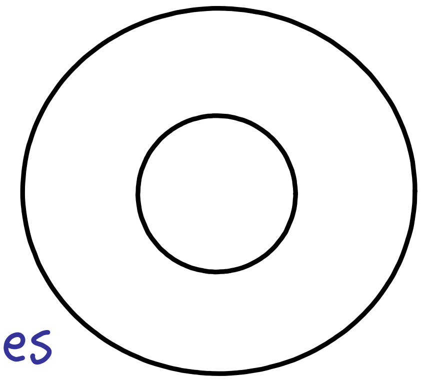
  Freedman / Walker

Possible generalizations?

Definition :

$(\varepsilon k, \eta \ell)$ affine tangle :

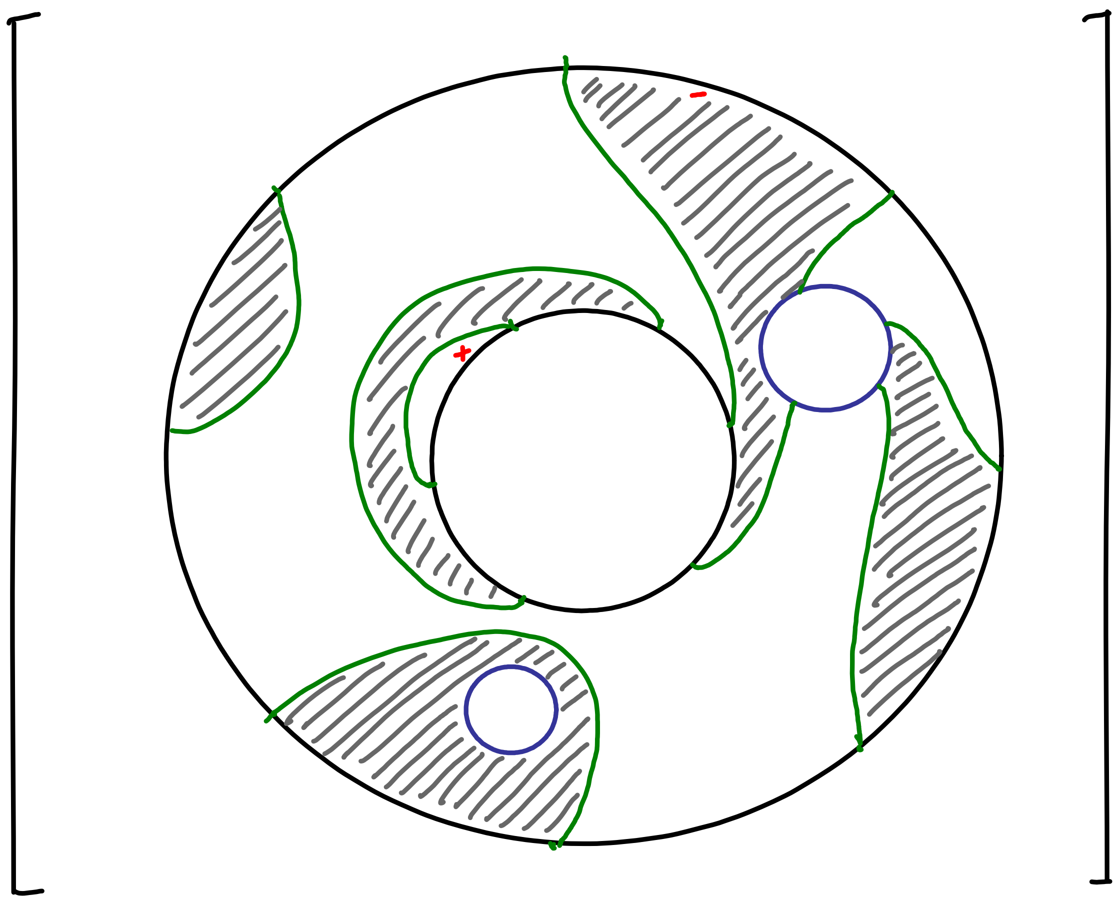
* Annulus with even # of marked points on the boundaries



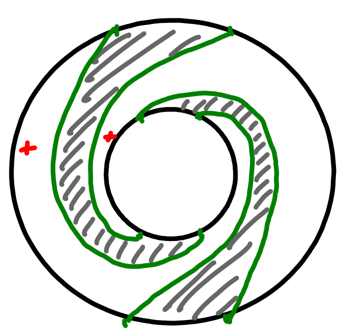
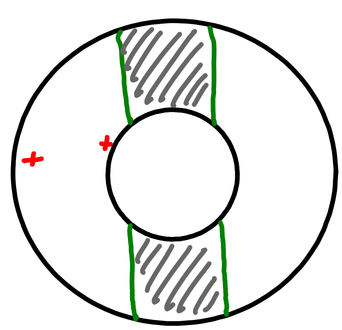
[Inner boundary: $2k$ points
Outer boundary: 2ℓ points]

* Finitely many non-intersecting discs in the interior of the annulus, each with even # of marked points on the boundary, counted clockwise.
* Smooth non-intersecting simple paths or strings with
  - each having two endpoints or none
  - each marked point is the endpoint of exactly one string
* Chequerboard shading + choice of a distinguished boundary component of inner and outer discs.

Example : $(+3, -4)$ affine tangle



* Affine isotopy : Annulus has rigid boundary

 $\not\approx$ 

2-rotation identity

Notation:

$\mathcal{T}_{\varepsilon k} = \{\text{Tangles whose ext. disc has color } \varepsilon k\}$

$P_{\varepsilon k} = \mathbb{C}\mathcal{T}_{\varepsilon k}$

$A\mathcal{T}_{\varepsilon k, \eta l} = \{(\varepsilon k, \eta l) \text{ affine tangles}\}$

$A_{\varepsilon k, \eta l} = \mathbb{C}\, A\mathcal{T}_{\varepsilon k, \eta l}$

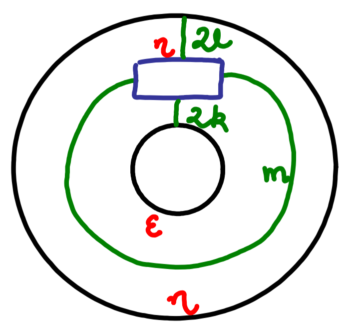
Composition:

$$A\mathcal{T}_{\varepsilon k, \eta l} \times A\mathcal{T}_{\zeta m, \varepsilon k} \longrightarrow A\mathcal{T}_{\zeta m, \eta l}$$

$$\begin{array}{ccc} T & S & T \circ S \\ \varepsilon k \to \eta l & \zeta m \to \varepsilon k & \zeta m \to \eta l \end{array}$$

A special affine tangle:

$$\Psi^{m}_{\varepsilon k, \eta l} : \mathcal{T}_{\eta(k+l+m)} \longrightarrow A\mathcal{T}_{\varepsilon k, \eta l}$$



Remark

* Any affine tangle $A \in A\mathcal{T}_{\varepsilon k, \eta l}$ can be expressed as $\Psi^{m}_{\varepsilon k, \eta l}(T)$ for some $m \in \mathbb{N}_0$, $T \in \mathcal{T}_{\eta(k+l+m)}$

* Same remark at the level of vector spaces, if we denote the linear extension also by $\Psi^m_{\varepsilon k, \eta l}$

<u>Notation</u> (contd.)

$P = \{P_{\varepsilon k}\}_{\varepsilon k}$ planar algebra with action Z.

$\mathcal{T}_{\varepsilon k}(P) = \{$Tangles in $\mathcal{T}_{\varepsilon k}$ with P-labelled internal discs$\}$

$\mathcal{P}_{\varepsilon k}(P) = \mathbb{C}\mathcal{T}_{\varepsilon k}(P)$

$\mathcal{AT}_{\varepsilon k, \eta l}(P) = \{A \in \mathcal{A}_{\varepsilon k, \eta l} : A$ is P-labelled$\}$

$\mathcal{A}_{\varepsilon k, \eta l}(P) = \mathbb{C}\mathcal{AT}_{\varepsilon k}(P)$

* Same remark for the space of P-labelled affine tangles.

$$\mathcal{W}_{\varepsilon k, \eta l} = \{\Psi^m_{\varepsilon k, \eta l}(x) : m \in \mathbb{N}_0,\ x \in \mathcal{P}_{\eta(l+k+m)}(P) \text{ s.t. } Z_x = 0\} \underset{\text{subsp.}}{\subset} \mathcal{A}_{\varepsilon k, \eta l}(P)$$

$$AP_{\varepsilon k, \eta l} := \frac{\mathcal{A}_{\varepsilon k, \eta l}(P)}{\mathcal{W}_{\varepsilon k, \eta l}}$$

$AP_{\varepsilon k, \eta l} = \{$Affine morphisms from εk to $\eta l\}$

Composition makes sense.

Restrict to: $k = 0 = l$, $\varepsilon = + = \eta$

$$\begin{tikzcd} P_{+2m} \arrow[r, "Id_{+2m}"] \arrow[rrr, bend right, dashed] & \mathcal{P}_{+2m}(P) \arrow[r, "\Psi^{2m}_{+0,+0}"] & A_{+0,+0}(P) \arrow[r, "q"] & AP_{+0,+0} \end{tikzcd}$$

$$\psi^{2m} := \psi^{2m}_{+0,+0}$$

To show:

$AP_{+0,+0} \cong$ Fusion ring of iso. class of irred N-N bimodules $= \mathbb{C}B$

$$\bigoplus_{l \in \mathbb{N}_0} P_{+2l} \xrightarrow{\psi^{2l}} AP_{+0,+0}$$

$$x, y \in P_{+2l} \Rightarrow \psi^{2l}(xy) = \psi^{2l}(yx)$$

Consequences:

(1) $AP_{+0,+0}$ is generated by ψ-images of min'l proj. of $\bigoplus_{l \in \mathbb{N}_0} P_{2l}$

(2) $\psi^{2l}(p) = \psi^{2l}(q)$ for p, q proj. in P_{+2l}, $p \sim q$

Relative commutants and intertwiners:

$$P_{+2\ell} = N' \cap M_{2\ell-1} \xrightarrow{\varphi^{\ell-1}} {}_N\mathcal{L}_N\left(L^2(M_{\ell-1})\right)$$

min'l proj $\longleftrightarrow$ irred bimodule

MvN eq. $\longleftrightarrow$ N-N iso.

$$\psi^{2\ell}(x e_{2\ell+1}) = \psi^{2\ell}(x) \quad \forall\ x \in P_{+2\ell}$$

(follows from def. of ψ)

Lemma: $\psi^{2\ell}(p) = \psi^{2(\ell+m)}(q)$ for p min'l proj. of $P_{+2\ell}$, q min'l proj of $P_{+2(\ell+m)}$ s.t. $\mathrm{Ran}\left(\varphi^{\ell-1}(p)\right) \underset{N-N}{\cong} \mathrm{Ran}\left(\varphi^{\ell+m-1}(q)\right)$

Pf: Note: p min'l proj. in $P_{+2\ell}$

$\Rightarrow pe_{2\ell+1}$ min'l proj. in $P_{+2\ell+1}$

E.T.S. $\mathrm{Ran}\ \varphi^{\ell}(pe_{2\ell+1}) \cong \mathrm{Ran}\ \varphi^{\ell-1}(p)$

Repeated application yields the result.

p - min'l proj of P_{+2k}

$wt(p) :=$ smallest 2ℓ s.t. $\exists\ q$ min'l proj. of $P_{+2\ell}$ satisfying $\mathrm{Ran}\ \varphi^{\ell-1}(q) \cong \mathrm{Ran}\ \varphi^{k-1}(p)$

$\{\psi^{2k}(p) \,/\, k \in \mathbb{N}_0,\ p \text{ min'l proj.},\ wt(p) = 2k\}$ generates $AP_{+0,+0}$

This induces $\alpha : \mathbb{C}B \longrightarrow AP_{+0,+0}$

We will construct an inverse of α, say $\beta : AP_{+0,+0} \longrightarrow \mathbb{C}B$

For this, define $\gamma_{2l} : P_{+2l} \longrightarrow \mathbb{C}B$ by

$$\gamma_{2l}(x) = \sum_{\substack{p \text{ min'l} \\ \text{central} \\ \text{in } P_{+2l}}} \frac{\tau(xp)}{\tau(p)} \left[\operatorname{Ran} \varphi^{l-1}(p)\right]$$

<u>Note</u>:

- $\gamma_{2l}(q) = \left[\operatorname{Ran} \varphi^{l-1}(q)\right]$ $\forall q$ min'l proj in P_{+2l}
- $\gamma_{2l}(xy) = \gamma_{2l}(yx)$
- $\gamma_{2l}(x) = \gamma_{2l+2}(x e_{2l+1})$

$$\begin{tikzcd}
AP_{+0,+0} \ni a \arrow[d, "\beta"'] & A \in \mathcal{A}_{+0,+0}(P) \arrow[l, "q"'] \\
\mathbb{C}B \ni \gamma_{2k}(Z_X) & \\
P_{+2k} \ni Z_X \arrow[u, "\gamma_{2k}"] & X \in \mathcal{P}_{+2k}(P) \arrow[l, "Z"'] \arrow[uu, "\Psi^{2k}_{+0,+0}"']
\end{tikzcd}$$

$$\beta(a) = \gamma_{2k}(Z_X) \quad (\text{Well-defined !})$$