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Aud 4

$M \subset B(H)$ v. N. alg

properties AF, P, injective

M is AF $:\Leftrightarrow$ $\exists\, (R_n)_{n\in\mathbb{N}}$ $R_n \subset R_{n+1}$ fin. dim subalg, $(\bigcup R_n)'' = M$

AF = approximately finite dimensional

M is P $:\Leftrightarrow$ $\forall x \in B(H)$:

$\overline{\mathrm{conv}\{uxu^* : u \in \mathcal{U}(M)\}}^{uw} \cap M' \neq \emptyset$

M is inj $:\Leftrightarrow$ $\exists\, \rho : B(H) \to M$ onto, $\rho^2 = \rho$

$\|\rho\| = 1$ (Tomiyama)

AF $\Rightarrow$ P $\Rightarrow$ inj „easy"

Connes 1995: inj $\Rightarrow$ AF

(got Fields medal for that (and other things))

R_n f.d. has prop P: ultra weak(?)

$x \in B(H)$, $y := \int uxu^* \, d_{\mathcal{U}(R_n)}(u)$

$y \in \overline{\mathrm{conv}\{uxu^* : u \in \mathcal{U}(M)\}}^{\text{norm}} \subset \overline{\mathrm{conv}\{\dots\}}^{uw}$

$vyv^* = \int vuxu^*v^* \, d(u) \,„=" \, y \quad \rightsquigarrow y \in R_n'$

an „averaging argument"

Open Problem: Suppose given $\gamma : B(H) \to M$ onto, $\gamma^2 = \gamma$. Is M inj?

(probably yes)

Sakai, Kadison

$\delta : M \to M$ is a derivation if $\forall x, y \in M$

$\delta(xy) = x\delta(y) + \delta(x)y$ (Leibnitz rule)

showed: $\forall \delta : M \to M$ derivation $\exists a \in M$ s.t.

$\delta(x) = [a, x] = ax - xa$

(any derivation is inner)

(M, α_t) $\alpha_t \in Aut(M)$, ~~$\alpha_t = \exp(itH)$~~

$\alpha_t = \exp(t\delta) = \exp(itH) \cdot \exp(itH)$

$\delta(x) = [iH, x]$

Given $\delta : M \to M$ find $c \in B(H)$ s.t. $\delta(x) = [c, x]$

Suppose property P, $u \in \mathcal{U}(M)$

$cu - uc \in M \rightsquigarrow c - ucu^* \in M$

$y \in \overline{\mathrm{conv}\{ucu^* : u \in \mathcal{U}(M)\}}^{uw} \cap M'$

$c - \sum \lambda_i u_i c u_i^* = \sum \lambda_i (c - u_i c u_i^*) \in M$

$\rightsquigarrow c - y \in M$

$[c - y, x] = [c, x] - [y, x] = [c, x] = \delta(x)$

$a := c - y \in M$

$\gamma : M \to A$ is completely bounded (cb)

(here A C^*-alg) if $\exists K > 0$ s.t.

$\| \gamma \otimes id_{M_n} \| \leq K \quad \forall n \qquad \gamma \otimes id_{M_n} : M \otimes M_n \to A \otimes M_n$

$A \subset B(H)$ Haagerup, Paulsen, Wittstock

γ cb $\Leftrightarrow$ $\exists \pi$ repr $M \to B(K)$, $\exists S \in B(K, L)$,

$T \in B(L, K)$ s.t. $\gamma(m) = S\,\pi(m)\,T$

$\|\gamma\|_{cb} = \|S\| \cdot \|T\|$

for cp mappings have similar result
with $T = S^*$

Thm: $M \subset B(H)$ v.N. alg. If $\gamma : B(H) \to M$
is c.b. projection, then M is inj

pf:
show that there ex a M-bimodule projection ρ
of $B(H)$ onto M ($\rho(m_1 x m_2) = m_1 \rho(x) m_2$)
Connes: M is inj

$\gamma(x) = S\pi(x)T$
$\gamma_u(x) = S\pi(xu)Tu^* = \gamma(xu)u^*$, $u \in \mathcal{U}(M)$
$\gamma_u(m) = \gamma(mu)u^* = muu^* = m$
$\rightsquigarrow \gamma_u$ proj onto M
$\gamma_u(x) = S\pi(x)(\pi(u)Tu^*)$
suppose $\hat{T} \in \overline{\text{conv}\{\pi(u)Tu^* \mid u \in \mathcal{U}(M)\}}^{uw} \cap$
$\cap \{x \in B(H) : \forall m \ \pi(m)x = xm\}$
$\forall m \in M : \hat{\gamma}(x) = S\pi(x)\hat{T}$
$\hat{\gamma}(xu) = S\pi(x)\pi(u)\hat{T} = S\pi(x)\hat{T}u$
$\rightsquigarrow \hat{\gamma}$ right M-module map
$\hat{\gamma}(m) = m$ c.b. proj

Setup: $M \subset B(H)$, $\gamma : B(H) \to M$ onto cb proj
$\gamma(x) = S\pi(x)T$
$\mathcal{S} = \{(m_i)_{i \in I} \mid m_i \in M, \sum m_i^* m_i = 1\}$
$A = (m_i) \in \mathcal{S}$, $B = (n_j) \in \mathcal{S}$
$A \cdot B = (m_i n_j) \in \mathcal{S}$
$\rightsquigarrow \mathcal{S}$ semigroup

$$A(T) = \underbrace{\begin{pmatrix} \pi(m_i^*) & \\ & 0 \end{pmatrix}}_{\in B(K \otimes \ell_2(J))} \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \underbrace{\begin{pmatrix} \vdots \\ m_i \\ \vdots \end{pmatrix}}_{\in B(H \otimes \ell^2(J))}$$

J - index set of A

$A(T) = \sum \pi(m_i^*) T m_i$

$\mathcal{C} := \overline{\mathrm{conv}\{A(T) : A \in \mathcal{S}\}}^{uw} \subset B(H, K)$

$(\pi : B(H) \to B(K))$

Try to replace T by $\hat{T} \in \mathcal{C}$ s.t.

$B(H) \ni x \mapsto S\pi(x)\hat{T}$ still is a proj

onto M, and is right M-module map

$A \rightsquigarrow$ affine transformation uw trans on $\mathcal{C}$

$\mathcal{C}_0 \subset \mathcal{C}$ minimal uw-cpct cvx set invariant

under $A \in \mathcal{S}$

$\hat{T} \in \mathcal{C}_0$ will do:

Lmma: $\forall \hat{T} \in \mathcal{C}_0$, $\forall x \in B(H), m \in M$

$S\pi(x)\pi(m)\hat{T} = S\pi(x)\hat{T}m$

$S\pi(m)\hat{T} = m$

$\forall \tilde{T} \in \mathcal{C}_0$ $\forall$ central proj z

$\|\tilde{T}z\| = \|\hat{T}z\|$

$A \in \mathcal{S}$, $A(\tilde{T}z) = A(\tilde{T})z$

$\|A(\tilde{T}z)\| = \|A(\tilde{T})z\| \geq \|A(\hat{T})z\|$

Claim: Let $e \in M$ be a proj, $c(e)$ its central support, then:

$\|\hat{T}c(e)\| = \|\hat{T}e\| = \|T(e)\hat{T}(e)\|$

$A \in \mathcal{J}$, $1-c(e)$, v_i partial isometries

$v_i v_i^* \leq e$, $\sum_i v_i^* v_i = c(e)$

$A = \{ (1-c(e)), (v_i)_{i\in I} \}$

$\tilde{T} = \pi(1-c(e)) \hat{T} (1-c(e)) + \sum_i \pi(v_i^*) \hat{T} v_i = A\hat{T}$

$\| \tilde{T} c(e) \| = \| \sum_i \pi(v_i^*) \hat{T} v_i \|$

$= \sup_i \| \pi(v_i^*) \hat{T} v_i \|$ since v_i orthogonal range + support

$= \sup_i \| \pi(v_i^* e) \hat{T} e v_i \|$

$\leq \| \pi(e) \hat{T} e \|$

other two inequalities easier

Claim: $\forall x \in B(H)$, $\forall e \in M$ proj

$S \pi(x) (1-\pi(e)) \hat{T} e = 0$

pf:

suppose not, then $a := S\pi(x)(1-\pi(e))\hat{T} e \neq 0$

but $a \in M$, $a^* a \in M$, $a^* a \neq 0$

f - spectral projection $E([\frac{1}{2}\|a^* a\|, \|a^* a\|]) \in M$

$a^* a f \geq \frac{\|a^* a\|}{2} f$

$\| \hat{T} f \|^2$ ~~$\geq$~~ $= \| \pi(f) \hat{T} f \|^2$?

$f^* \hat{T}^* \hat{T} f = f \hat{T}^* (1-\pi(f)) \hat{T} f + f \hat{T}^* \pi(f) \hat{T} f$

$\geq f \hat{T}^* (1-\pi(f)) \underbrace{\pi(x^*) S^* S \pi(x)}_{\leq 1} (1-\pi(f)) \hat{T} f$

$+ f \hat{T}^* \pi(f) \hat{T} f$

$\geq \frac{\|a\|^2}{2} f + f \hat{T}^* \pi(f) \hat{T} f$ [assume $1 \geq \| S\pi(x)(1-\pi(e))\| > 0$?]

both is cutdown by f

$\rightsquigarrow \| f \hat{T}^* \hat{T} f \| \geq \frac{\|a\|^2}{2} + \| f \hat{T}^* \pi(f) \hat{T} f \|$

$= \frac{\|a\|^2}{2} + \| \pi(f) \hat{T} f \|^2 = \|a\|^2/2 + \| \hat{T} f \|^2$

$= \|a\|^2/2 + \| f \hat{T}^* \hat{T} f \|$ $\lightning$

$\leadsto \quad S\pi(x)\,\pi(m)\,\hat{T} = S\pi(x)\,\hat{T}\,m$

$S\,\pi(m)\,A(T) = m \quad , \quad S\pi(m)\,\hat{T} = m$

Connes: $\hat{S}\,\pi(x)\,\hat{T}$ proj onto M which is M-bimodule $\leadsto$ M inj

Thm: $M \subset B(H)$, M has a c.b. proj onto M

Then M is inj.

Thm: $M \subset B(H)$ v.N. alg which is complemented

(so there is some proj $B(H)$ onto M)

If $M = N \bar{\otimes} R$ where R is an inj factor, then M is inj

pf

$\varphi : B(H) \to M$ onto, $\varphi^2 = \varphi$, φ bdd

case where M prop. inf, $M = N \bar{\otimes} B(\ell^2\mathbb{N})$

$B(H) = B(K) \bar{\otimes} B(\ell^2\mathbb{N})$

$\varphi : B(K) \bar{\otimes} B(\ell^2\mathbb{N}) \to N \bar{\otimes} B(\ell^2\mathbb{N})$

replace φ by $\hat{\varphi}$

$\hat{\varphi}(x) =$ mean of $\varphi(x(1\otimes u))(1\otimes u)^*$, $u \in B\ell^2\mathbb{N}$

$\hat{\hat{\varphi}}(x) =$ mean of $(1\otimes v)\,\hat{\varphi}((1\otimes v)^* x)$

$\hat{\hat{\varphi}} : B(H) \to M$ proj onto M

$\hat{\hat{\varphi}}$ is $I_K \otimes B(\ell^2\mathbb{N})$ bimod

$\leadsto \hat{\hat{\varphi}} = \tilde{\varphi} \otimes id \qquad \tilde{\varphi}$ proj $B(K)$ onto N

$\tilde{\varphi}$ c.b. $\leadsto N$ inj $\leadsto M = N \otimes B(\ell^2\mathbb{N})$ inj

M inj $\Leftrightarrow H^n(M, X^*) = 0 \qquad X^*$ - dual nml module

(Connes)