The Little Grothendieck Inequality - From Classical to Non-Commutative

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Resumé Dette bachelorprojekt i matematik omhandler den lille Grothendieck ulighed, som er et af hovedresultaterne fra Grothendiecks "Résumé de la théorie métrique des produits tensoriels topologiques". Den klassiske lille Grothendieck ulighed vises i det reelle tilfælde, og efter en gennemgang af relevant teori om blandt andet ultraprodukter af Banach rum gives et bevis af den ikke-kommutative lille Grothendieck ulighed.

Abstract This bachelor thesis in mathematics deals with one of the fundamental results from Grothendieck's "Résumé de la théorie métrique des produits tensoriels topologiques", known as the little Grothendieck inequality. The classical little Grothendieck inequality is proved in the real case, and after a brief overview on techniques of ultraproducts of Banach spaces as well as some other relevant background theory, a proof of the non-commutative little Grothendieck inequality is presented.

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Introduction

In 1956 Grothendieck published the celebrated "Résumé de la théorie métrique des produits tensoriels topologiques", containing a general theory of tensor norms on tensor products of Banach spaces, describing several operations to generate new norms from known ones and studying the duality theory between these norms. Since 1968 it has had a considerable influence on the dvelopment of Banach space theory (see, e.g, [7]).

The Résumé ends with a list of six problems which are linked together, and revolve around the following two fundamental questions: Given Banach spaces X and Y, when does a bounded linear operator $u : X \longrightarrow Y$ factor through a Hilbert space? For which Banach spaces X and Y does this happen for all such operators u?

The purpose of this thesis is to discuss one of the fundamental results from the Résumé, known as "The little Grothendieck inequality", both in its classical and non-commutative version.

The material is organized as follows: The first chapter is devoted to the classical little Grothendieck inequality, which asserts that any bounded linear operator $T : C(K) \longrightarrow H$, where K is a compact topological space and H a (separable) Hilbert space, does factor through a Hilbert space. We will give the proof of this result in the real case, following Pisier's approach from [7].

The proof uses a refinement of the Grothendieck-Pietsch factorization theorem for p-absolutely summing operators from $C(K)$ into H, which is discussed in section 1.2. The section 1.1 contains the necessary background on p-absolutely summing operators.

We end chapter 1 with a proof of the fact that the best constant in the little Grothendieck inequality in the real case is $\sqrt{\frac{\pi}{2}}$.

The second part of the thesis is devoted to the non-commutative little Grothendieck inequality, as proved by Haagerup in [4]. The non-commutative analogue to of $C(K)$ is a C^*-algebra A, while the non-commutative analogue of a probability measure on K is a state on A.

The non-commutative version of the little Grothendieck inequality asserts that if A is a C^*-algebra and H is a (separable) Hilbert space, then given a bounded linear operator $T : A \longrightarrow H$ there exist states φ and ψ on A such that

$$\|T(x)\|^2 \leq \|T\|^2(\varphi(x^*x) + \psi(xx^*)), \qquad x \in A.$$

The proof, for which we will follow the presentation in [4], is achieved in several steps. First we show that the assertion holds under the assumption that there exists a unitary U in A such that $1 = \|T\| = \|T(U)\|$. The passage from here to the general case requires a result from operator algebra, namely the fact that the convex hull of the unitary elements in A is dense in the closed unit ball of A, which we will discuss in section 3.1.

We end chapter 3 with few corollaries of the non-commutative little Grothendieck inequality.

In the proof of the non-commutative little Grothendieck inequality, techniques of ultraproducts of Banach spaces are needed. As this theory was new to me, I have chosen to devote chapter 2 to a brief oveview on ultrafilters and ultraproducts.

Aknowledgements I would like to thank my supervisor Magdalena Musat for many useful discussions, as well as all her patience.

- Julie Pedersen.

Contents

1 The Classical Little Grothendieck Inequality

Theorem 1.1 (The Little Grothendieck Inequality). *Let H be a (separable) Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and K a compact topological space. Let $T : C(K) \longrightarrow H$ be a bounded linear operator. Then there exists a probability measure μ on K such that*

$$\|Tf\| \leq C_{\mathbb{F}} \|T\| \left(\int_K |f(t)|^2 d\mu(t) \right)^{\frac{1}{2}}$$

for all $f \in C(K)$, where $C_{\mathbb{F}}$ is a universal constant.

The best value of $C_{\mathbb{F}}$ is $\sqrt{\frac{\pi}{2}}$ when $\mathbb{F} = \mathbb{R}$, and $\sqrt{\frac{4}{\pi}}$ when $\mathbb{F} = \mathbb{C}$

We will present the proof of the little Grothendieck inequality in the real case. For the exposition we will follow Pisier's approach from [7].

First we will need some preliminaries on p-absolutely summing operators.

1.1 p-Absolutely Summing Operators

Definition 1.2. *Let V and W be Banach spaces, $1 \leq p < \infty$. A linear operator $T : V \longrightarrow W$ is called p-absolutely summing if there exists $K > 0$ such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in V$ we have*

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{\frac{1}{p}} \leq K \sup \left\{ \left(\sum_{i=1}^{n} |x^*(x_i)|^p \right)^{\frac{1}{p}} : \|x^*\|_{V^*} \leq 1 \right\}.$$

We define $\pi_p(T)$ to be the infimum over all such K's, and use the notation

$$\Pi_p(V, W) := \{T \in B(V, W) : \pi_p(T) < \infty\}.$$

Remark 1. *Note that if $T \in \Pi_p(V, W)$ then T is bounded with $\|T\| \leq \pi_p(T)$. Indeed, for all $x \in V$ we have*

$$\|Tx\| \leq \pi_p(T) \sup\{|x^*(x)| : \|x^*\|_{V^*} \leq 1\} = \pi_p(T)\|x\|,$$

which proves the assertion.

Proposition 1.3. *Let V and W be Banach spaces, $1 \leq p < \infty$. Then $(\Pi_p(V, W), \pi_p(\cdot))$ is a Banach space and*

$$\pi_p(STU) \leq \|S\| \pi_p(T) \|U\|$$

for all $T \in B(V, W)$, $S, U \in B(W, V)$.

Proof. We will omit the proof of the fact that $(\Pi_p(V, W), \pi_p(\cdot))$ is a Banach space. The last statement follows from the definition. Indeed, for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in V$, we have that

$$
\begin{aligned}
& \left(\sum_{i=1}^{n} \|STUx_i\|^p\right)^{\frac{1}{p}} \\
\leq \quad & \|S\| \left(\sum_{i=1}^{n} \|T(Ux_i)\|^p\right)^{\frac{1}{p}} \\
\leq \quad & \|S\|\pi_p(T) \sup \left\{ \left(\sum_{i=1}^{n} |x^*(Ux_i)|^p\right)^{\frac{1}{p}} : \|x^*\|_{V^*} \leq 1 \right\} \\
\leq \quad & \|S\|\pi_p(T) \sup \left\{ \left(\sum_{i=1}^{n} \left| \left(x^* \circ \frac{U}{\|U\|}\right)(x_i) \right|^p \right)^{\frac{1}{p}} : \|x^*\|_{V^*} \leq 1 \right\} \|U\|. \qquad (1)
\end{aligned}
$$

But $x^* \circ \frac{U}{\|U\|} \in W^*$ and $\left\| x^* \circ \frac{U}{\|U\|} \right\|_{W^*} \leq 1$, so we get

$$
(1) \leq \|S\|\pi_p(T) \sup \left\{ \left(\sum_{i=1}^{n} |y^*(x_i)|^p\right)^{\frac{1}{p}} : \|y^*\|_{W^*} \leq 1 \right\} \|U\|.
$$

This implies that STU is p-absolutely summing and

$$
\pi_p(STU) \leq \|S\|\pi_p(T)\|U\|,
$$

as wanted. □

Remark 2. *To give an example of a p-absolutely summing operator, let Ω be a compact topological space and μ a probability measure on Ω. Then the identity map $I_p : C(\Omega) \longrightarrow L_p(\Omega, \mu)$ is p-absolutely summing for all $1 \leq p < \infty$, with $\pi_p(I_p) \leq 1$.*

1.2 The Grothendieck-Pietsch Factorization Theorem

The following fundamental result, known as the Grothendieck-Pietsch factorization theorem, is a characterization of p-absolutely summing operators:

Theorem 1.4 (The Grothendieck-Pietsch Factorization Theorem)**.** *Let $1 \leq p < \infty$ and V, W be Banach spaces. A linear operator $T : V \longrightarrow W$ is p-absolutely summing if and only if there exists a probability measure μ on $\overline{B}_{V^*}(0,1)$ (the closed unit ball of V^*) and a constant $K > 0$ such that for all $x \in V$*

$$
\|Tx\| \leq K \left(\int_{\overline{B}_{V^*}(0,1)} |x^*(x)|^p d\mu(x^*) \right)^{\frac{1}{p}}.
$$

Moreover, $\inf\{K : K \textit{ as above}\} = \pi_p(T)$.

For the proof, see for example [7] or [9].

The following is a very useful refinement of the Grothendieck-Pietsch factorization theorem for linear operators on $C(K)$. We will also include a proof.

Theorem 1.5. *Let K be a compact topological space and W a Banach space. If the operator $T : C(K) \longrightarrow W$ is p-absolutely summing for some $1 \leq p < \infty$, then there exists a probability measure μ on K such that*

$$\|Tf\| \leq \pi_p(T) \left(\int_K |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}$$

for all $f \in C(K)$.

Proof. This proof follows the stategy of the proof of the Grothendieck-Pietsch Theorem in the general case, but with some appropriate modifications. We consider $\mathbb{F} = \mathbb{R}$. The proof in the complex case is similar.

Since the p-absolutely summing norm of an operator is homogeneous, i.e., if $c > 0$ then $\pi_p(cT) = c\pi_p(T)$, we may assume without loss of generality that $\pi_p(T) = 1$. Let

$$F_1 := \{f \in C(K) : \sup_{x \in K} f(x) < 1\}.$$

For every $v \in C(K)$ we define $f_v : K \longrightarrow \mathbb{R}$ by

$$f_v(x) := |v(x)|^p$$

for all $x \in K$. Then $f_v \in C(K)$. Set

$$F_2 := \operatorname{conv}\{f_v : v \in C(K), \|Tv\| = 1\}.$$

It is obvious that F_1 and F_2 are convex sets in $C(K)$. Also, F_1 is seen to be open since it contains the open unit ball of $C(K)$. Furthermore, we claim that $F_1 \cap F_2 = \emptyset$. To see this, let $f \in F_2$. We will show that $f \notin F_1$. By definition of F_2 there exist $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$ and $v_1, \ldots, v_n \in C(K)$ with $\|Tv_i\| = 1$ for all $1 \leq i \leq n$, such that

$$f = \sum_{i=1}^n \alpha_i f_{v_i}.$$

The goal is now to show that

$$\sup \left\{ \sum_{i=1}^n \alpha_i f_{v_i}(x) : x \in K \right\} \geq 1, \tag{2}$$

which shows that $f \notin F_1$, as claimed.

We observe that

$$\sum_{i=1}^n \alpha_i f_{v_i}(x) = \sum_{i=1}^n \alpha_i |v_i(x)|^p = \sum_{i=1}^n |\alpha_i^{\frac{1}{p}} v_i(x)|^p,$$

so showing (2) is equivalent to showing

$$\sup\left\{\sum_{i=1}^{n}|\alpha_i^{\frac{1}{p}}v_i(x)|^p : x\in K\right\}\geq 1. \tag{3}$$

To simplify notation, set $\omega_i := \alpha_i^{\frac{1}{p}}v_i$ for all $1\leq i\leq n$. Then (3) becomes

$$\sup\left\{\sum_{i=1}^{n}|\omega_i(x)|^p : x\in K\right\}\geq 1. \tag{4}$$

Now, since $\pi_p(T)=1$ we have by definition

$$\left(\sum_{i=1}^{n}\|T\omega_i\|^p\right)^{\frac{1}{p}}\leq \sup\left\{\left(\sum_{i=1}^{n}|x^*(\omega_i)|^p\right)^{\frac{1}{p}} : x^*\in C(K)^*, \|x^*\|\leq 1\right\}, \tag{5}$$

and we will proceed to show that we in fact have

$$\sup\left\{\sum_{i=1}^{n}|x^*(\omega_i)|^p : x^*\in C(K)^*, \|x^*\|\leq 1\right\}=\sup\left\{\sum_{i=1}^{n}|\omega_i(x)|^p : x\in K\right\}. \tag{6}$$

For any $x\in K$, define $\delta_x(f):=f(x)$ for all $f\in C(K)$. Note that $\|\delta_x\|=1$ and therefore $\delta_x\in\overline{B}_{C(K)^*}(0,1)$. It is not difficult to show that $\pm\delta_x$ are extreme points in $\overline{B}_{C(K)^*}(0,1)$, for all $x\in K$. It is significantly more difficult to prove that any extreme point of $\overline{B}_{C(K)^*}(0,1)$ is of this form. (See, e.g., [2]). Then, by the Krein-Milman Theorem (see, e.g., Theorem 3.23 in [8]) it follows that

$$\overline{B}_{C(K)^*}(0,1)=\overline{\text{conv}}\{\pm\delta_x, x\in K\}.$$

Hence, the lefthand side of (6) can be rewritten as

$$\begin{aligned}
&\sup\left\{\sum_{i=1}^{n}|x^*(\omega_i)|^p : x^*\in C(K)^*, \|x^*\|\leq 1\right\} && (7)\\
=\;&\sup\left\{\sum_{i=1}^{n}|x^*(\omega_i)|^p : x^*\in \overline{B}_{C(K)^*}(0,1)\right\}\\
=\;&\sup\left\{\sum_{i=1}^{n}|x^*(\omega_i)|^p : x^*\in \overline{\text{conv}}\{\pm\delta_x : x\in K\}\right\},
\end{aligned}$$

and since the supremum of a continuous function over a set is equal to the supremum over the closure of that set,

$$(7)=\sup\left\{\sum_{i=1}^{n}|x^*(\omega_i)|^p : x^*\in \text{conv}\{\pm\delta_x : x\in K\}\right\}.$$

For $f \in C(K)$ the function $x^* \mapsto |x^*(f)|^p$ is easily seen to be convex, so since the supremum of a convex function over a set A is equal to the supremum over $\mathrm{conv}(A)$ we obtain

$$\begin{aligned}(7) \quad &= \sup\left\{\sum_{i=1}^{n}|x^*(\omega_i)|^p : x^* \in \{\pm\delta_x : x \in K\}\right\}\\ &= \sup\left\{\sum_{i=1}^{n}|\delta_x(\omega_i)|^p : x \in K\right\}\\ &= \sup\left\{\sum_{i=1}^{n}|\omega_i(x)|^p : x \in K\right\},\end{aligned}$$

which proves (6).

Now we put together (5) and (6) to obtain

$$\begin{aligned}\sup\left\{\sum_{i=1}^{n}|\omega_i(x)|^p : x \in K\right\} &= \sup\left\{\sum_{i=1}^{n}|x^*(\omega_i)|^p : x^* \in C(K)^*, \|x^*\| \leq 1\right\}\\ &\geq \sum_{i=1}^{n}\|T\omega_i\|^p\\ &= \sum_{i=1}^{n}\alpha_i\|Tv_i\|^p\\ &= \sum_{i=1}^{n}\alpha_i\\ &= 1,\end{aligned}$$

which proves (4). So indeed $F_1 \cap F_2 = \emptyset$, as wanted. Now we can apply the Hahn-Banach Separation Theorem (see, e.g., Theorem 3.4 (a) in [8]) to conclude that there exists a functional $\varphi : C(K) \longrightarrow \mathbb{R}$ and a $\lambda \in \mathbb{R}$ such that

$$\varphi(f_1) < \lambda \leq \varphi(f_2) \tag{8}$$

for all $f_1 \in F_1$ and all $f_2 \in F_2$. Note that $\lambda > 0$ because $f_1 = 0$ belongs to F_1.

We now show that $\varphi \in C(K)^*$ is positive. For this, let $f \in C(K)_+$. Then $-f \in F_1$, so $\varphi(-f) < \lambda$, whence $\varphi(f) > -\lambda$ by linearity. Further, since

$$tf \in C(K)_+ \quad \text{for all } t > 0$$

it follows that

$$\varphi(tf) > -\lambda \quad \text{for all } t > 0$$

which implies

$$\varphi(f) > -\frac{1}{t}\lambda \quad \text{for all } t > 0.$$

By letting $t \longrightarrow \infty$ we get $\varphi(f) \geq 0$, which shows that φ is positive.

Now we can apply the Riesz Representation Theorem (see, e.g., [2], Appendix) to get the existence of a positive measure μ on K such that

$$\varphi(f) = \int_K f(x) d\mu(x)$$

for all $f \in C(K)$. Note that $\mu(K) = \varphi(f_0)$ where $f_0 = 1$, so if we normalize μ to become a probability measure on K, this will correspond to taking $\lambda = 1$ in (8). Indeed, by what we showed before we can conclude that $\varphi(f_2) \geq 1$ for all $f_2 \in F_2$. Moreover, under the assumption that $\mu(K) = 1$ it follows that $\varphi(f_1) < 1$ for all $f_1 \in F_1$.

Now that we have proved that (8) holds with $\lambda = 1$ we are ready to finnish the proof of the Theorem. Let $f \in C(K)$. It suffices to assume that $\|Tf\| = 1$, otherwise we can just rescale. An application of (8) gives us

$$\|Tf\| = 1 \leq \int_K |f(x)|^p d\mu(x),$$

which implies that

$$\|Tf\| = 1 \leq \left(\int_K |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}$$

for all $f \in C(K)$, and the proof is complete. □

In particular, if W is a Hilbert space we obtain the following result, which will be of great use to us in proving the Little Grothendieck Inequality.

Theorem 1.6. *Let H be a (separable) Hilbert space and K a compact topological space. If $T : C(K) \longrightarrow H$ is a 2-absolutely summing operator, then there exists a probability measure μ on K so that*

$$\|Tf\| \leq \pi_2(T) \left(\int_K |f(t)|^2 d\mu(t) \right)^{\frac{1}{2}}$$

for all $f \in C(K)$.

1.3 Proving the Little Grothendieck Inequality in the Real Case

Having Theorem 1.6 at hand, in order to obtain the little Grothendieck inequality in the real case it will suffice to prove the following :

Theorem 1.7. *Let H be a (separable) real Hilbert space and K a compact topological space. If $T : C(K) \longrightarrow H$ is a bounded linear operator, then T is 2-absolutely summing with*

$$\pi_2(T) \leq \sqrt{\frac{\pi}{2}} \|T\|.$$

The proof of this will be achieved in several steps. For this, we will prove a number of intermediate results.

Proposition 1.8. *Let (M,m) be a measure space and H a real Hilbert space. Then any bounded linear operator $u : H \longrightarrow L_1(M,m)$ satisfies*

$$\int_M \left(\sum_{i=1}^n |(u(x_i))(t)|^2\right)^{\frac{1}{2}} dm(t) \leq \sqrt{\frac{\pi}{2}}\|u\| \left(\sum_{i=1}^n \|x_i\|^2\right)^{\frac{1}{2}}$$

for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in H$.

Proof. Let $n \in \mathbb{N}$ and consider $g_1, \dots, g_n$ independent standard real-valued Gaussian stochastic variables on some probability space $(\Omega, F, \mathbb{P})$. Let $x_1, \dots, x_n \in H$. Then, by definition of $\|u\|$, we have

$$\left\| u\left(\sum_{i=1}^n g_i(\omega)x_i\right)\right\|_{L_1(M,m)} \leq \|u\|\left\|\sum_{i=1}^n g_i(\omega)x_i\right\|$$

for any fixed $\omega \in \Omega$. Integrating both sides with respect to ω, we get

$$\begin{aligned} \int_\Omega \left\| u\left(\sum_{i=1}^n g_i(\omega)x_i\right)\right\|_{L_1(M,m)} d\mathbb{P}(\omega) \quad &\leq \quad \|u\| \int_\Omega \left\|\sum_{i=1}^n g_i(\omega)x_i\right\| d\mathbb{P}(\omega) \qquad (9)\\ &\leq \quad \|u\| \left(\int_\Omega \left\|\sum_{i=1}^n g_i(\omega)x_i\right\|^2 d\mathbb{P}(\omega)\right)^{\frac{1}{2}}, \end{aligned}$$

where we have used the fact that $\|\cdot\|_{L_1(\Omega,\mathbb{P})} \leq \|\cdot\|_{L_2(\Omega,\mathbb{P})}$. On the other hand, we can rewrite the left-hand side of (9) further as

$$\begin{aligned} &\int_\Omega \left\| u\left(\sum_{i=1}^n g_i(\omega)x_i\right)\right\|_{L_1(M,m)} d\mathbb{P}(\omega)\\ = \quad &\int_\Omega \int_M \left| u\left(\sum_{i=1}^n g_i(\omega)x_i\right)(t)\right| dm(t)d\mathbb{P}(\omega)\\ = \quad &\int_M \int_\Omega \left| u\left(\sum_{i=1}^n g_i(\omega)x_i\right)(t)\right| d\mathbb{P}(\omega)dm(t)\\ = \quad &\int_M \int_\Omega \left|\sum_{i=1}^n g_i(\omega)\left(u(x_i)(t)\right)\right| d\mathbb{P}(\omega)dm(t), \end{aligned}$$

and letting $\alpha_i(t) := (u(x_i))(t), 1 \le i \le n$ to ease notation, we further obtain

$$
\begin{aligned}
& \int_M \int_\Omega \left| \sum_{i=1}^n g_i(\omega)\, (u(x_i)(t)) \right| d\mathbb{P}(\omega) dm(t) \\
= \; & \int_M \int_\Omega \left| \sum_{i=1}^n g_i(\omega) \alpha_i(t) \right| d\mathbb{P}(\omega) dm(t) \\
= \; & \int_M \left\| \sum_{i=1}^n g_i \alpha_i(t) \right\|_{L_1(\Omega,\mathbb{P})} dm(t) \\
= \; & \int_M \left(\sum_{i=1}^n |\alpha_i(t)|^2 \right)^{\frac{1}{2}} \|g_i\|_{L_1(\Omega,\mathbb{P})} dm(t) && (10) \\
= \; & \sqrt{\frac{2}{\pi}} \int_M \left(\sum_{i=1}^n |\alpha_i(t)|^2 \right)^{\frac{1}{2}} dm(t). && (11)
\end{aligned}
$$

To get from (10) to (11) we have used that

$$\|g_i\|_{L_1(\Omega,\mathbb{P})} = \int_\Omega |g_i(\omega)| d\mathbb{P}(\omega) = \int_{\mathbb{R}} |x| e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} dx = \sqrt{\frac{2}{\pi}},$$

and to obtain the equality giving (10) we need the following observation: If $1 \le p < \infty$ then for all $a_i, \ldots, a_n \in \mathbb{R}$ we have

$$\left\| \sum_{i=1}^n g_i a_i \right\|_{L_p(\Omega,\mathbb{P})} = \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}} \|g_1\|_{L_p(\Omega,\mathbb{P})}. \tag{12}$$

This follows from the fact that if X, Y are independent $N(0,1)$ stochastic variables on $(\Omega, F, \mathbb{P})$, then for all $a, b \in \mathbb{R}$, the stochastic variable $aX + bY$ is $N(0, a^2 + b^2)$. Hence, the stochastic variable $\sum_{i=1}^n g_i a_i$ has the same distribution as $\left(\sum_{i=1}^n |a_i|^2\right)^{\frac{1}{2}} g_1$, which implies (12).

With an argument similar to this, where we use the fact that $\|g_1\|_{L_2(\Omega,\mathbb{P})} = 1$, we can also show that

$$\int_\Omega \left\| \sum_{i=1}^n g_i(\omega) x_i \right\|^2 d\mathbb{P}(\omega) = \sum_{i=1}^n \|x_i\|^2. \tag{13}$$

Replacing $\alpha_i(t)$ by $(u(x_i)(t))$ back into (11), inequality (9) becomes

$$\sqrt{\frac{2}{\pi}} \int_M \left(\sum_{i=1}^n |(u(x_i))(t)|^2 \right)^{\frac{1}{2}} dm(t) \le \|u\| \left(\int_\Omega \left\| \sum_{i=1}^n g_i(\omega) x_i \right\|^2 d\mathbb{P}(\omega) \right)^{\frac{1}{2}},$$

so using (13) yields

$$\int_M \left(\sum_{i=1}^n |(u(x_i))(t)|^2 \right)^{\frac{1}{2}} dm(t) \le \sqrt{\frac{\pi}{2}} \|u\| \left(\sum_{i=1}^n \|x_i\|^2 \right)^{\frac{1}{2}},$$

which is what we wanted to show. $\square$

Proposition 1.9. *Let $N \in \mathbb{N}$ and H be a (separable) real Hilbert space. Then any bounded linear operator $v : \ell_\infty^N \longrightarrow H$ is 2-absolutely summing and satisfies, moreover,*

$$\pi_2(v) \le \sqrt{\frac{\pi}{2}}\|v\|.$$

Proof. Equip $S := \{1, 2, \dots, N\}$ with the measure m given by $m(\{i\}) = 1$ for all $1 \le i \le N$. Then we can identify ℓ_∞^N with $L_\infty(S, m)$ in the way that every element $y \in \ell_\infty^N$ is viewed as a function $y : S \longrightarrow \mathbb{R}$.

We have to prove that for all $n \ge 1$ and all $y_1, \dots, y_n \in \ell_\infty^N$ we have

$$\left(\sum_{i=1}^n \|vy_i\|^2\right)^{\frac{1}{2}} \le \sqrt{\frac{\pi}{2}}\|v\| \sup\left\{\left(\sum_{i=1}^n |\langle y_i, f\rangle|^2\right)^{\frac{1}{2}} : f \in \ell_1^N, \|f\|_{\ell_1^N} \le 1\right\}. \tag{14}$$

Considering the right-hand side of (14) we observe that

$$\sup\left\{\left(\sum_{i=1}^n |\langle y_i, f\rangle|^2\right)^{\frac{1}{2}} : f \in \ell_1^N, \|f\|_{\ell_1^N} \le 1\right\} = \sup\left\{\left(\sum_{i=1}^n |y_i(t)|^2\right)^{\frac{1}{2}} : t \in S\right\}. \tag{15}$$

To see that this is true, let, for all $t \in S$, $\delta_t : S \longrightarrow \mathbb{R}$ be given by

$$\delta_t(s) := \begin{cases} 0 & \text{if } s \in S \\ 1 & \text{if } s = t \end{cases}.$$

Then $f = \delta_t$ satisfies $f \in \ell_1^N$ with $\|f\|_{\ell_1^N} = 1$, and for all $1 \le i \le n$ we have that $\langle y_i, \delta_t\rangle = y_i(t)$. Moreover, the extreme points of the closed unit ball $\overline{B}_{\ell_1^N}(0, 1)$ is exactly the set $\{\delta_t : t \in S\}$. Since the mapping $f : \overline{B}_{\ell_1^N}(0, 1) \longrightarrow \mathbb{R}$ given by

$$f \mapsto \sum_{i=1}^n |\langle y_i, f\rangle|^2$$

is continuous and convex we can use the arguments we used in the proof of Theorem 1.6

to get that

$$\begin{aligned}
\sup\left\{\sum_{i=1}^{n}|\langle y_i, f\rangle|^2 : f\in \ell_1^N, \|f\|_{\ell_1^N}\le 1\right\} &= \sup\left\{\sum_{i=1}^{n}|\langle y_i, f\rangle|^2 : f\in \overline{B}_{\ell_1^N}(0,1)\right\}\\
&= \sup\left\{\sum_{i=1}^{n}|\langle y_i, f\rangle|^2 : f\in \overline{\mathrm{conv}}\{\delta_t : t\in S\}\right\}\\
&= \sup\left\{\sum_{i=1}^{n}|\langle y_i, f\rangle|^2 : f\in \mathrm{conv}\{\delta_t : t\in S\}\right\}\\
&= \sup\left\{\sum_{i=1}^{n}|\langle y_i, f\rangle|^2 : f\in \{\delta_t : t\in S\}\right\}\\
&= \sup\left\{\sum_{i=1}^{n}|y_i(t)|^2 : t\in S\right\}\\
&= \max\left\{\sum_{i=1}^{n}|y_i(t)|^2 : t\in S\right\},
\end{aligned}$$

and by taking square roots on both sides, (15) follows.

Inserting in (14), what we have to prove now becomes:

$$\left(\sum_{i=1}^{n}\|vy_i\|^2\right)^{\frac{1}{2}} \le \sqrt{\frac{\pi}{2}}\|v\| \max\left\{\sum_{i=1}^{n}|y_i(t)|^2 : t\in S\right\}. \tag{16}$$

Let $V : L_\infty((S,m),\ell_2^n) \longrightarrow \ell_2^n(H)$ be the linear operator defined by

$$(z_1,\ldots,z_n)\mapsto (v(z_1),\ldots,v(z_n))$$

for all $(z_1,\ldots,z_n)\in L_\infty((S,m),\ell_2^n)$. Note that we identify an element of $L_\infty((S,m),\ell_2^n)$ with an n-tuple of elements in $L_\infty(S,m)$, and recall the identification $L_\infty(S,m)=\ell_\infty^N$. Now we can see that the left-hand side of (16) is equal to

$$\|V(y_1,\ldots,y_n)\|_{\ell_2^n(H)},$$

and furthermore, since

$$\begin{aligned}
\|(y_1,\ldots,y_n)\|_{L_\infty((S,m),\ell_2^n)} &= \max\{\|(y_1(t),\ldots,y_n(t))\|_{\ell_2^n} : t\in S\}\\
&= \max\left\{\left(\sum_{i=1}^{n}|y_i(t)|^2\right) : t\in S\right\},
\end{aligned}$$

the right-hand side of (16) is equal to

$$\sqrt{\frac{\pi}{2}}\|v\|\|(y_1,\ldots,y_n)\|_{L_\infty((S,m),\ell_2^n)}.$$

Because of this, in order to prove (16) it suffices to prove

$$\|V\| \leq \sqrt{\frac{\pi}{2}}\|v\|. \tag{17}$$

Let V^*, v^* be the adjoints of V, v respectively. Then proving

$$\|V^*\| \leq \sqrt{\frac{\pi}{2}}\|v^*\| \tag{18}$$

is the same as proving (17), since $\|V^*\| = \|V\|$ and $\|v^*\| = \|v\|$. Since Hilbert spaces are reflexive, we have that $H^* = H$ and $(\ell_2^n(H))^* = \ell_2^n(H)$. Furthermore, $(\ell_\infty^N)^*$ is isometrically isomorphic to ℓ_1^N and similarly, $(L_\infty((S,m),\ell_2^n))^* = (L_1((S,m),\ell_2^n)$. Thus V^*, v^* can be viewed as mappings

$$V^* : \ell_2^n(H) \longrightarrow L_1((S,m),\ell_2^n)$$

and

$$v^* : H \longrightarrow \ell_1^N.$$

Moreover, for all $(x_1,\ldots,x_n) \in \ell_2^n(H)$ and all $(z_1,\ldots,z_n) \in L_\infty((S,m),\ell_2^n)$ we get

$$\begin{aligned}
\langle V^*(x_1,\ldots,x_n),(z_1,\ldots,x_n)\rangle &= \langle (x_1,\ldots,x_n), V(z_1,\ldots,z_n)\rangle \\
&= \langle (x_1,\ldots,x_n),(v(z_1),\ldots,v(z_n))\rangle \\
&= \sum_{i=1}^n \langle x_i, v(z_i)\rangle \\
&= \sum_{i=1}^n \langle v^*(x_i), z_i\rangle \\
&= \langle (v^*(x_1),\ldots,v^*(x_n)),(z_1,\ldots,z_n)\rangle,
\end{aligned}$$

whence $V^*(x_1,\ldots,x_n) = (v^*(x_1),\ldots,v^*(x_n)) \in \ell_1^N \oplus \cdots \oplus \ell_1^N$. Then we get, by the definition of the norms $\|\cdot\|_{L_1((S,m),\ell_2^n)}$ and $\|\cdot\|_{\ell_2^n(H)}$, that

$$\begin{aligned}
\|V^*(x_1,\ldots,x_n)\|_{L_1((S,m),\ell_2^n)} &= \sum_{t=1}^N \|(v^*(x_1)(t),\ldots,v^*(x_n)(t))\|_{\ell_2^n(H)} \\
&= \sum_{t=1}^N \left(\sum_{j=1}^n |v^*(x_j)(t)|^2\right)^{\frac{1}{2}} \\
&= \int_S \left(\sum_{j=1}^n |v^*(x_j)(t)|^2\right)^{\frac{1}{2}} dm(t),
\end{aligned}$$

and if we apply Proposition 1.8 to v^* we obtain

$$\int_S \left(\sum_{j=1}^n |v^*(x_j)(t)|^2\right)^{\frac{1}{2}} dm(t) \leq \sqrt{\frac{\pi}{2}}\|v^*\| \left(\sum_{j=1}^n \|x_j\|^2\right)^{\frac{1}{2}}.$$

Thus we now have that

$$\|V^*(x_1,\dots,x_n)\|_{L_1((S,m),\ell_2^n)} \le \sqrt{\frac{\pi}{2}}\|v^*\|\|(x_1,\dots,x_n)\|_{\ell_2^n(H)},$$

which implies (18), thus completing the proof. □

The following property of p-absolutely summing operators (cf. Proposition 5.2 in [7]) will be very useful in the proof of our Theorem 1.7. We will also include a proof.

Proposition 1.10. *Let V and W be Banach spaces and let $(V_i)_{i\in I}$ be a net of subspaces of V directed by inclusion such that $\overline{\cup_{i\in I}V_i} = V$. Let $u : V \longrightarrow W$ be a linear operator and set $u_i = u|_{V_i}$ for all $i \in I$. Let $1 \le p < \infty$. Then u is p-absolutely summing if and only if for all $i \in I$*

$$u_i \in \Pi_p(V_i, W) \qquad \textit{and} \qquad \sup_{i\in I} \pi_p(u_i) < \infty. \tag{19}$$

Moreover, we then have

$$\pi_p(u) = \sup_{i\in I} \pi_p(u_i).$$

Proof. Suppose that $u \in \Pi_p(V,W)$. Then it follows straight from the definition that $u|_{V_i} \in \Pi_p(V_i, W)$ and that $\pi_p(u|_{V_i}) \le \pi_p(u)$.

Conversely, assume that we have (19). We will show that then u is p-absolutely summing and that $\pi_p(u) \le \sup\{\pi_p(u|_{V_i}) : i \in I\}$. Let $c = \sup\{\pi_p(u|_{V_i}) : i \in I\}$ and let $x_1,\dots,x_n \in \cup_{i\in I}V_i$. Since $(V_i)_{i\in I}$ is directed by inclusion, $\{x_1,\dots,x_n\}$ must be included in at least one V_i. Therefore we have, by the definition of p-absolutely summing operators,

$$\left(\sum_{j=1}^n \|ux_j\|^p\right)^{\frac{1}{p}} \le c \sup\left\{\left(\sum_{j=1}^n |x^*(x_j)|^p\right)^{\frac{1}{p}} : \|x^*\|_{V_i^*} \le 1\right\}.$$

Now, since $\cup_{i\in I}V_i$ is dense in V this must remain true for any $x_1,\dots,x_n$ in V, but that means exactly that $u \in \Pi_p(V,W)$, as wanted. Moreover, by the definition of $\pi_p(u)$ we must have $\pi_p(u) \le c$. This completes the proof. □

We are now ready to give the proof of Theorem 1.7:

Proof of Theorem 1.7. Recall that if X, Y are Banach spaces, then $d(X,Y)$ denotes the Banach-Mazur distance between X and Y, i.e.,

$$d(X,Y) := \inf\{\|S\|\cdot\|S^{-1}\| : S : S \longrightarrow Y \text{ linear isomorphism}\}.$$

It is proved in [6] that $C(K)$ has the following property: For any $\lambda > 1$, there exists a directed net $(V_i)_{i\in I}$ of finite dimensional subspaces of $C(K)$ with $\overline{\cup_{i\in I}V_i} = C(K)$ such that for all $i \in I$,

$$d(V_i, \ell_\infty) \le \lambda, \tag{20}$$

where $N_i = \dim V_i$ for all $i \in I$. This is the property of $C(K)$ of being a so-called $\mathcal{L}_\infty^\lambda$-space for all $\lambda > 1$.

Let $\lambda > 1$ and consider the associated directed net $(V_i)_{i\in I}$ of finite dimensional subspaces of $C(K)$ with $\overline{\cup_{i\in I} V_i} = C(K)$ satisfying (20). For any $i \in I$, let $T_i := T|_{V_i}$ and let $S_i : \ell_\infty^{N_i} \longrightarrow V_i$ be a linear isomorphism such that $\|S_i\|\|S_i^{-1}\| \leq \lambda$. Then we have the following diagram of linear operators:

$$\begin{tikzcd} V_i \arrow[r, "T_i"] & H \\ \ell_\infty^{N_i} \arrow[u, "S_i"] \arrow[ur, dashed, "T_i \circ S_i"'] & \end{tikzcd}$$

By Proposition 1.9, $T_i \circ S_i$ is 2-absolutely summing with

$$\pi_2(T_i \circ S_i) \leq \sqrt{\frac{\pi}{2}}\|T_i \circ S_i\| \leq \sqrt{\frac{\pi}{2}}\|T_i\|\|S_i\|. \tag{21}$$

Since $T_i = (T_i \circ S_i) \circ S_i^{-1}$, by the ideal-property of Proposition 1.3 it follows that T_i is 2-absolutely summing with

$$\pi_2(T_i) \leq \pi_2(T_i \circ S_i)\|S_i^{-1}\|. \tag{22}$$

Combining (21) and (22) we deduce that

$$\begin{aligned} \pi_2(T_i) &\leq \sqrt{\frac{\pi}{2}}\|T_i\|\|S_i\|\|S_i^{-1}\| \\ &\leq \lambda\sqrt{\frac{\pi}{2}}\|T_i\| \\ &\leq \lambda\sqrt{\frac{\pi}{2}}\|T\|. \end{aligned}$$

Since this holds for all $i \in I$, an application of Proposition 1.10 shows that T is 2-absolutely summing with

$$\pi_2(T) \leq \lambda\sqrt{\frac{\pi}{2}}\|T\|,$$

and since $\lambda > 1$ was arbitrary, it follows that

$$\pi_2(T) \leq \sqrt{\frac{\pi}{2}}\|T\|,$$

which completes the proof. $\square$

Remark 3. *We can actually prove that the best constant in the little Grothendieck inequality in the real case is* $\sqrt{\frac{\pi}{2}}$. *This we will do by showing that the best constant in the inequality in Proposition 1.8 is* $\sqrt{\frac{\pi}{2}}$.

Proof. Let $(\Omega, F, \mathbb{P})$ be a probability space and let $(g_n)_{n\in\mathbb{N}}$ be a sequence of independant $N(0,1)$ stochastic variables on $(\Omega, F, \mathbb{P})$. Let $u : \ell_2(\mathbb{N}) \longrightarrow L_1(\Omega, \mathbb{P})$ be defined by

$$u(e_n) := g_n$$

for all $n \in \mathbb{N}$, where $(e_n)_{n\in\mathbb{N}}$ is the sequence of canonical unit vectors in $\ell_2(\mathbb{N})$.

We start by showing that $\|u\| = \sqrt{\frac{2}{\pi}}$. Indeed, for all $x \in \ell_2(\mathbb{N})$, $x = \sum_{n=1}^{\infty} \alpha_n e_n$, we have

$$u(x) = \sum_{n=1}^{\infty} \alpha_n u(e_n) = \sum_{n=1}^{\infty} \alpha_n g_n,$$

thus

$$\left\| \sum_{n=1}^{\infty} \alpha_n g_n \right\|_{L_1(\Omega,\mathbb{P})} = \left(\sum_{n=1}^{\infty} \alpha_n^2 \right)^{\frac{1}{2}} \|g_1\|_{L_1(\Omega,\mathbb{P})} \tag{23}$$

$$= \sqrt{\frac{\pi}{2}} \left(\sum_{n=1}^{\infty} \alpha_n^2 \right)^{\frac{1}{2}}, \tag{24}$$

where the equality (23) follows as explained in the proof of Proposition 1.8. On the other hand, as $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis for $\ell_2(\mathbb{N})$,

$$\|x\|_{\ell_2(\mathbb{N})} = \left\| \sum_{n=1}^{\infty} \alpha_n e_n \right\|_{\ell_2(\mathbb{N})} = \left(\sum_{n=1}^{\infty} \alpha_n^2 \right)^{\frac{1}{2}}.$$

Hence $\|u(x)\|_{L_1(\Omega,\mathbb{P})} = \sqrt{\frac{\pi}{2}} \|x\|_{\ell_2(\mathbb{N})}$ for all $x \in \ell_2$ and this implies that $\|u\| = \sqrt{\frac{2}{\pi}}$, as wanted.

Now, for all $n \in \mathbb{N}$ we have

$$\int_\Omega \left(\sum_{j=1}^{n} |u(e_j)|^2 \right)^{\frac{1}{2}} d\mathbb{P} = \int_\Omega \left(\sum_{j=1}^{n} |g_j|^2 \right)^{\frac{1}{2}} d\mathbb{P}.$$

Using classical facts from probability theory, the distribution of $\sqrt{g_1^2 + \cdots + g_n^2}$ is the so-called chi-distribution with n degrees of freedom, denoted by $\chi(n)$. Its mean is

$$\mathbb{E}(\chi(n)) = \sqrt{2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})},$$

whence

$$\int_\Omega \left(\sum_{j=1}^{n} |g_j|^2 \right)^{\frac{1}{2}} d\mathbb{P} = \sqrt{2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}.$$

On the other hand,

$$\left(\sum_{j=1}^{n}\|e_j\|_{\ell_2(\mathbb{N})}^2\right)^{\frac{1}{2}} = \sqrt{n}.$$

Our task is now to show that for n large,

$$\sqrt{2}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \cong \sqrt{n}.$$

More precisely, if we let $\beta_n := \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$ for all $n \in \mathbb{N}$ to ease notation, we will prove that

$$\lim_{n\to\infty}\frac{\beta_n}{\sqrt{\frac{n}{2}}} = 1.$$

First, observe that for all $n \in \mathbb{N}$ we have

$$\beta_n\beta_{n+1} = \frac{n}{2}. \tag{25}$$

Indeed, using the properties of the Γ-function,

$$\beta_n\beta_{n+1} = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})} = \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2})} = \frac{n}{2}.$$

Moreover, we claim that the sequence $(\beta_n)_{n\in\mathbb{N}}$ is nondecreasing, i.e.,

$$\beta_n \le \beta_{n+1} \tag{26}$$

for all $n \in \mathbb{N}$. This follows from the fact that the mapping $t \mapsto \log\Gamma(t)$, $t > 0$ is convex (cf. the Bohr-Mollerup theorem). Indeed, let $n \in \mathbb{N}$ be given. Then convexity yields that

$$\log\Gamma\left(\frac{n}{2}+\frac{1}{2}\right) \le \frac{1}{2}\left(\log\Gamma\left(\frac{n}{2}\right) + \log\Gamma\left(\frac{n}{2}+1\right)\right),$$

which implies

$$2\log\Gamma\left(\frac{n+1}{2}\right) \le \log\Gamma\left(\frac{n}{2}\right) + \log\Gamma\left(\frac{n+2}{2}\right),$$

whence

$$\log\Gamma\left(\frac{n+1}{2}\right) - \log\Gamma\left(\frac{n}{2}\right) \le \log\Gamma\left(\frac{n+2}{2}\right) - \log\Gamma\left(\frac{n+1}{2}\right),$$

i.e.,

$$\log\beta_n \le \log\beta_{n+1},$$

which proves (26).

Therefore, by (25) and (26) we get for all $n \ge 1$ that

$$\beta_n^2 \le \beta_n\beta_{n+1} = \frac{n}{2},$$

whence $\beta_n \leq \sqrt{\frac{n}{2}}$ for all $n \geq 1$. Respectively, for all $n \geq 2$ we get

$$\beta_n^2 \geq \beta_n \beta_{n-1} = \frac{n-1}{2},$$

which implies that $\beta_n \geq \sqrt{\frac{n+1}{2}}$ for all $n \geq 2$.

Now we have, for all $n \geq 2$, that

$$\sqrt{\frac{n+1}{2}} \leq \beta_n \leq \sqrt{\frac{n}{2}},$$

which implies exactly that $\lim_{n\to\infty} \frac{\beta_n}{\sqrt{\frac{n}{2}}} = 1$, thus completing the proof. □

2 On Ultrafilters and Ultraproducts

Before we can prove the non-commutative version of the little Grothendieck inequality we need some theory about ultrafilters and ultraproducts. We will start with some definitions and then give some propositions that show some of the important properties.

2.1 Filters and Ultrafilters

Definition 2.1. *Let $I \neq \emptyset$ be a set. A nonempty collection $\mathscr{F}$ of subsets of I is called a filter on I if and only if*

1. $\emptyset \notin \mathscr{F}$
2. $A, B \in \mathscr{F} \Longrightarrow A \cap B \in \mathscr{F}$
3. $A \in \mathscr{F}, A \subseteq B \Longrightarrow B \in \mathscr{F}$.

Note that since $\mathscr{F}$ is nonempty it follows from condition 3 that $I \in \mathscr{F}$. Among filters on I there is a natural partial order given by inclusion.

Definition 2.2. *An ultrafilter on I is a maximal filter on I (with respect to the partial order $\subseteq$). That is, $\mathcal{U}$ is an ultrafilter on I if and only if*

1. *$\mathcal{U}$ is a filter on I*
2. *If S is another filter on I such that $\mathcal{U} \subseteq S$, then $\mathcal{U} = S$.*

An ultrafilter on I is called trivial if it is on the form $\mathcal{U}_x = \{A \subseteq I : x \in A\}$. A nontrivial ultrafilter is called a free ultrafilter. Note that if $\mathcal{U}$ is a free ultrafilter then $\bigcap_{A\in\mathcal{U}} A = \emptyset$.

Definition 2.3 (Convergence along ultrafilters)**.** *Let I be an infinite set and $\mathscr{F}$ be an (ultra)filter on I. Let (X, τ) be a Hausdorff topological space and $\{x_\alpha\}_{\alpha\in I} \subseteq X$. We say* *that $\{x_\alpha\}_{\alpha\in I}$ converges to some element $x \in X$ along the (ultra)filter $\mathscr{F}$ (and we write $\lim_{\mathscr{F}} x_\alpha = x$) if and only if for every open neighborhood O_x of x we have*

$$\{\alpha \in I : x_\alpha \in O_x\} \in \mathscr{F}.$$

If $\lim_{\mathscr{F}} x_\alpha$ exists then it is unique.

Now we will give some properties of ultrafilters and convergence along ultrafilters.

Proposition 2.4. *Let I be an infinite set and let $\mathscr{F}$ be a free ultrafilter on I. Then for any $A \subset I$, A finite, we have*

(a) $A \notin \mathscr{F}$.

(b) $I \backslash A \in \mathscr{F}$.

Proof. a) Suppose by contradiction that $A \in \mathscr{F}$. Set

$$\mathscr{B} := \{A \cap F : F \in \mathscr{F}\} \subset \mathscr{F}.$$

Note that $\mathscr{B}$ is a finite family of subsets of $\mathscr{F}$, since they are all subsets of A which is finite. Therefore there exists a minimal element $B \in \mathscr{B}$, $B = A \cap F_0$ for some $F_0 \in \mathscr{F}$. Here minimal means that there is no proper subset of B which is contained in $\mathscr{F}$. Now we have

- $B \in \mathscr{F}$.
- $B \subseteq A$, hence B is finite.
- $B \cap F = B$ for all $F \in \mathscr{F}$ by minimality, and hence $B \subseteq F$ for all $F \in \mathscr{F}$.

This contradicts with the fact that $\mathscr{F}$ is free. Thus $A \notin \mathscr{F}$.

b) With the sets and constructions as above, we have

$$B = A \cap F_0 \subseteq A \cap F, \qquad F \in \mathscr{F}$$

because otherwise we would have that the set

$$(A \cap F_0) \cap (A \cap F) = A \cap (F \cap F_0) \in \mathscr{F}$$

would be a proper subset of $B = A \cap F_0$. But that contradicts the minimality of B.

Therefore, since $\mathscr{F}$ is free, it follows that

$$B = A \cap F_0 \subset \bigcap_{F\in\mathscr{F}} (A \cap F) = A \cap (\bigcap_{F\in\mathscr{F}} F) = \emptyset.$$

Hence $A \cap F_0 = \emptyset$, which implies that $F_0 \subset I \backslash A$. Since $F_0 \in \mathscr{F}$ it now follows that $I \backslash A \in \mathscr{F}$, as we wanted. $\square$

The following is a very useful result concerning convergence along a filter that we will use repeatedly in the sequel. For a proof see, e.g., [1].

Theorem 2.5. *Let X be a compact topological space. For any (infinite) index set I and any ultrafilter $\mathcal{U}$ on I, every family $\{x_\alpha\}_{\alpha\in I} \subseteq X$ converges along the ultrafilter $\mathcal{U}$ to some point $x \in X$.*

Let $\ell_\infty(\mathbb{N}, \mathbb{F})$ be the space of all bounded sequences indexed on $\mathbb{N}$ with entries in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, that is,

$$\ell_\infty(\mathbb{N}, \mathbb{F}) := \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{F}, \|(x_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}}|x_n| < \infty\}.$$

This corollary is an immediate consequence of the above Theorem.

Corollary 2.6. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and $(x_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N}, \mathbb{F})$. Then $\lim_{\mathcal{U}} x_n$ exists.*

Proof. For every sequence $(x_n)_{n\in N} \in \ell_\infty(\mathbb{N}, \mathbb{F})$ we have for all $n \in \mathbb{N}$ that $|x_n| \leq \|(x_n)_{n\in\mathbb{N}}\|_\infty < \infty$. This means that $\{x_n\}_{n\in\mathbb{N}}$ is contained in the closed unit ball of radius $\|(x_n)_{n\in\mathbb{N}}\|$. Since this is compact Theorem 2.5 gives us that $\lim_{\mathcal{U}} x_n$ exists. □

Proposition 2.7. *Let $\mathcal{U}$ be a (free) ultrafilter on $\mathbb{N}$ and define $\varphi_u : \ell_\infty(\mathbb{N}, \mathbb{F}) \longrightarrow \mathbb{F}$ by*

$$\varphi_u((x_n)_{n\in\mathbb{N}}) := \lim_{\mathcal{U}} x_n$$

for all $(x_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N}, \mathbb{F})$. Then φ_u is linear, multiplicative, positive (that is, if $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\varphi_u((x_n)_{n\in\mathbb{N}}) \geq 0$) and bounded with $\|\varphi_u\| = 1$.

Proof. We consider $\mathbb{F} = \mathbb{R}$. The proof is similar in the complex case, with obvious modifications.

Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N}, \mathbb{R})$ and suppose that $\lim_{\mathcal{U}} x_n = x$ and $\lim_{\mathcal{U}} y_n = y$. Let O be an open neighborhood of $x + y$. Then there exist open neighborhoods O_x and O_y of x and y respectively, such that $O_x + O_y \subseteq O$. Since $\lim_U x_n = x$ and O_x is an open neighborhood of x we have that $\{n \in \mathbb{N} : x_n \in O_x\} \in \mathcal{U}$, and similarly $\{n \in \mathbb{N} : y_n \in O_y\} \in \mathcal{U}$. Therefore the intersection

$$\{n \in \mathbb{N} : x_n \in O_x\} \cap \{n \in \mathbb{N} : y_n \in O_y\}$$

belongs to $\mathcal{U}$, and since

$$\{n \in \mathbb{N} : x_n \in O_x\} \cap \{n \in \mathbb{N} : y_n \in O_y\} \subseteq \{n \in \mathbb{N} : x_n + y_n \in O_x + O_y \subseteq O\}$$

we have that $\{n \in \mathbb{N} : x_n + y_n \in O\} \in \mathcal{U}$. Since O is an open neighborhood of $x + y$ this means exactly that

$$\lim_{\mathcal{U}}(x_n + y_n) = x + y.$$

Now let $c \in \mathbb{R}$ and let $\varepsilon > 0$ be given. Since $\lim_{\mathcal{U}} x_n = x$ we have that $\left\{n \in \mathbb{N} : |x_n - x| < \frac{\varepsilon}{|c|}\right\} \in \mathcal{U}$, but

$$\left\{n \in \mathbb{N} : |x_n - x| < \frac{\varepsilon}{|c|}\right\} = \{n \in \mathbb{N} : |cx_n - cx| < \varepsilon\},$$

so $\lim_{\mathcal{U}}(cx_n) = cx$. Thus φ_u is linear.

We proceed to show that φ_u is multiplicative. As before, suppose that $\lim_{\mathcal{U}} x_n = x$ and $\lim_{\mathcal{U}} y_n = y$ and let $\varepsilon > 0$ be given. Our goal is to show that $\{n \in \mathbb{N} : |x_n y_n - xy| < \varepsilon\} \in \mathcal{U}$. We observe that

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&\leq |x_n||y_n - y| + |x_n - x||y| \\
&\leq \|(x_n)_{n\in\mathbb{N}}\|_\infty |y_n - y| + |x_n - x||y|,
\end{aligned}$$

and from this it follows that

$$\begin{aligned}
\left\{n \in \mathbb{N} : |y_n - y| < \frac{\varepsilon}{2\|(x_n)_{n\in\mathbb{N}}\|_\infty}\right\} \quad &\cap \quad \left\{n \in \mathbb{N} : |x_n - x| < \frac{\varepsilon}{2|y|}\right\} \\
&\subseteq \quad \{n \in \mathbb{N} : |x_n y_n - xy| < \varepsilon\}.
\end{aligned} \tag{27}$$

Since $\lim_{\mathcal{U}} y_n = y$ and $(x_n)_{n\in\mathbb{N}} \in \ell_\infty$, so that $\|(x_n)_{n\in\mathbb{N}}\|_\infty < \infty$, we obtain

$$\left\{n \in \mathbb{N} : |y_n - y| < \frac{\varepsilon}{2\|(x_n)_{n\in\mathbb{N}}\|_\infty}\right\} \in \mathcal{U}$$

and since $\lim_{\mathcal{U}} x_n = x$

$$\left\{n \in \mathbb{N} : |x_n - x| < \frac{\varepsilon}{2|y|}\right\} \in \mathcal{U},$$

so by (27) we conclude that

$$\{n \in \mathbb{N} : |x_n y_n - xy| < \varepsilon\} \in \mathcal{U},$$

since $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. This completes the proof of φ_u being multiplicative.

Now let $(x_n)_{n\in\mathbb{N}}$ be such that $x_n \geq 0$ for all n. We wish to show that this implies $\lim_{\mathcal{U}} x_n = x \geq 0$. Suppose that $x < 0$. Then there exists an $\varepsilon > 0$ such that $x \in (-2\varepsilon, -\varepsilon)$. This implies that

$$\{n \in \mathbb{N} : x_n \in (-2\varepsilon, -\varepsilon)\} \in \mathcal{U}$$

but the set on the left-hand side is empty. This contradicts with $\mathcal{U}$ being an ultrafilter. Thus $x \geq 0$, and this shows that φ_u is positive.

To show the boundedness of φ_u, take $(x_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N}, \mathbb{R})$ with $\|(x_n)_{n\in\mathbb{N}}\|_\infty \leq 1$. Then we get

$$|\varphi_u((x_n)_{n\in\mathbb{N}})| = |\lim_{\mathcal{U}} x_n| \leq \lim_{\mathcal{U}} |x_n| \leq \|(x_n)_{n\in\mathbb{N}}\|_\infty \leq 1.$$

Suppose now that $(x_n)_{n\in\mathbb{N}}$ is such that $x_n = 1$ for all n. Then it follows from the definition of convergence along an ultrafilter that $\lim_{\mathcal{U}} x_n = 1$, since $\mathbb{N} \in \mathcal{U}$. Hence $\|\varphi_u\| = 1$, which completes the proof. $\square$

2.2 Ultraproducts of Banach spaces

We now define the ultraproduct of Banach spaces. First we need some constructions.

Let I be an (infinite) index set and $(V_i, \|\cdot\|)_{i\in I}$ be a family of Banach spaces indexed on I. Consider

$$\ell_\infty(I, V_i) := \{(x_i)_{i\in I} : x_i \in V_i \,,\ \|(x_i)_{i\in I}\|_\infty = \sup_{i\in I} \|x_i\| < \infty\}.$$

Note that $\|\cdot\|_\infty$ is a norm on $\ell_\infty(I, V_i)$ and that $(\ell_\infty(I, V_i), \|\cdot\|_\infty)$ is a Banach space.

Now let $\mathcal{U}$ be an ultrafilter on I and set

$$N_{\mathcal{U}} := \{(x_i)_{i\in I} \in \ell_\infty(I, V_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

Proposition 2.8. *1. $N_{\mathcal{U}}$ is a closed subspace of $\ell_\infty(I, V_i)$.*

2. The quotient norm on $\ell_\infty(I, V_i)/N_{\mathcal{U}}$ is given by

$$\|[(x_i)_{i\in I}]\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\| \tag{28}$$

for all $[(x_i)_{i\in I}] \in \ell_\infty(I, V_i)/N_{\mathcal{U}}$.

Proof. 1) First we show that $N_{\mathcal{U}}$ is a subspace of $\ell_\infty(I, V_i)$. Given $(x_i)_{i\in I}, (y_i)_{i\in I} \in N_{\mathcal{U}}$ we have that $(x_i + y_i)_{i\in I} \in \ell_\infty(I, V_i)$ and

$$0 \le \lim_{\mathcal{U}} \|x_i + y_i\| \le \lim_{\mathcal{U}} (\|x_i\| + \|y_i\|) = \lim_{\mathcal{U}} \|x_i\| + \lim_{\mathcal{U}} \|y_i\| = 0$$

so this means that $(x_i + y_i)_{i\in I} \in N_{\mathcal{U}}$. Similarly, if $\alpha \in \mathbb{F}$, then $\alpha(x_i)_{i\in I} = (\alpha x_i)_{i\in I} \in N_{\mathcal{U}}$. Hence $N_{\mathcal{U}}$ is a subspace of $\ell_\infty(I, V_i)$.

Now we show that $N_{\mathcal{U}}$ is closed. Let $((x_i^n)_{i\in I})_{n\in\mathbb{N}}$ be a sequence of elements in $N_{\mathcal{U}}$ converging to some $(x_i)_{i\in I} \in \ell_\infty(I, V_i)$. Our goal is then to show that $(x_i)_{i\in I} \in N_{\mathcal{U}}$, or equivalently, that $\lim_{\mathcal{U}} \|x_i\| = 0$.

Let $\varepsilon > 0$ be given. Because the sequence converges there exists n_ε such that for all $n \ge n_\varepsilon$ we have

$$\|(x_i^n)_{i\in I} - (x_i)_{i\in I}\|_\infty < \frac{\varepsilon}{2}$$

but $\|(x_i^n)_{i\in I} - (x_i)_{i\in I}\|_\infty = \sup_{i\in I} \|x_i^n - x_i\|$ and hence $\|x_i^{n_\varepsilon} - x_i\| < \frac{\varepsilon}{2}$ for all $i \in I$. Note that $\|x_i\| \le \|x_i^{n_\varepsilon} - x_i\| + \|x_i^{n_\varepsilon}\|$, thus

$$\left\{i \in I : \|x_i^{n_\varepsilon}\| < \frac{\varepsilon}{2}\right\} \subseteq \{i \in i : \|x_i\| < \varepsilon\}.$$

We see that $\lim_{\mathcal{U}} \|x_i^{n_\varepsilon}\| = 0$ because $(x_i^{n_\varepsilon})_{i\in I} \in N_{\mathcal{U}}$, and hence

$$\left\{i \in I : \|x_i^{n_\varepsilon}\| < \frac{\varepsilon}{2}\right\} \in \mathcal{U}$$

which implies that $\{i \in I : \|x_i\| < \varepsilon\} \in \mathcal{U}$. It then follows that $\lim_{\mathcal{U}} \|x_i\| = 0$, i.e., $(x_i)_{i\in I} \in N_{\mathcal{U}}$ as we wanted.

2) Since $N_{\mathcal{U}}$ is a closed subspace of $\ell_\infty(I, V_i)$ we can consider the quotient space $\ell_\infty(I, V_i)/N_{\mathcal{U}}$, with quotient norm given by

$$\|[(x_i)_{i\in I}]\|_{\mathcal{U}} = \inf\{\|(x_i)_{i\in I} + (y_i)_{i\in I}\|_\infty : (y_i)_{i\in I} \in N_{\mathcal{U}}\}.$$

We wish to show that

$$\|[(x_i)_{i\in I}]\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\|.$$

Given $[(x_i)_{i\in I}] \in \ell_\infty(I, V_i)/N_{\mathcal{U}}$ we have that for all $(y_i)_{i\in i} \in N_{\mathcal{U}}$

$$\lim_{\mathcal{U}} \|x_i\| = \lim_{\mathcal{U}} \|x_i + y_i\|.$$

On the other hand,

$$\lim_{\mathcal{U}} \|x_i + y_i\| \leq \sup_{i\in I} \|x_i + y_i\| = \|(x_i + y_i)_{i\in I}\|_\infty.$$

This gives us that

$$\lim_{\mathcal{U}} \|x_i\| \leq \inf\{\|(x_i + y_i)_{i\in I}\|_\infty : (y_i)_{i\in I} \in N_{\mathcal{U}}\} = \|[x_i]\|_{\mathcal{U}}.$$

Now, suppose by contradiction that there exists an $r > 0$ such that $\lim_{\mathcal{U}} \|x_i\| < r < \|[x_i]\|_{\mathcal{U}}$ and define

$$A_r := \{i \in I : \|x_i\| < r\} = \{i \in I : \|x_i\| \in (-\infty, r)\}.$$

By the definition on ultralimit we have that $A_r \in \mathcal{U}$. Consider the element $(y_i)_{i\in I} \in \ell_\infty(I, V_i)$ defined for $i \in I$ by

$$y_i = \begin{cases} -x_i & \text{if } i \notin A_r \\ 0 & \text{if } i \in A_r \end{cases}.$$

We show that $(y_i)_{i\in I} \in N_{\mathcal{U}}$, i.e., that $\lim_{\mathcal{U}} \|y_i\| = 0$. Let $\varepsilon > 0$ be given. Then

$$A_r \subseteq \{i \in I : \|y_i\| < \varepsilon\} \in \mathcal{U}$$

because $\mathcal{U}$ is an ultrafilter and $A_r \in \mathcal{U}$, and we also have that

$$\lim_{\mathcal{U}} \|x_i\| = \lim_{\mathcal{U}} \|x_i + y_i\|.$$

Now we get

$$\begin{aligned} \|[(x_i)_{i\in I}]\|_{\mathcal{U}} &\leq \|(x_i)_{i\in I} + (y_i)_{i\in I}\|_\infty \\ &= \sup_{i\in A_r} \|x_i\| \\ &\leq r \\ &< \|[(x_i)_{i\in I}]\|_{\mathcal{U}} \\ &= \inf\{\|(x_i)_{i\in I} + (y_i)_{i\in I}\|_\infty : (y_i)_{i\in I} \in N_{\mathcal{U}}\}, \end{aligned}$$

which is impossible. Thus we have reached our contradiction. $\square$

As a consequence, the quotient space $\ell_\infty(I, V_i)/\,N_{\mathcal{U}}$ is a Banach space, with norm given by (28). We are now ready for the following definition:

Definition 2.9. *The Banach space $\ell_\infty(I, V_i)/\,N_{\mathcal{U}}$ is denoted by $(\prod_{i\in I} V_i)/\,\mathcal{U}$ and it is called the ultraproduct of $(V_i)_{i\in I}$ with respect to $\mathcal{U}$. If $V_i = V$ for all $i \in I$ then the space $(\prod_{i\in I} V_i)/\,\mathcal{U}$ is called the ultrapower of V with respect to $\mathcal{U}$, and it is usually denoted by $(V)_{\mathcal{U}}$ or $V_{\mathcal{U}}$.*

Remark 4. *Let V be a Banach space, I an (infinite) set and let $\mathcal{U}$ be a free ultrafilter on I. Then there is a canonical isometric embedding $\varphi : V \hookrightarrow (V)_{\mathcal{U}}$ given by*

$$\varphi(x) = [(x_i)_{i\in I}],$$

where $x_i := x$ for all $i \in I$. Indeed, for every $x \in V$,

$$\|\varphi(x)\|_{\mathcal{U}} = \|[(x_i)_{i\in I}]\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x\| = \|x\|,$$

as wanted.

Proposition 2.10. *Let $\mathcal{U}$ be a (free) ultrafilter on $\mathbb{N}$. Let $(V_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ be families of Banach spaces over $\mathbb{F}$. For every $n \in \mathbb{N}$ let $T_n \in B(V_n, W_n)$ and assume that $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. Define $(T_n)_{\mathcal{U}} : (\prod_{n\in\mathbb{N}} V_n)/\mathcal{U} \longrightarrow (\prod_{n\in\mathbb{N}} W_n)/\mathcal{U}$ by*

$$(T_n)_{\mathcal{U}}([(x_n)_{n\in\mathbb{N}}]) := [(T_n(x_n))_{n\in\mathbb{N}})]$$

for all $[(x_n)_{n\in\mathbb{N}}] \in (\prod_{n\in\mathbb{N}} V_n)/\mathcal{U}$.
The operator $(T_n)_{\mathcal{U}}$ is called the ultraproduct of the family of operators $(T_n)_{n\in N}$, and it is a well defined bounded linear operator satisfying

$$\|(T_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|T_n\|.$$

Proof. First we make sure that $(T_n)_{\mathcal{U}}$ is in fact well defined. Suppose that $[(x_n)_{n\in\mathbb{N}}] = [(x'_n)_{n\in\mathbb{N}}]$. This is equivalent to saying that $\lim_{\mathcal{U}} \|x_n - x'_n\| = 0$. From Proposition 2.7 we know that φ_u is linear and positive, so since $\|T_n(x_n - x'_n)\| \le \|T_n\|\|x_n - x'_n\|$ we get

$$\begin{aligned}
\lim_{\mathcal{U}} \|T_n(x_n - x'_n)\| &\le \lim_{\mathcal{U}}(\|T_n\|\|x_n - x'_n\|) \\
&\le \lim_{\mathcal{U}}((\sup_{n\in\mathbb{N}} \|T_n\|)\|x_n - x'_n\|) \\
&= (\sup_{n\in\mathbb{N}} \|T_n\|) \lim_{\mathcal{U}} \|x_n - x'_n\| \\
&= 0,
\end{aligned}$$

so $(T_n)_{\mathcal{U}}$ is well defined. Now, let $[(x_n)_{n\in\mathbb{N}}] \in (\prod_{n\in\mathbb{N}} V_n)/\mathcal{U}$. Then, using the multiplicativity of φ_u, we see that

$$\begin{aligned}
\|(T_n)_{\mathcal{U}}([(x_n)_{n\in\mathbb{N}}])\|_{\mathcal{U}} &= \|[(T_n(x_n))_{n\in\mathbb{N}}]\|_{\mathcal{U}} \\
&= \lim_{\mathcal{U}} \|T_n(x_n)\| \\
&\le \lim_{\mathcal{U}}(\|T_n\|\|x_n\|) \\
&= (\lim_{\mathcal{U}} \|T_n\|)(\lim_{\mathcal{U}} \|x_n\|) \\
&= (\lim_{\mathcal{U}} \|T_n\|)\|[(x_n)_{n\in\mathbb{N}}]\|_{\mathcal{U}},
\end{aligned}$$

so $(T_n)_\mathcal{U}$ is bounded, with $\|(T_n)_\mathcal{U}\| \leq \lim_\mathcal{U} \|T_n\|$. To show that we have equality, let $\varepsilon > 0$ be given. By the definition of the norm of an operator there exists an x_n with $\|x_n\| = 1$, such that $\|T_n(x_n)\| \geq \|T_n\| - \varepsilon$. Then by positivity of φ_u,

$$\|(T_n)_\mathcal{U}([(x_n)_{n\in\mathbb{N}}])\|_\mathcal{U} = \lim_\mathcal{U} \|T_n(x_n)\| \geq \lim_\mathcal{U}(\|T_n\| - \varepsilon) = \lim_\mathcal{U} \|T_n\| - \varepsilon.$$

Since this holds for arbitrary $\varepsilon > 0$ we conclude $\|(T_n)_\mathcal{U}\| = \lim_\mathcal{U} \|T_n\|$, as wanted. $\square$

Proposition 2.11. *Let $\mathcal{U}$ be a (free) ultrafilter on $\mathbb{N}$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of C^*-algebras. Then the ultraproduct $(\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$ is a C^*-algebra and moreover, if A_n is unital for every $n \in \mathbb{N}$ then $(\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$ will be unital as well.*

Proof. Since $(A_n)_{n\in\mathbb{N}}$ are C^*-algebras they are in particular Banach spaces, so $((\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}, \|\cdot\|_\mathcal{U})$ is a Banach space. Define a multiplication on $(\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$ by

$$[(x_n)_{n\in\mathbb{N}}] \cdot [(y_n)_{n\in\mathbb{N}}] := [(x_n y_n)_{n\in\mathbb{N}}].$$

This is welldefined. Indeed, assume that $[(x_n)_{n\in\mathbb{N}}] = [(x'_n)_{n\in\mathbb{N}}]$ and $[(y_n)_{n\in\mathbb{N}}] = [(y'_n)_{n\in\mathbb{N}}]$. That is, $\lim_\mathcal{U} \|x_n - x'_n\| = \lim_\mathcal{U} \|y_n - y'_n\| = 0$. We observe that

$$\|x_n y_n - x'_n y'_n\| = \|x_n y_n - x_n y'_n + x_n y'_n - x'_n y'_n\| \leq \|x_n\|\|y_n - y'_n\| + \|x_n - x'_n\|\|y'_n\|,$$

and using the properties of φ_u we get

$$\begin{aligned}
\lim_\mathcal{U} \|x_n y_n - x'_n y'_n\| &\leq \lim_\mathcal{U}(\|x_n\|\|y_n - y'_n\| + \|x_n - x'_n\|\|y'_n\|) \\
&\leq \lim_\mathcal{U}(\|(x_n)_{n\in\mathbb{N}}\|_\infty \|y_n - y'_n\|) + \lim_\mathcal{U}(\|x_n - x'_n\|\|(y_n)_{n\in\mathbb{N}}\|_\infty) \\
&= \|(x_n)_{n\in\mathbb{N}}\|_\infty \lim_\mathcal{U} \|y_n - y'_n\| + \|(y_n)_{n\in\mathbb{N}}\|_\infty \lim_\mathcal{U} \|x_n - x'_n\| \\
&= 0,
\end{aligned}$$

i.e., $[(x_n y_n)_{n\in\mathbb{N}}] = [(x'_n y'_n)_{n\in\mathbb{N}}]$.

One can check that that $(\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$ is an algebra, but we will omit the proof of that. Now we define an involution on $(\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$ by

$$([(x_n)_{n\in\mathbb{N}}])^* := [(x_n^*)_{n\in\mathbb{N}}]$$

for all $[(x_n)_{n\in\mathbb{N}}] \in (\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$. This is easily seen to be well-defined. We observe that for all $[(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}] \in (\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$ we have

$$\begin{aligned}
\|[(x_n)_{n\in\mathbb{N}}] \cdot [(y_n)_{n\in\mathbb{N}}]\|_\mathcal{U} &= \|[(x_n y_n)_{n\in\mathbb{N}}]\|_\mathcal{U} \\
&= \lim_\mathcal{U} \|x_n y_n\| \\
&\leq \lim_\mathcal{U}(\|x_n\|\|y_n\|) \\
&= (\lim_\mathcal{U} \|x_n\|)(\lim_\mathcal{U} \|y_n\|) \\
&= \|[(x_n)_{n\in\mathbb{N}}]\|_\mathcal{U} \|[(y_n)_{n\in\mathbb{N}}]\|_\mathcal{U},
\end{aligned}$$

so $(\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$ is in fact a Banach algebra. To show that it is a C^*-algebra it now only remains to check that the C^*-identity holds. We get, for all $[(x_n)_{n\in\mathbb{N}}] \in (\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$,

$$\begin{aligned}
\|[(x_n)_{n\in\mathbb{N}}]^*[(x_n)_{n\in\mathbb{N}}]\|_{\mathcal{U}} &= \|[(x_n^* x_n)_{n\in\mathbb{N}}]\|_{\mathcal{U}} \\
&= \lim_{\mathcal{U}} \|x_n^* x_n\| \\
&= \lim_{\mathcal{U}} (\|x_n\|^2) \\
&= (\lim_{\mathcal{U}} \|x_n\|)^2 \\
&= \|[(x_n)_{n\in\mathbb{N}}]\|_{\mathcal{U}}^2,
\end{aligned}$$

as wished.

Now suppose that A_n is unital for every $n \in \mathbb{N}$, with unit 1_n. Set $1_{\mathcal{U}} := [(1_n)_{n\in\mathbb{N}}]$. This makes sense since $\|(1_n)_{n\in\mathbb{N}}\|_\infty = 1 < \infty$, so $(1_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N}, A_n)$. We wish to show that $1_{\mathcal{U}}$ is the unit in $(\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$. Let $[(x_n)_{n\in\mathbb{N}}] \in (\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$. We get

$$[(x_n)_{n\in\mathbb{N}}] \cdot 1_{\mathcal{U}} = [(x_n 1_n)_{n\in\mathbb{N}}] = [(x_n)_{n\in\mathbb{N}}]$$

and similarly $1_{\mathcal{U}} \cdot [(x_n)_{n\in\mathbb{N}}] = [(x_n)_{n\in\mathbb{N}}]$. Thus $(\prod_{n\in\mathbb{N}} A_n)/\mathcal{U}$ is indeed unital, with unit $1_{\mathcal{U}}$. □

Proposition 2.12. *Let $\mathcal{U}$ be a (free) ultrafilter on $\mathbb{N}$. If $(H_n)_{n\in\mathbb{N}}$ is a family of Hilbert spaces over $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), then $(\prod_{n\in\mathbb{N}} H_n)/\mathcal{U}$ is a Hilbert space (over $\mathbb{F}$), as well.*

Proof. In particular $(H_n)_{n\in\mathbb{N}}$ are Banach spaces, so $(\prod_{n\in\mathbb{N}} H_n)/\mathcal{U}$ is a Banach space. We can define an inner product on $(\prod_{n\in\mathbb{N}} H_n)/\mathcal{U}$ by

$$\big\langle [(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}] \big\rangle_{\mathcal{U}} := \lim_{\mathcal{U}} \langle x_n, y_n \rangle_{H_n}.$$

This is well-defined. Indeed, note that

$$|\langle x_n, y_n \rangle_{H_n}|^2 \leq \|x_n\|_{H_n} \|y_n\|_{H_n} \leq \|(x_n)_{n\in\mathbb{N}}\|_\infty \|(y_n)_{n\in\mathbb{N}}\|_\infty < \infty,$$

which implies that $(\langle x_n, y_n \rangle_{H_n})_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N}, \mathbb{F})$. This shows that $\lim_{\mathcal{U}} \langle x_n, y_n \rangle_{H_n}$ exists, as explained before. Moreover, suppose that $[(x_n)_{n\in\mathbb{N}}] = [(x_n')_{n\in\mathbb{N}}]$ and $[(y_n)_{n\in\mathbb{N}}] = [(y_n')_{n\in\mathbb{N}}]$. We then wish to show that $\lim_{\mathcal{U}} \langle x_n, y_n \rangle_{H_n} = \lim_{\mathcal{U}} \langle x_n', y_n' \rangle_{H_n}$. We get

$$\begin{aligned}
|\lim_{\mathcal{U}} \langle x_n, y_n \rangle_{H_n} - \lim_{\mathcal{U}} \langle x_n', y_n' \rangle_{H_n}| &= |\lim_{\mathcal{U}} (\langle x_n, y_n \rangle_{H_n} - \langle x_n', y_n' \rangle_{H_n})| \\
&\leq \lim_{\mathcal{U}} |\langle x_n - x_n', y_n - y_n' \rangle_{H_n}| \\
&\leq (\lim_{\mathcal{U}} \|x_n - x_n'\|)(\lim_{\mathcal{U}} \|y_n - y_n'\|) \\
&= 0,
\end{aligned}$$

as wanted.

Now we show that $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ defined as above is indeed an inner product on $(\prod_{n\in\mathbb{N}} H_n)/\mathcal{U}$. It is obvious that $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ is linear in the first variable and conjugate linear in the second, because $\langle\cdot,\cdot\rangle_{H_n}$ are inner products and from Proposition 2.7 we know that ultralimit is linear. Moreover we see that

$$\begin{aligned}
\overline{\big\langle[(x_n)_{n\in\mathbb{N}}],[(y_n)_{n\in\mathbb{N}}]\big\rangle_{\mathcal{U}}} &= \overline{\lim_{\mathcal{U}}\langle x_n,y_n\rangle}_{H_n} && (29)\\
&= \lim_{\mathcal{U}}\overline{\langle x_n,y_n\rangle}_{H_n} && (30)\\
&= \lim_{\mathcal{U}}\langle y_n,x_n\rangle_{H_n}\\
&= \big\langle[(y_n)_{n\in\mathbb{N}}],[(x_n)_{n\in\mathbb{N}}]\big\rangle_{\mathcal{U}},
\end{aligned}$$

where the equality from (29) to (30) is a consequence of linearity of the map φ_u from Proposition 2.7 in the complex case.

Also, by positivity of φ_u, we know that $\lim_{\mathcal{U}}\langle x_n,x_n\rangle_{H_n} > 0$ for all $(x_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N},\mathbb{F})$. We conclude that $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ above defined is, indeed, an inner product on the ultraproduct $(\prod_{n\in\mathbb{N}} H_n)/\mathcal{U}$. Further we observe that for all $(x_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N},\mathbb{F})$,

$$\begin{aligned}
\|[(x_n)_{n\in\mathbb{N}}]\|_{\mathcal{U}} &= \lim_{\mathcal{U}}\|x_n\|_{H_n}\\
&= \lim_{\mathcal{U}}\langle x_n,x_n\rangle_{H_n}^{\frac{1}{2}}\\
&= \left(\lim_{\mathcal{U}}\langle x_n,x_n\rangle_{H_n}\right)^{\frac{1}{2}}\\
&= \big\langle[(x_n)_{n\in\mathbb{N}}],[(x_n)_{n\in\mathbb{N}}]\big\rangle_{\mathcal{U}}^{\frac{1}{2}},
\end{aligned}$$

where we have used the multiplicativity of φ_u from Proposition 2.7.

This argument shows that the norm induced by $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ is exactly $\|\cdot\|_{\mathcal{U}}$. Since $(\prod_{n\in\mathbb{N}} H_n)/\mathcal{U}$ is complete with respect to $\|\cdot\|_{\mathcal{U}}$ the assertion follows. $\square$

3 The Non-Commutative Little Grothendieck Inequality

The goal in this section is to give a proof of the non-commutative version of the little Grothendieck inequality. We will follow Uffe Haagerup's approach from [4]. First we will need some results from operator algebra theory.

3.1 Some Operator Algebra Results

Definition 3.1. *For any T in a unital C^*-algebra A, the set*

$$\sigma(T) := \{\lambda\in\mathbb{C} : T-\lambda I \text{ is not invertible in } A\}$$

is called the spectrum of T (relative to A).

Remark 5. *For T bounded, the spectrum $\sigma(T)$ is a nonempty compact subset of $\mathbb{C}$ and, moreover, $\sigma(T)\subseteq\overline{B}(0,\|T\|)$. If T is self-adjoint then $\sigma(T)\subseteq\mathbb{R}$.*

In the sequel we will frequently use the so-called continuous functional calculus for normal elements. The following is a very brief introduction in order to recall some of the important properties. This is treated in greater detail in most introductory volumes on Operator Algebra, for example [10].

The Continuous Functional Calculus For any $T \in A$, the smallest unital C^*-subalgebra containing T is denoted by $C^*(T)$, and it is the closure of all non-commutative polynomials in T, T^* and I. We say that $C^*(T)$ is the C^*-algebra generated by T. If T is normal, that is, if $T^*T = TT^*$, then $C^*(T)$ is commutative. Moreover, we then have a $*$-isomorphism

$$\begin{aligned} C(\sigma(T)) &\cong C^*(T) \\ f &\mapsto f(T) \end{aligned} \tag{31}$$

defined by

$$f(T) := \lim_{n\to\infty} p_n(T),$$

where $(p_n)_{n\in\mathbb{N}}$ are polynomials such that $\lim_{n\to\infty} \|f - p_n\|_{(C(\sigma(T)),\|\cdot\|_\infty)} = 0$. This is well-defined and isometric because for every polynomial p, $p(T)$ is normal in A and the spectral mapping theorem (see, e.g., Corollary I.3.3 in [3]) implies

$$\begin{aligned} \|p(T)\| &= \sup\{|\lambda| : \lambda \in \sigma(p(T))\} \\ &= \sup\{|p(\lambda)| : \lambda \in \sigma(T)\} \\ &= \|p\|_{C(\sigma(T))}. \end{aligned}$$

In the correspondence (31), the constant function $1 \in C(\sigma(T))$ is mapped to the identity $I \in A$, and the identity function $id : z \mapsto z$ is mapped to T.

The algebra of all bounded operators on a Hilbert space H is denoted $B(H)$, and it is a C^*-algebra with the usual adjoint operation.

The polar decomposition of an operator T in $B(H)$ is a factorization $T = U|T|$ where $|T|$ is positive and U is a partial isometry with initial space $\overline{\mathrm{Ran}(|T|)}$ and range space $\overline{\mathrm{Ran}(T)}$.

Theorem 3.2. *Every bounded operator T on a Hilbert space H has a unique polar decomposition $T = U|T|$. The positive operator $|T| := (T^*T)^{1/2}$ lies in $C^*(T)$ and the partial isometry U belongs to $W^*(T)$ (the von Neumann algebra generated by T). If T is invertible, then U is a unitary element of $C^*(T)$.*

Proof. Define an operator U on $\mathrm{Ran}(|T|)$ by $U(|T|x) = Tx$. The range of U is then precisely $\mathrm{Ran}(T)$. Extend U by continuity to the closure $\overline{\mathrm{Ran}(|T|)}$, and then define U to be 0 on $\mathrm{Ran}(|T|)^\perp = \ker(|T|) = \ker(T)$.

Since H is a Hilbert space we have that $H = \overline{\mathrm{Ran}(|T|)} \oplus \mathrm{Ran}(|T|)^\perp$ so we can extend U by linearity to all of H. By construction, U is a partial isometry with initial space equal to $\overline{\mathrm{Ran}(|T|)}$ and range $\overline{\mathrm{Ran}(T)}$. The choice of U will be uniquely determined since we require that $U(|T|x) = Tx$ and that the range of U^*U equals $\overline{\mathrm{Ran}(|T|)}$. By definition we have that $W^*(T) = \overline{C^*(T)}^{wot}$ (the weak operator topology closure of $C^*(T)$). The von Neumann Double Commutant Theorem (see, e.g., Theorem I.7.1 in [3]) tells us that $\overline{C^*(T)}^{wot} = C^*(T)''$ (the double commutant of $C^*(T)$). Hence $W^*(T) = C^*(T)''$, so to verify that $U \in W^*(T)$ we show $U \in C^*(T)''$.

Let $X \in C^*(T)'$ (the commutant of $C^*(T)$) and $x \in \ker(T)$. Then $TXx = XTx = 0$ so we see that $Xx \in \ker(T)$. Since U equals 0 on $\ker(T)$, we have $UXx = 0 = XUx$ for every $x \in \ker(T)$ so X and U commute on $\ker(T)$.

Now let $x \in \mathrm{Ran}(|T|)$, that is, $x = |T|y$ for some y. Then

$$UXx = UX|T|y = (U|T|)Xy = TXy = XTy = XU|T|y = XUx,$$

because $|T| \in C^*(T)$ and X commutes with everything in $C^*(T)$.

So now we have that X and U commute on $\ker(T) = \mathrm{Ran}(|T|)^\perp$ and on $\mathrm{Ran}(|T|)$, so again using the fact that $H = \mathrm{Ran}(|T|)^\perp \oplus \overline{\mathrm{Ran}(|T|)}$ it will follow by continuity and linearity that they commute on all of H. Thus $U \in C^*(T)'' = W^*(T)$.

When T is invertible it implies that $|T|(= \sqrt{T^*T})$ is invertible, because when T is invertible then T^*T is also invertible, so $0 \notin \sigma(T^*T)$. Since T^*T is normal we get, by continuous functional calculus and the spectral mapping theorem (see, e.g., Corollary I.3.3 in [3]), that $0 \notin \sqrt{\sigma(T^*T)} = \sigma(\sqrt{T^*T})$.

So since $T = U|T|$ we have, when T is invertible, that $U = T|T|^{-1} = T(\sqrt{T^*T})^{-1}$ so $U \notin C^*(T)$ and we see that

$$U^*U = \sqrt{T^*T}^{-1} T^*T \sqrt{T^*T}^{-1} = T^*T(T^*T)^{-1} = I$$

and similarly $UU^* = I$, so U is unitary. □

The following is Theorem I.8.4 in [3], and we will use it in the proof of the non-commutive little Grothendieck inequality.

Theorem 3.3. *In any unital C^*-algebra A, the closed convex hull of the set of unitary elements is the whole closed unit ball.*

Proof. Let $S \in A$ be invertible with $\|S\| < 1$. We start by showing that S is the average of two unitaries U_1 and V_1 from A, i.e., $S = \frac{U_1+V_1}{2}$.

By Theorem 3.2, S has a unique polar decomposition $S = U|S|$ where $U \in C^*(S) \subseteq A$ is unitary and $|S| = \sqrt{S^*S}$. Now define $V_\pm := |S| \pm i\sqrt{I - S^*S}$. These are unitaries and we see that

$$S = \frac{UV_+ + UV_-}{2}.$$

To realize that V_+ is in fact a unitary, define a function $f_+ : [0,1] \longrightarrow D = \{z \in \mathbb{C} : |z| = 1\}$ by

$$f_+(x) = x + i\sqrt{1-x^2} \qquad , \qquad x \in [0,1].$$

Then f_+ is continuous on $[0,1]$, and since $|S|$ is positive and $\|S\| < 1$ it follows that $\sigma(S) \subset [0,1]$. Since $|S|$ is normal we can now use continuous functional calculus, which yields $f_+(|S|) = V_+$.
By definition of continuous functional calculus, $f_+(|S|) \in C^*(|S|)$, which is a commutative C^*-algebra. So $f_+(|S|)$ is normal. In fact we see that

$$(f_+(|S|))^*(f_+(|S|)) = \overline{f_+}(|S|)f_+(|S|) = (\overline{f_+}f_+)(|S|) = |f_+|^2(|S|) = I,$$

so $f_+(|S|) = V_+$ is unitary. Showing that V_- is unitary is similar to this.
Thus we have

$$S = \frac{U_1 + V_1}{2} \tag{32}$$

where $U_1 = UV_+$ and $V_1 = UV_-$, as we wanted.

Now let $S, U \in A$ be such that U is unitary and $\|S\| < 1$. We observe that

$$\frac{U+S}{2} = \frac{U(I+U^*S)}{2}.$$

Since $\|S\| < 1$ we have $\|U^*S\| < 1$, so $I+U^*A$ is invertible and $\frac{U(I+U^*S)}{2}$ is a contraction. Furthermore, $\frac{U(I+U^*S)}{2}$ is invertible as a product of invertibles. Applying what we proved above it now follows that there exist unitaries U_1 and V_1 such that

$$\frac{U(I+U^*S)}{2} = \frac{U_1+V_1}{2},$$

i.e.,

$$U + S = U_1 + V_1. \tag{33}$$

Now let $S, U \in A$ be as before. We will show by induction that for all $n \in \mathbb{N}$ there exist unitaries $U_1, \ldots, U_n, V_n$ such that

$$U + nS = U_1 + \ldots + U_n + V_n.$$

We have already in (33) seen that this is true for $n = 1$. Assume it holds for some n. We will show it holds for $n+1$. By (33) there exist unitaries $U_{n+1}, V_{n+1} \in A$ such that we can write $V_n + S = U_{n+1} + V_{n+1}$. We then obtain

$$\begin{aligned} U + (n+1)S &= U + nS + S \\ &= (U + nS) + S \\ &= U_1 + \ldots + U_n + V_n + S \\ &= U_1 + \ldots + U_n + U_{n+1} + V_{n+1}, \end{aligned}$$

which is exactly what we wanted to show.

Now, using these results we have just verified, we proceed to prove the statement. Let $T \in A$ be given. It suffices to assume $\|T\| < 1$. This implies that there exists $n \in \mathbb{N}$ such that $\|T\| < 1 - \frac{2}{n} = \frac{n-2}{n}$. Set

$$S := \frac{n}{n-1}T - \frac{1}{n-1}I.$$

We observe that

$$\begin{aligned}\|S\| &\leq \left\|\frac{n}{n-1}T\right\| + \left\|\frac{1}{n-1}I\right\| \\ &= \frac{n}{n-1}\|T\| + \frac{1}{n-1} \\ &\leq \frac{n}{n-1}\frac{n-2}{n} + \frac{1}{n-1} \\ &= 1,\end{aligned}$$

so from what we have just proved by induction there exist unitaries $U_1, \ldots, U_{n-1}, V_{n-1}$ such that

$$I + (n-1)S = U_1 + \ldots + U_{n-1} + V_{n-1}.$$

But seeing how we defined S we have that

$$I + (n-1)S = I + nT - I = nT.$$

Thus

$$nT = U_1 + \ldots + U_{n-1} + V_{n-1},$$

i.e.,

$$T = \frac{U_1 + \ldots + U_{n-1} + V_{n-1}}{n},$$

so T is in the convex hull of unitary elements from A. $\square$

3.2 Proving the Non-Commutative Little Grothendieck Inequality

We first prove a lemma which will be of great use to us in the following. This is Lemma 3.1 in [4].

Lemma 3.4. *Let T be a bounded linear map from a C^*-algebra A into a Hilbert space H and assume there exists a unitary element $u \in A$ such that $\|T(u)\| = \|T\|$. Consider the functionals φ, ψ on A defined by*

$$\varphi(x) = \frac{1}{\|T\|^2}\langle T(ux), T(u)\rangle \quad , \quad x \in A$$

and

$$\psi(x) = \frac{1}{\|T\|^2}\langle T(xu), T(u)\rangle \quad , \quad x \in A.$$

Then φ and ψ are states on A and

$$\|T(x)\|^2 \leq \|T\|^2(\varphi(x^*x) + \psi(xx^*)) \quad , \quad x \in A.$$

Proof. It is sufficient to consider $\|T\| = 1$ because otherwise we can always just normalize T by dividing by the norm.

We divide the proof into several steps. First assume that $u = 1$. Then $T(1) = \|T\| = 1$ and so $\varphi(x) = \psi(x) = \langle T(x), T(1)\rangle = \langle T(x), 1\rangle$ for all $x \in A$. That φ is linear follows directly from the linearity of T because the inner product is linear in the first coordinate.

We see that $\|\varphi\| = \varphi(1) = 1$ and from this it also follows that φ is positive (see, e.g., Theorem 13.5 in [10]). Hence φ is a state on A.

Let $a \in A$ be self-adjoint and consider for any $t \in \mathbb{R}$ the function $f : \lambda \mapsto e^{it\lambda}$, $\lambda \in \mathbb{R}$. This is continuous, so by continuous functional calculus we can define $f(a) = e^{ita}$. Then

$$\|e^{ita}\| = \|f(a)\| = \|f\| = \sup\{|f(\lambda)| : \lambda \in \sigma_A(a)\} = 1$$

so e^{ita} is unitary for any $t \in \mathbb{R}$.

The power series for the exponetial function is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$$

so by continuous functional calculus

$$e^{ita} = 1 + ita - \frac{t^2a^2}{2!} - i\frac{t^3a^3}{3!} + \frac{t^4a^4}{4!} \dots \tag{34}$$

Define h to be the first three terms in this expansion, that is, $h = 1 + ita - \frac{t^2a^2}{2!}$. Then (34) gives, by rearranging,

$$h = e^{ita} + i\frac{t^3a^3}{3!} - \frac{t^4a^4}{4!} \dots$$

Now applying T and using that T is linear we get

$$T(h) = T(e^{ita}) + t^3T\left(i\frac{a^3}{3!} - t\frac{a^4}{4!} \dots\right).$$

To ease notation let us define $g(a) := i\frac{a^3}{3!} - t\frac{a^4}{4!} \dots$, so that $T(h) = T(e^{ita}) + t^3T(g(a))$. This yields

$$\begin{aligned}
\|T(h)\|^2 &= \|T(e^{ita}) + t^3T(g(a))\|^2 \\
&\leq (\|T(e^{ita})\| + |t^3|\|T(g(a))\|)^2 \\
&= \|T(e^{ita})\|^2 + |t|^6\|T(g(a))\|^2 + 2|t|^3\|T(e^{ita})\|\|T(g(a))\| \\
&\leq 1 + t^6\|T(g(a))\|^2 + 2|t|^3\|T(g(a))\| \\
&= 1 + |t|^3(|t|^3\|T(g(a))\|^2 + 2\|T(g(a))\|) \\
&= 1 + O(|t|^3) \text{ as } t \searrow 0.
\end{aligned} \tag{35}$$

On the other hand, by the definition of h, we get

$$\begin{aligned}
T(h) &= T\left(1 + ita - \frac{t^2a^2}{2!}\right) \\
&= T(1) + itT(a) - \frac{t^2}{2}T(a^2) \\
&= 1 + itT(a) - \frac{t^2}{2}T(a^2)
\end{aligned}$$

and using this gives us

$$\begin{aligned}
\|T(h)\|^2 &= \langle T(h), T(h)\rangle \\
&= \left\langle 1 + itT(a) - \frac{t^2}{2}T(a^2), 1 + itT(a) - \frac{t^2}{2}T(a^2)\right\rangle.
\end{aligned}$$

By trivial calculations, using that $z + \overline{z} = 2\mathrm{Re}(z)$ and that a self-adjoint implies a^2 to be self-adjoint as well, and the fact that $\varphi(a) = \overline{\varphi(a)}$ because φ is positive, we obtain

$$\|T(h)\|^2 = 1 + t^2(\|T(a)\|^2 - \varphi(a^2)) + t^3\left(\frac{t}{4}\|T(a^2)\|^2 - \mathrm{Im}\langle T(a^2), T(a)\rangle\right).$$

By inserting into the inequality (35) we get

$$1 + t^2(\|T(a)\|^2 - \varphi(a^2)) + t^3\left(\frac{t}{4}\|T(a^2)\|^2 - \mathrm{Im}\langle T(a^2), T(a)\rangle\right) \leq 1 + O(|t|^3) \text{ as } t \searrow 0.$$

Now, using the properties of O-notation we obtain

$$t^2(\|T(a)\|^2 - \varphi(a^2) \leq O(|t|^3) \text{ as } t \searrow 0.$$

Letting $t \searrow 0$ this yields $\|T(a)\|^2 - \varphi(a^2) \leq 0$, i.e.,

$$\|T(a)\|^2 \leq \varphi(a^2). \tag{36}$$

Now, let $x \in A$ be arbitrary. Then x can be written as $x = a + b$ where

$$a := \frac{x + x^*}{2} \text{ and } b := \frac{x - x^*}{2i}.$$

Clearly a and b are self-adjoints. Using the linearity of T and that, in general, $a^2 + b^2 \geq 2ab$ we get

$$\begin{aligned}
\|T(x)\|^2 &= \|T(a) + iT(b)\|^2 \\
&\leq (\|T(a)\| + \|T(b)\|)^2 \\
&\leq 2((\|T(a)\|^2 + \|T(b)\|^2) && (37) \\
&\leq 2(\varphi(a^2) + \varphi(b^2)) && (38) \\
&= 2\varphi\left(\left(\frac{x + x^*}{2}\right)^2 + \left(\frac{x - x^*}{2i}\right)^2\right) \\
&= \varphi(xx^* + x^*x),
\end{aligned}$$

where the step from (37) to (38) follows from (36). This completes the proof. $\square$

We will now proceed with proving the non-commutative version of the Little Grothendieck inequality, as it is presented in [4].

Theorem 3.5. *Let A be a C^*-algebra and H a Hilbert space, and let $T : A \longrightarrow H$ be a bounded linear map. Then there exist two states φ and ψ on A such that*

$$\|T(x)\|^2 \leq \|T\|^2(\varphi(x^*x) + \psi(xx^*)), \quad x \in A.$$

Proof. First assume that A is unital. It suffices to show the statement for $\|T\| = 1$, since it is just a scaling. So assume without loss of generality that $\|T\| = 1$.

In Lemma 3.4 we have proven that the statement holds under the assumption that there exists a unitary $U \in A$ such that $1 = \|T\| = \|T(U)\|$. So our goal is now to find such a unitary and then apply the lemma.

By Theorem 3.3, the closed unit ball in A is the closed convex hull of the unitary operators in A. Therefore there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of unitary operators in A such that $\|T(U_n)\| \to \|T\|$ for $n \to \infty$.

Let ω be a free ultrafilter on $\mathbb{N}$ and let A_ω and H_ω be the ultrapowers of A and H respectively, with respect to ω. By Proposition 2.11 we know that A_ω is a unital C^*-algebra (with unit $1_\omega = [(1_n)_{n\in\mathbb{N}}]$, where $1_n = 1_A$ for all $n \in \mathbb{N}$), and from Proposition 2.12 that H_ω is a Hilbert space. Define $T_\omega : A_\omega \longrightarrow H_\omega$ by

$$T_\omega((x_n)_{n\in\mathbb{N}}) = (T(x_n))_{n\in\mathbb{N}}.$$

Then $\|T_\omega\| = \|T\| = 1$. Let $U_\omega \in A_\omega$ be the operator which has $(U_n)_{n\in\mathbb{N}}$ as representing sequence, that is, $U_\omega := [(U_n)_{n\in\mathbb{N}}]$. Note that $\|(U_n)_{n\in\mathbb{N}}\|_\infty = \sup_n \|U_n\| = 1 < \infty$, i.e., $(U_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N}, A)$, so this definition makes sense. We see that

$$(U_\omega)^* U_\omega = [(U_n^*)_{n\in\mathbb{N}}][(U_n)_{n\geq 1}] = [(1_A)_{n\in\mathbb{N}}] = 1_{A_\omega}$$

and similarly

$$U_\omega(U_\omega)^* = 1_{A_\omega},$$

so U_ω is a unitary in A_ω. Moreover,

$$\|T_\omega(U_\omega)\|_\omega = lim_\omega \|T(U_n)\| = \|T\| = 1,$$

so U_ω is indeed the unitary we were looking for. We can now apply lemma 3.4. Hence there exist states φ_ω and ψ_ω on A_ω such that for all $x \in A_\omega$,

$$\|T_\omega(x)\|^2 \leq \|T\|^2(\varphi_\omega(x^*x) + \psi_\omega(xx^*)).$$

Let $j_A : A \hookrightarrow A_\omega$ be the canonical imbedding of A into A_ω. By Remark 4, j_A is an isometry. Let moreover $j_H : H \hookrightarrow H_\omega$ be the canonical embedding of H into H_ω. We then have that the diagram

$$\begin{tikzcd} A \arrow[r, hook, "j_A"] \arrow[d, "T"'] & A_\omega \arrow[d, "T_\omega"] \\ H \arrow[r, hook, "j_H"] & H_\omega \end{tikzcd}$$

commutes, so $T_\omega \circ j_A = j_H \circ T$.

Now, let $\varphi, \psi : A \longrightarrow \mathbb{F}$ be the restrictions of φ_ω resp. ψ_ω to A, that it,

$$\varphi := \varphi_\omega|_A = \varphi_\omega \circ j_A$$

and

$$\psi := \psi_\omega|_A = \psi_\omega \circ j_A.$$

Then φ and ψ are linear as compositions of linear maps; we see that $\|\varphi\| \leq \|\varphi_\omega\|\|j_A\| = 1$ and

$$\varphi(1_A) = \varphi_\omega(j_A(1_A)) = \varphi_\omega(1_\omega) = 1$$

since φ_ω is a state, and similar for ψ. Hence φ and ψ are states on A and since $T = T_\omega|_A$ we have that for all $x \in A$,

$$\|Tx\|^2 \leq \varphi(x^*x) + \psi(xx^*).$$

This concludes the proof in the unital case.

Suppose that A is not unital. Then $A \hookrightarrow A^{**}$ isometrically. Again, let $T : A \longrightarrow H$ be a bounded linear operator with $\|T\| = 1$, and consider the double dual $T^{**} : A^{**} \longrightarrow H^{**}$. We have that $\|T^{**}\| = 1$ and $H^{**} = H$ since Hilbert spaces are reflexive, and we know from, e.g., Proposition 10.1.21 in [5], that A^{**} is a unital C^*-algebra. So by what we proved before there exist states $\tilde{\varphi}$ and $\tilde{\psi}$ on A^{**} such that for all $x \in A^{**}$,

$$\|T^{**}(x)\|^2 \leq \tilde{\varphi}(x^*x) + \tilde{\psi}(xx^*).$$

Then we can choose states φ and ψ on A such that $\varphi \geq \tilde{\varphi}$, $\psi \geq \tilde{\psi}$ and we get

$$\|Tx\|^2 \leq \varphi(x^*x) + \psi(xx^*), \qquad x \in A,$$

as wanted. □

In the following we discuss a couple of corollaries which follow easily from the non-commutative version of the little Grothendieck inequality that we have just proved. They are, respectively, Corollary 3.3 and 3.4 in [4].

Corollary 3.6. *Let $T : A \longrightarrow H$ be a bounded linear map of a C^*-algebra A into a Hilbert space H. Then there exist a state φ on A such that*

$$\|T(x)\|^2 \leq 4\|T\|^2 \varphi\left(\frac{x^*x + xx^*}{2}\right), \qquad x \in A.$$

Proof. By Theorem 3.5 we can find states φ_1 and φ_2 on A such that

$$\|T(x)\|^2 \leq \|T\|^2(\varphi_1(x^*x) + \varphi_2(xx^*)), \qquad x \in A.$$

If we then put $\varphi := \frac{\varphi_1+\varphi_2}{2}$ we get that

$$\begin{aligned}
\|T(x)\|^2 &\leq \|T\|^2((2\varphi - \varphi_2)(x^*x) + (2\varphi - \varphi_1)(xx^*)) \\
&= \|T\|^2(2\varphi(x^*x) + 2\varphi(xx^*) - \varphi_2(x^*x) - \varphi_1(xx^*)) \\
&\leq \|T\|^2(2\varphi(x^*x) + 2\varphi(xx^*)) \\
&= 2\|T\|^2\varphi(x^*x + xx^*),
\end{aligned}$$

exactly as we wanted. □

Corollary 3.7. *Let A and B be C^*-algebras and let $S : A \longrightarrow B$ be a bounded linear map. Then for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in A$,*

$$\left\| \sum_{k=1}^n S(a_k)^* S(a_k) + S(a_k) S(a_k)^* \right\| \leq 4\|S\|^2 \left\| \sum_{k=1}^n a_k^* a_k + a_k a_k^* \right\|.$$

Proof. We can assume that $B \subseteq B(H)$ for some Hilbert space H. Let $\xi \in H$ be a unit vector. Applying Corollary 3.6 to the map $x \mapsto T(x)\xi$, $x \in A$ we get the existence of a state φ on A, such that

$$\|T(x)\xi\|^2 \leq 4\|T\|^2 \varphi\left(\frac{x^* x + x x^*}{2}\right).$$

Hence for $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A$ we get

$$\sum_{k=1}^n \|T(a_k)\xi\|^2 \leq 2\|T\|^2 \varphi\left(\sum_{k=1}^n a_k^* a_k + a_k a_k^*\right) \leq 2\|T\|^2 \left\| \sum_{k=1}^n a_k^* a_k + a_k a_k^* \right\|.$$

We can apply the same argument to the map $x \mapsto T(x^*)^* \xi$ and the operators $a_1^*, \ldots, a_n^* \in A$, and that gives us

$$\sum_{k=1}^n \|T(a_k)^* \xi\|^2 \leq 2\|T\|^2 \left\| \sum_{k=1}^n a_k a_k^* + a_k^* a_k \right\|.$$

Hence

$$\sum_{k=1}^n (\|T(a_k)\xi\|^2 + \|T(a_k)^* \xi\|^2) \leq 4\|T\|^2 \left\| \sum_{k=1}^n a_k a_k^* + a_k^* a_k \right\|,$$

so, since

$$\left\| \sum_{k=1}^n T(a_k)^* T(a_k) + T(a_k) T(a_k)^* \right\| = \sup\left\{ \sum_{k=1}^n \|T(a_k)\xi\|^2 + \|T(a_k)^* \xi\|^2 : \xi \in H, \|\xi\| = 1 \right\}$$

this concludes the proof. $\square$

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