7.6 Softic groups and Metric Ultrapowers of Finite Symmetric groups

Recall

Def. 7.5.1 A group $K \subseteq G$ finite subset, $\varepsilon > 0$, $F$ a non-empty finite set. A map $\varphi : K \to \text{Sym}(F)$

is called an $(K, \varepsilon)$-almost-homomorphism if

1. $\forall k_1, k_2 \in K: d_F(\varphi(k_1k_2), \varphi(k_1)\varphi(k_2)) \leq \varepsilon$
2. $\forall k_1, k_2 \in K, k_1k_2^{-1}: d_F(\varphi(k_1), \varphi(k_2)) \leq 1 - \varepsilon$

Def. 7.5.2 A group $G$ is called softic if for every finite subset $K \subseteq G$ and every $\varepsilon > 0$, there exist $F \neq \emptyset$, $|F| < \infty$ and a $K$-almost-homomorphism $\varphi : G \to \text{Sym}(F)$.

Back to 7.6: let $\mathcal{F} = (I, \omega, \mathcal{F})$, $\mathcal{F} = (F_i)_{i \in I}$ be a family of finite groups, then

$P = \prod_{i \in I} \text{Sym}(F_i)$

is a softic group. Let $\omega$ be an ultrafilter on $I$.

For $\alpha, \beta \in P$, we define

$d_\omega(\alpha, \beta) = \lim_{i \to \omega} d_{F_i}(\alpha_i, \beta_i) \in [0, 1]$

$d_\omega$ is a pseudo-pseudo-metric on $P$.
i) $d_ω(α, α) = 0$

ii) $d_ω(α, β) = d_ω(β, α)$

iii) $d_ω(α, γ) ≤ d_ω(α, β) + d_ω(β, γ)$

Moreover, $d_ω$ is left and right invariant.

iv) $d_ω(γα, γβ) = d_ω(α, β)$

v) $d_ω(αγ, βγ) = d_ω(α, β)$

for all $α, β, γ ∈ P$.

Proposition 7.6.2

The set $N_ω = \{α ∈ P : d_ω(α, e) = 0\}$

is a normal subgroup of $P$.

Proof

$α, β ∈ N_ω \Rightarrow αβ ∈ N_ω$, because

\[
d_ω(αβ, e) ≤ d_ω(αβ, β) + d_ω(β, e) = d_ω(α, e) + d_ω(β, e)
\]

$α ∈ N_ω \Rightarrow α^{-1} ∈ N_ω$, because

\[
d_ω(α^{-1}, e) = d_ω(αα^{-1}, α) = d_ω(e, α) = d_ω(α, e).
\]

Hence $N_ω$ is a subgroup.

$α, γ ∈ N_ω \Rightarrow γαγ^{-1} ∈ N_ω$

\[
d_ω(γαγ^{-1}, e) = d_ω(γα, γ) = d_ω(α, e).
\]
Remark,
\[ \alpha N_w = \beta N_w \iff \beta^{-1} \alpha \in N_w \]
\[ \iff d_w(\beta^{-1} \alpha, e) = 0 \iff d_w(\alpha, \beta) = 0 . \]

Prove the following for any metrics \( \rho \) on \( P/N_w \).

**Theorem 7.6.3**

With \( T = (F_i)_{i \in I} \) and \( w \) as above, then the group \( G_T = P/N_w \) (\( T = (I, \omega, \Sigma) \)) is simple.

**Proof**

**Inductive Step** \((0 < \varepsilon < 1)\)

For \( K \subseteq G_T \) and \( \varepsilon \geq 0 \). For each \( g \in G_T \) choose a representative \( \tilde{g} = (\tilde{g}_i)_{i \in I} \) in \( P \)

\[ (\tilde{g}_i)_{i \in I} = g \]

i.e.

\[ g(\tilde{g}) = g \quad \forall g \in G_T, \] where

\[ g : P \to G_T = P/N_w \] is the quotient map.

If \( K_1, K_2 \subseteq G_T \) and \( k_1 \neq k_2 \), then \( k_1 N_w \neq k_2 N_w \) and hence \( d_w(k_1, k_2) > 0 \). Therefore

\[ \eta = \frac{1}{2} \min_{2 \leq k_1, k_2 \in K} d_w(k_1, k_2) > 0 . \]

Then \( 0 < \eta \leq \frac{1}{2} \). Choose an integer \( m \),

\[ m \geq \frac{\log \frac{1}{\eta}}{\log \frac{1}{\eta}} \quad (0 < \log \frac{1}{\eta} < \log 2) . \]

Then

\[ (\log \frac{1}{\eta})^m \leq (\log \frac{1}{\eta})^m \leq \log 2 \leq (1 - \eta)^m . \]
Hence, \( 1 - (1-\varepsilon)^m \geq 1 - \varepsilon \).

Finally, choose \( \delta \in \mathcal{G}(0,1) \), s.t.
\[
1 - (1-\delta)^m \leq \varepsilon.
\]

For \( h, k \in \mathcal{K} \) we have \( g(\tilde{h}\tilde{k}) = h^*k^*g(h^*)g(k^*) \).

Hence,
\[
\hat{h}^*N_\omega \hat{N}_\omega = (\hat{h}^*N_\omega)(\hat{k}N_\omega) = h^*kN_\omega.
\]

Thus,
\[
d_{\omega}(\tilde{h}, h, k) = 0.
\]

Therefore,
\[
A(h, k) := \{ i \in I : d_{\omega}(h^*_i, k^*_j) \leq \varepsilon \}
\]
is in the ultrafilter \( \mathcal{W} \).

If \( h, k \in \mathcal{K} \), \( h \neq k \), then \( d_{\omega}(h, k) \geq 2\eta \).

Define \( \eta > 0 \) s.t.
\[
C(h, k) := \{ i \in I : d_{\omega}(h^*_i, k^*_j) \geq \eta \},
\]
it is in the ultrafilter \( \mathcal{W} \). Hence,
\[
S = \bigcap_{h, k \in \mathcal{K}} A(h, k) \cap \bigcap_{h, k \in \mathcal{K}} C(h, k) \subset I.
\]

In particular, \( S \neq \emptyset \). Choose \( j \in S \).

Then,
\[
\forall h, k \in \mathcal{K}, \quad j \in A(h, k)
\]
\[
\quad \forall h, k \in \mathcal{K}, \quad j \in C(h, k)
\]

\( \forall h, k \in \mathcal{K} \),
\[
d_{\omega}(h^*_j, k^*_j) \leq \varepsilon
\]
\[
d_{\omega}(h^*_j, k^*_j) \geq \eta
\]
Consider now the map \( \psi: G \to G_j \) given by
\[
\psi(g) = \tilde{g}_j
\]
The
\[
d_F^j(\psi(hk), \psi(h)\psi(k)) \leq \varepsilon \quad \forall h, k \in K
\]
\[
d_F^j(\psi(h), \psi(k)) \geq \eta \quad \forall h, k \in K, h \neq k.
\]
Consider now the Cartesian product
\[
F = F \times F \times \cdots \times F
\]
and
\[
\Phi: \text{Sym}(F_j) \to \text{Sym}(F), \quad \Phi(\pi)(x_1, \ldots, x_m) = (\pi(x_1), \ldots, \pi(x_m))
\]
\[
\equiv x_1 \cdots x_m \in F_j
\]
The \( \Phi \) is a homomorphism and by Corollary 7.4.1,
\[
d_F^m(\Phi(\sigma), \Phi(\sigma')) = (1 - (1 - d_F^j(\sigma, \sigma')))^m
\]
for \( \sigma, \sigma' \in \text{Sym}(F_j) \).
Hence with \( \phi = \Phi \circ \psi : G \to G_j \),
\[
d_F^m(\phi(hk), \phi(h)\phi(k)) = 1 - (1 - d_F^j(\psi(hk), \psi(h)\psi(k)))^m
\]
\[
\leq 1 - (1 - \varepsilon)^m
\]
and for \( h, k, k' \in K \),
\[
d_F^m(\phi(h), \phi(k)) = 1 - (1 - d_F^j(\psi(h), \psi(k)))^m
\]
\[
\leq 1 - (1 - \eta)^m
\]
\[
\geq 1 - \varepsilon
\]
Hence \( \phi \) is a \((K, k)\)-almost homomorphism of \( G_T \) into \( \text{Sym}(F) \). This shows that \( G_T \) is \( k \)-finite.

**Remark 7.6.4**

If \( \omega \) is a principal (i.e. non-free) ultrafilter, then \( \omega = \omega^* = \{ F \in \mathcal{P}(I) : \text{C} \in F \} \)

for some \( i \in I \). Hence \( G_T \approx \text{Sym}(F_i) \) is finite.

**Remark 7.6.5**

\[ \Delta_\omega(g, h) = \delta_\omega(g, h) , \quad g, h \in G_T \]
defines a bi-invariant metric on \( G_T \).

**Theorem 7.6.6**

If \( G \) is a group, then \( T \)-\( \Lambda \)-finite \( G \)

(a) \( G \) is \( k \)-finite

(b) there exists a triple \( T = (I, \omega, \mathcal{F}) \),

(I being set \( \omega = \beta \mathcal{P}(I), \mathcal{F} = (F_i)_{i \in I}, \) each \( F_i \) non-empty and finite)
such that \( G \) is isomorphic to a subgroup of \( G_T \approx \text{Sym}(F_i) \) of \( G_T \).
proof

(b) ⇒ (a) since "soft" is preserved by going to subgroups.

(a) ⇒ (b). Assume \( G \) is a soft group. The set

\[ I = \\{ (K, e) : K \subseteq G \text{ finite}, e \geq 0 \} \]

is directed w.r.t. the ordering

\[(K, e) \leq (K', e') \text{ if } K \subseteq K' \text{ and } e \geq e'.\]

For each \( (K, e) \in I \) put

\[ F_{(K, e)} = \{ (K', e') \in I : K' \supseteq K, e \geq e' \} = \{ i \in I : i \geq i \} .\]

Note that \( I \) is directed for all \( i \in I \). Moreover

\[ I(K, e) \cap F_{(K', e)} = I(K \cup K', \min(e, e')) \]

hence \( (I, \leq) \) is has the finite intersection property, so there exists an ultrafilter \( U \) containing \( I \), i.e.

\[ I(K, e) \in \omega \forall K \subseteq G \text{ finite and } e \geq 0 .\]

Since \( G \) is soft we have for all \( (K, e) \in I \),

if finite non-empty set \( F_i \) and a

\[(K, e) \text{-almost - homomorphism } \phi : G \to F_i ,\]

consider now \( T(I, U, \phi) \) where \( \phi = (F_i)_{i \in I} \) and let

\[ \phi^* = \prod_{i \in I} \phi_i \]

be the product map \( \phi^* : G \to \prod_{i \in I} \text{sym}(F_i) = \prod_{i \in I} \).

(Sym(\( F_i \)))
Claim: \( \Phi \) is an injective homomorphism of \( G \) into \( 6_T \) (prove Thm. 7.6.6).

Proof of Claim:

Let \( g, h \in G \) and let \( \eta > 0 \).

Put \( \eta_0 = (h, h, \eta) \in I \).

If \( i = (k, e) \in I \) and \( i \preceq \eta_0 \), then \( k \in B, e \in \) and \( 0 < e \leq \eta \). Moreover,

\[ d\varphi_i (\varphi_i (gh), \varphi_i (g) \varphi_i (h)) \leq \varepsilon \leq \eta \]

\( \forall i, j \) is a \((K, \varepsilon)\) -almost homomorphism. Thus

\[ I_\eta \subseteq \{ i \in I : d\varphi_i (\varphi_i (gh), \varphi_i (g) \varphi_i (h)) \leq \eta \}. \]

Hence, \( \forall \eta > 0 \), \( \forall i \in I \),

\[ i \in I : d\varphi_i (\varphi_i (gh), \varphi_i (g) \varphi_i (h)) \leq \eta \]

belongs to \( I_\eta \). For fixed \( g, h \in G \), we therefore have

\[ \lim_{i \to m} d\varphi_i (\varphi_i (gh), \varphi_i (g) \varphi_i (h)) = 0, \]

i.e.,

\[ d\varphi (\varphi (gh), \varphi (g) \varphi (h)) = 0 \]

which is equivalent to

\[ \overline{\varphi} (\overline{\varphi} (gh)) = \overline{\varphi} (\overline{\varphi} (g) \overline{\varphi} (h)) = \overline{\varphi} (\overline{\varphi} (g)) \overline{\varphi} (\overline{\varphi} (h)) \]

i.e.,

\[ \overline{\varphi} (gh) = \overline{\varphi} \circ \overline{\varphi} (h). \]
Moreover, if \( g \neq h \), and \( i' \geq \alpha_k = (2g, h^k, \gamma) \).

The

\[
\delta_{E_k}(\phi_i(g), \phi_i(h)) \geq (1 - \varepsilon_\gamma) \geq 1 - \gamma.
\]

Since \( I_{\varepsilon_\gamma} \in \mathcal{U} \), it follows that

\[
\{ i \in I : \delta_{E_k}(\phi_i(g), \phi_i(h)) \geq 1 - \gamma \} \in \mathcal{U}
\]

and hence \( \delta_{E_k}(\phi_i(g), \phi_i(h)) \leq 1 \) always.

Thus

\[
\lim_{l \to \infty} \delta_{E_k}(\phi_l(g), \phi_l(h)) = 1 \quad \forall g, h \in G, \ g \neq h.
\]

The

\[
\delta_{\mathcal{U}}(\phi(g), \phi(h)) = 1.
\]

In particular \( \phi(g) \neq \phi(h) \), this proves the claim as well as Theorem 2.6.6.