The Haar measure

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Resumé I nærværende projekt behandles Haarmålet, som er et translationsinvariant ikke-trivielt Radonmål på en lokalkompakt gruppe. Hovedmålet i dette projekt er at vise at enhver lokalkompakt gruppe besidder et venstre såvel som højre Haarmål og at disse er entydige op til multiplikation af en positiv konstant. For at nå dette resultat, bevises Carathéodorys sætning, som sammen med basale topologiske resultater anvendes til at vise Riesz' repræsentationssætning for positive lineære funktionaler på $C_c(X)$, mængden af kontinuerte funktioner med kompakt støtte, hvor X er et lokalkompakt Hausdorff topologisk rum. Dette resultat anvendes så til at vise eksistensen og entydigheden af Haarmålet. Der gives to beviser for eksistensen af Haarmålet; et bevis i det generelle tilfælde, og ét i tilfældet hvor den betragtede gruppe er abelsk og kompakt.

Ydermere diskuteres sammenhængen mellem venstre og højre Haarmål ved hjælp af den modulære funktion, og der gives eksempler på lokalkompakte grupper og deres venstre og højre Haarmål.

Til sidst, som anvendelse af den teori, der bliver opbygget i løbet af projektet, udnyttes eksistensen af Haarmålet på kompakte grupper til at vise en generalisering af Peter-Weyls sætning om unitære repræsentationer af en kompakt gruppe.

Kommentarer vedrørende den litteratur, som er anvendt i dette Bachelorprojekt, kan findes på s.51

Abstract In the present thesis the Haar measure is discussed, which is a translation invariant nonzero Radon measure on a locally compact group. The main goal of this thesis is to show that every locally compact group posseses a left, as well as a right Haar measure and that they are unique up to multiplication by a positive constant. The first chapter contains a proof of the Carathéodory theorem. This theorem is used together with basic basic topological results to show the Riesz representation theorem for positive, linear functionals on $C_c(X)$, the space of compactly supported continuous functions on X, where X is a locally compact Hausdorff topological space. This result is then used to prove the existence and uniqueness of the Haar measure. Two proofs are presented of the existence of the Haar measure; one proof in the general case, and one in the case where the considered group is compact and abelian.

Furthermore, this thesis contains a discussion about the connection between the left and the right Haar measure using the modular function, as well as examples of locally compact groups and their left and right Haar measures.

Last, as an application of the theory developed in this thesis, the existence of the Haar measure on compact groups is used to show a generalization of the Peter-Weyl theorem about unitary representations of locally comapact groups.

For comments on the litterature used in this thesis, see p.51

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Chapter 1

The Carathéodory theorem

In this chapter we shall present a standard way of constructing measures on a set X, given that we want the measure to take certain values on a collection of subsets $\mathcal{E} \subseteq P(X)$, where $P(X)$ is the power set of X. This can, for example, be used to construct the Lebesgue measure on $\mathbb{R}$. The same procedure will be used to construct the measure in the Riesz representation theorem, and this is why we prove the following proposition and theorem.

We will need the notion of an *outer measure*, defined as follows:

Definition 1.1. *Let X be a set. An* ***outer measure*** *on X is a map $\mu^* : P(X) \to [0, \infty]$ which satisfies*

1. $\mu^*(\emptyset) = 0$.
2. *If $A \subseteq B \subseteq X$ then $\mu^*(A) \leq \mu^*(B)$.*
3. *If $\{A_n\}_{n\in\mathbb{N}} \subseteq P(X)$ then $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.*

We commence by proving a proposition that we can use for constructing an outer measure, provided that we know some of the values it should take:

Proposition 1.2. *Let X be a set and $\mathcal{E} \subseteq P(X)$. Let $\rho : \mathcal{E} \to [0, \infty]$. Assume $\emptyset, X \in \mathcal{E}$ and that $\rho(\emptyset) = 0$. Define $\mu^* : P(X) \to [0, \infty]$ by*

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \rho(E_n) \mid E_n \in \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} E_n\right\}.$$

Then μ^* is an outer measure.

Proof. First of all, note that μ^* is well-defined. Indeed, let $A \subseteq X$. Since $X \in \mathcal{E}$, the set $\{\sum_{n=1}^{\infty} \rho(E_n) \mid E_n \in \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} E_n\}$ is nonempty since A is covered by the family $E_n = X \in \mathcal{E}$, for all $n \in \mathbb{N}$. Moreover, it is clear that $\mu^*(A) \geq 0$ since $\rho(E) \geq 0$ for each $E \in \mathcal{E}$.

Now, we prove that μ^* is an outer measure:

1. It is clear that $\mu^*(\emptyset) = 0$ since $\emptyset$ is covered by the family $E_n = \emptyset \in \mathcal{E}$ for $n \in \mathbb{N}$, and by assumption $\rho(\emptyset) = 0$.

2. Let $A \subseteq B \subseteq X$. Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{E}$ be a family that covers B. Then $A \subseteq B \subseteq \bigcup_{n=1}^{\infty} E_n$ and hence

$$\left\{\sum_{n=1}^{\infty} \rho(E_n) \mid E_n \in \mathcal{E}, B \subseteq \bigcup_{n=1}^{\infty} E_n\right\} \subseteq \left\{\sum_{n=1}^{\infty} \rho(E_n) \mid E_n \in \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} E_n\right\}.$$

We conclude that $\mu^*(A) \leq \mu^*(B)$.

3. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq P(X)$. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, we can choose a family $\{E_{n,k}\}_{k=1}^{\infty} \subseteq \mathcal{E}$ such that $A_n \subseteq \bigcup_{k=1}^{\infty} E_{n,k}$ and $\sum_{k=1}^{\infty} \rho(E_{n,k}) \leq \mu^*(A_n) + 2^{-n}\varepsilon$. It is clear that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n,k=1}^{\infty} E_{n,k}$ and hence

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n,k=1}^{\infty} \rho(E_{n,k}) \leq \sum_{n=1}^{\infty}(\mu^*(A_n) + 2^{-n}\varepsilon) = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

Hence μ^* is an outer measure and the proof is complete. □

We are now ready for some more definitions:

Definition 1.3. *Let X be a set and $\mu^* : P(X) \to [0, \infty]$ be an outer measure. A set $A \subseteq X$ is called* ***μ^*-measurable*** *if the following holds for every $E \subseteq X$:*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Here, A^c is the complement of A in X.

It is easy to see that a set A is μ^*-measurable if and only if $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$ such that $\mu^*(E) < \infty$.

Definition 1.4. *Let X be a set equipped with a σ-algebra $\mathcal{E}$. Let $\mu : \mathcal{E} \to [0, \infty]$ be a measure. A subset $A \subseteq X$ is called a* ***μ-null-set*** *if there is a $B \in \mathcal{E}$ such that $A \subseteq B$ and $\mu(B) = 0$. A measure μ on X is called* ***complete****, if every μ-null-set is measurable.*

In fact, in the Riesz representation theorem, which is the main reason for which we prove the next theorem, we will not use the fact that the measure is actually complete, but nevertheless, this is a nice feature of the Carathéodory theorem, so we will include it anyways. Let us now turn to the main theorem of this chapter:

Theorem 1.5 (Carathéodory)**.** *Let μ^* be an outer measure on a set X. Let $\mathcal{M}$ be the set of μ^*-measurable sets and denote by μ the restriction of μ^* to $\mathcal{M}$. Then $\mathcal{M}$ is a σ-algebra and μ is a complete measure.*

Proof. We will go through the following steps in order to complete the proof:

1. $\mathcal{M}$ is an algebra.
2. For all $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$ we have $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.
3. $\mathcal{M}$ is a σ-algebra.
4. μ is a measure.
5. μ is complete.

1. Let $E \subseteq X$. Since $\mu^*(\emptyset) = 0$ we obtain the equality $\mu^*(E) = \mu^*(E \cap X) + \mu^*(E \cap X^c)$. Hence $X \in \mathcal{M}$. It follows directly from the definition that $A^c \in \mathcal{M}$ whenever $A \in \mathcal{M}$. Now, let $A, B \in \mathcal{M}$. Then for $E \subseteq X$,

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A \cap B) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B^c) \\
&\geq \mu^*((E \cap A \cap B) \cup (E \cap A^c \cap B) \cup (E \cap A \cap B^c)) + \mu^*(E \cap (A \cup B)^c).
\end{aligned}$$

It is not hard to see that $(E\cap A\cap B)\cup(E\cap A^c\cap B)\cup(E\cap A\cap B^c) = E\cap(A\cup B)$, and hence
$$\mu^*(E) \geq \mu^*(E\cap(A\cup B)) + \mu^*(E\cap(A\cup B)^c).$$
As previously remarked, we conclude that $A\cup B\in\mathcal{M}$. Hence $\mathcal{M}$ is an algebra.

2. Assume A, $B\in\mathcal{M}$ and that $A\cap B=\emptyset$. Since A is μ^*-measurable it follows that
$$\mu^*(A\cup B) = \mu^*((A\cup B)\cap A) + \mu^*((A\cup B)\cap A^c) = \mu^*(A)+\mu^*(B).$$

3. Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{M}$. First assume that all the sets are pairwise disjoint, i.e., $A_i\cap A_j=\emptyset$ for $i\neq j$.

Now, for $n\in\mathbb{N}$, define $B_n := \bigcup_{i=1}^n A_i$, and set $B := \bigcup_{i=1}^\infty A_i$. Let $E\subseteq X$. Then, by assumption, for all $n>1$,
$$\mu^*(E\cap B_n) = \mu^*(E\cap B_n\cap A_n)+\mu^*(E\cap B_n\cap A_n^c) = \mu^*(E\cap A_n)+\mu^*(E\cap B_{n-1}).$$
Thus it follows inductively that for $n\in\mathbb{N}$,
$$\mu^*(E\cap B_n) = \sum_{i=1}^n \mu^*(E\cap A_i)$$
Since $\mathcal{M}$ is an algebra, we know that $B_n\in\mathcal{M}$. Furthermore, since $B_n\subseteq B$ we have that $E\cap B^c\subseteq E\cap B_n^c$ and since μ^* is an outer measure, we get that $\mu^*(E\cap B^c)\leq\mu^*(E\cap B_n^c)$. Therefore
$$\mu^*(E) = \mu^*(E\cap B_n)+\mu^*(E\cap B_n^c) \geq \sum_{i=1}^n \mu^*(E\cap A_i)+\mu^*(E\cap B^c).$$
This holds for every $n\in\mathbb{N}$, and hence
$$\begin{aligned}\mu^*(E) &\geq \sum_{i=1}^\infty \mu^*(E\cap A_i)+\mu^*(E\cap B^c)\\ &\geq \mu^*\left(\bigcup_{i=1}^\infty E\cap A_i\right)+\mu^*(E\cap B^c)\\ &= \mu^*(E\cap B)+\mu^*(E\cap B^c)\geq\mu^*(E).\end{aligned}$$

First of all, we note that all the inequalities above are in fact equalities, and hence in particular we have that
$$\mu^*\left(\bigcup_{i=1}^\infty E\cap A_i\right) = \sum_{i=1}^\infty \mu^*(E\cap A_i). \tag{1.1}$$
Furthermore, we see that $\bigcup_{i=1}^\infty A_i = B\in\mathcal{M}$. Hence $\mathcal{M}$ is stable under countable, pairwise disjoint unions. In the general case, when $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{M}$ is not necessarily consisting of pairwise disjoint sets, consider the family $\{C_i\}_{i\in\mathbb{N}}$ where $C_1 := A_1$ and $C_i := A_i\backslash\left(\bigcup_{j=1}^{i-1}A_j\right)$. Using that $\mathcal{M}$ is an algebra, we conclude that $\{C_i\}_{i\in\mathbb{N}}\subseteq\mathcal{M}$. Since the C_i's are pairwise disjoint and $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty C_i$, we conclude that $\bigcup_{i=1}^\infty A_i\in\mathcal{M}$.

4. First note that since $\emptyset \in \mathcal{M}$, it follows that $\mu(\emptyset) = \mu^*(\emptyset) = 0$ by assumption. Now we show that μ is σ-additive. Let $\{A_i\}_{i \in \mathbb{N}}$ and assume that $A_i \cap A_j = \emptyset$ for $i \neq j$. It follows from the preceeding, using the equality (1.1) with $E = \bigcup_{i=1}^{\infty} A_i$ that:

$$\mu^* \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu^*(A_i),$$

and hence $\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i)$.

5. First, let $B \subseteq X$ and assume $\mu^*(B) = 0$. Then for an arbitrary $E \subseteq X$,

$$\mu^*(E) \leq \mu^*(E \cap B) + \mu^*(E \cap B^c) = \mu^*(E \cap B^c) \leq \mu^*(E),$$

using that μ^* is an outer measure. So $B \in \mathcal{M}$.

Now, for $A \in \mathcal{M}$ assume that $\mu(A) = \mu^*(A) = 0$ and that $B \subseteq A$. Then $\mu^*(B) = 0$ by monotonicity of μ^* and therefore $B \in \mathcal{M}$, and we conclude that μ is complete. □

By now it should be clear how Proposition 1.2 and Theorem 1.5 work together in order to construct a measure. We have some idea of what values our measure should take on the collection $\mathcal{E}$. Therefore, we define ρ with these values and by Proposition 1.2 we construct an outer measure μ^* which by Theorem 1.5 we restrict to the set of μ^*-measurable sets in order to get a measure. If the values of ρ are chosen reasonably, then our measure μ is actually an extension of ρ, and our goal is reached. As already mentioned, we shall use this strategy later on, in Chapter 3.

Chapter 2

Topology

Our main topic, the Haar measure is defined on a *locally compact* group. "Locally compact" is a topological notion, and we need to know something about topology in order to get any further.

The reader is assumed to know some basic topology, and hopefully, much of the following will be repetition of well-known facts. We will skip the notion of a basis and subbasis, etc., together with the defition of the product topology, which can be looked up in any book about topology (e.g. [1] Chapter 4).

Throughout this thesis, the notion **neighbourhood** of x is used for an *open* set U such that $x \in U$; in the litterature a neighbourhood is not always open. We will also encounter the notion of a *compact neighbourhood*, and it will be precisely defined later.

Definition 2.1. *Let X be a topological space. Then*

1. *X is called* $\mathbf{T_1}$ *if every one-point set is closed.*
2. *X is called **Hausdorff** if for every $x \neq y$ in X there exist disjoint, open sets U and V such that $x \in U$ and $y \in V$.*
3. *X is called **normal** if X is T_1 and if for every disjoint closed sets $A, B \subseteq X$ there exist disjoint, open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

We need the following notion of an open cover in order to formulate the next definition. Let $Y \subseteq X$. An **open cover** of Y is a family of open sets $\{U_\alpha\}_{\alpha \in A}$ such that $Y \subseteq \bigcup_{\alpha \in A} U_\alpha$. An open cover is called **finite** if the index set A is finite. A **subcover** of the open cover $\{U_\alpha\}_{\alpha \in A}$ is simply an open cover $\{U_\beta\}_{\beta \in B}$ of Y such that $B \subseteq A$.

Definition 2.2. *$K \subseteq X$ is called **compact** if every open cover of K contains a finite subcover.*

Definition 2.3. *Let $\{F_\alpha\}_{\alpha \in A}$ be a family of subsets of X. We say that the family has the **finite intersection property** if $\bigcap_{\beta \in B} F_\beta \neq \emptyset$ whenever B is finite.*

It is time to refresh some basic properties of compact spaces. The proofs of the following propositions should be well-known, and will be omitted.

Proposition 2.4. *A topological space X is compact if and only if every family of closed subsets of X has the finite intersection property.*

Proposition 2.5. *Let X be compact. Then every closed subspace of X is compact.*

Proposition 2.6. *Every compact Hausdorff space is normal.*

Now we turn our attention to *locally compact* topological spaces. This kind of topological spaces will be of main interest to us. Recall that the Haar measure is defined on a *locally compact* group.

Definition 2.7. *For $x \in X$ we call $C \subseteq X$ a* ***compact neighbourhood*** *of x if C is compact and there exists a neighbourhood U of x such that $U \subseteq C$.*

Definition 2.8. *A topological space X is called* ***locally compact*** *if every point has a compact neighbourhood.*

Definition 2.9. *A subspace $Y \subseteq X$ of a topological space is called* ***precompact*** *if its closure $\bar{Y}$ is compact.*

Proposition 2.10. *Let X be a locally compact Hausdorff space and let $x \in X$. Then every neighbourhood U of x contains a compact neighbourhood N of x.*

Proof. First, assume U is precompact. Since $x \notin \partial U (:= \bar{U} \backslash U)$, by Proposition 2.6 there exist disjoint open sets V and W in the subspace topology of $\bar{U}$ such that $x \in V$ and $\partial U \subseteq W$.

Since $V \subseteq U$, we conclude that V is open in X. Furthermore, clearly $\bar{V} \subseteq \bar{U} \backslash W$ and since $\partial U \subseteq W$, we conclude that $\bar{V} \subseteq U \backslash W$. $\bar{V}$ is a closed hence compact subset of $\bar{U}$, and letting $N = \bar{V}$, we are done.

Now, let U be an arbitrary open set. Since X is locally compact, there exists a compact neighbourhood F of x. Let F° denote the interior of F, which is clearly a neighbourhood of x. Then $U \cap F^\circ$ is a precompact neighbourhood of x, so by the preceeding argument, it contains a compact neigbourhood of x which is clearly contained in U. $\square$

To proceed, we need a nontrivial but nevertheless well-known result due to Urysohn, which we will state without proof. The proof can be found in [1] (Lemma 4.15).

Lemma 2.11. *Let X be a normal space. If A and B are disjoint, closed sets in X, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for each $x \in A$ and $f(x) = 1$ for each $x \in B$.*

We shall reformulate Urysohn's Lemma so it fits the case where our space is locally compact and Hausdorff. For this, we need a proposition.

Proposition 2.12. *Let X be a locally compact Hausdorff space and let $K \subseteq U \subseteq X$ where K is compact and U is open. Then there exists a precompact V such that $K \subseteq V \subseteq \bar{V} \subseteq U$.*

Proof. For every $x \in K$, choose, using Proposition 2.10, a compact neighbourhood N_x of x such that $N_x \subseteq U$. It is clear that the interior N_x° is a neighbourhood of x. Hence $\{N_x^\circ\}_{x \in K}$ is an open cover of K, and since K is compact, there is a finite subcover $\{N_{x_i}^\circ\}_{i=1}^n$ of K. Letting $V = \bigcup_{i=1}^n N_{x_i}^\circ$ we see that $K \subseteq V$, and since $\bar{V} = \bigcup_{i=1}^n N_{x_i}$ and $N_{x_i} \subseteq U$ for all i, we conclude that $\bar{V}$ is compact and $\bar{V} \subseteq U$. $\square$

Lemma 2.13 (Urysohn, the locally compact version)**.** *Let X be a locally compact Hausdorff space and let $K \subseteq U \subseteq X$ where K is compact and U is open. Then there exist a compact subset $C \subseteq U$ and a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \in C^c$.*

Proof. By Proposition 2.12 there exist a precompact open set V such that $K \subseteq V \subseteq \bar{V} \subseteq U$. Let $C := \bar{V}$ Since C is compact and Hausdorff, we know that C is normal. Hence, by Urysohn's lemma, there exists a continuous function $\tilde{f} : C \to [0, 1]$ such that $\tilde{f}(x) = 1$ for $x \in K$ and $\tilde{f}(x) = 0$ for $x \in \partial V$. Extend $\tilde{f}$ to $f : X \to [0, 1]$ by letting $f(x) = 0$ for $x \in C^c$.

It remains to show that f is continuous. Let $E \subseteq [0, 1]$ be a closed set. If $0 \notin E$, then $f^{-1}(E) = \tilde{f}^{-1}(E)$ which is closed by continuity of f. If $0 \in E$ then $f^{-1}(E) = f^{-1}(E \backslash \{0\}) \cup f^{-1}(\{0\}) = \tilde{f}^{-1}(E \backslash \{0\}) \cup V^c$ which is a union of two closed sets, and we are done. □

Now we are ready to prove the Riesz representation theorem, but in order to reach our main goal, the existence of the Haar measure, we will need a deep topological result later, namely Tychonoff's theorem. Again, we shall omit the proof, which can be found in [1] (Theorem 4.43).

Theorem 2.14 (Tychonoff)**.** *Let $\{X_\alpha\}_{\alpha \in A}$ be any family of compact topological spaces and let $X = \prod_{\alpha \in A} X_\alpha$ equipped with the product topology. Then X is compact.*

Chapter 3

Radon Measures

3.1 The Riesz representation theorem

The Riesz representation theorem is actually a collection of thereoms that share the common name. This chapter is devoted to one of these theorems which is about integration on locally compact spaces. In order to understand the content of this theorem, we will start out with some definitions:

Definition 3.1. *Let X be a locally compact Hausdorff space. For a function $f : X \to \mathbb{C}$, we define the **support** of f as the set*

$$\operatorname{supp} f := \overline{\{x \in A \mid f(x) \neq 0\}} = \overline{f^{-1}(\mathbb{C}\backslash\{0\})}$$

We let $C_c(X)$ denote the set of continuous, complex-valued functions on X with compact support.

Definition 3.2. *A linear functional $I : C_c(X) \to \mathbb{C}$ is called **positive** if $I(f) \geq 0$ whenever $f \geq 0$.*

It is clear that integration with respect to some measure on X gives rise to a positive linear functional, and the Riesz representation theorem states that every positive linear functional on $C_c(X)$ arises in this way, that is, if I is a positive linear functional, then $I(f) = \int f \, d\mu$ for some measure μ on X. But it actually gives us more than that; the measure μ has some really nice regularity properties, namely, it is a *Radon* measure.

Definition 3.3. *Let X be a locally compact Hausdorff space. A **Radon measure** μ on X is a Borel measure with the following properties:*

1. *$\mu(K) < \infty$ for all compact $K \subseteq X$.*
2. *(Outer regularity) For each Borel set E,*

$$\mu(E) = \inf\{\mu(U) \mid U \text{ open}, E \subseteq U\}.$$

3. *(Inner regularity) For each open set U,*

$$\mu(U) = \sup\{\mu(K) \mid K \text{ compact}, K \subseteq U\}.$$

To ease the notation, we need the following definition:

Definition 3.4. *Let X be a locally compact Hausdorff space and let $U \subseteq X$ be open. We call $f \in C_c(X)$* ***subordinate*** *to U, and we write $f \prec U$, if $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subseteq U$.*

The next lemma is a handy reformulation of Urysohn's lemma, that we will use over and over in the following chapters.

Lemma 3.5 (Urysohn)**.** *Let X be a locally compact Hausdorff space and let $K \subseteq U \subseteq X$ where K is compact and U is open. Then there exists $f \in C_c(X)$ such that $f \prec U$ and $f \geq 1_K$.*

Proof. It follows directly from Urysohn's lemma in the locally compact version (Lemma 2.13). □

Proposition 3.6. *Let X be a locally compact Hausdorff space and let $K \subseteq X$ be a compact subset. Then for every finite open cover $\{U_i\}_{i=1}^n$ of K, there exist $g_1, g_2, \ldots, g_n \in C_c(X)$ such that $g_i \prec U_i$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n g_i(x) = 1$ for all $x \in K$.*

Proof. For each $x \in K$ choose a compact neighbourhood N_x such that $N_x \subseteq U_i$ for some i. This can be obtained by using Proposition 2.10 since the U_i's cover K.

Clearly $\{N_x^\circ\}_{x \in K}$ is an open cover of K and hence it contains a finite subcover $\{N_{x_j}^\circ\}_{j=1}^m$. Hence $K \subseteq \bigcup_{j=1}^m N_{x_j}$.

Let $A_i := \{j \in \{1, 2, \ldots, m\} \mid N_{x_j} \subseteq U_i\}$ and set $F_i := \bigcup_{j \in A_i} N_{x_j}$, which is compact. Then $F_i \subseteq U_i$. For each i, by Urysohn's lemma (Lemma 3.5) there exists $h_i \in C_c(X)$ such that $h_i \prec U_i$ and $h_i \geq 1_{F_i}$. Clearly $\sum_{i=1}^n h_i \geq 1_K$.

Let $U := \{x \in X | \sum_{i=1}^n h_i(x) > 0\}$. Since the h_i's are continuous, U is open. By Urysohn's lemma again, there is an $f \in C_c(X)$ such that $f \prec U$ and $f \geq 1_K$.

Let $h_{n+1} := 1 - f \in C_c(X)$. Then $\sum_{i=1}^{n+1} h_i(x) > 0$ for all $x \in X$. For $i = 1, 2, \ldots, n$ we then define

$$g_i = \frac{h_i}{\sum_{j=1}^{n+1} h_j}.$$

Clearly $g_i \in C_c(X)$ with $\operatorname{supp}(g_i) = \operatorname{supp}(h_i)$. Furthermore, $g_i \prec U_i$ and for $x \in K$, we have that $f(x) = 1$ (note that $f \prec U$), so $h_{n+1}(x) = 0$ and hence

$$\sum_{i=1}^n g_i(x) = \frac{\sum_{i=1}^n h_i(x)}{\sum_{i=1}^{n+1} h_i(x)} = 1.$$

The proof is complete. □

Using these results, we are now ready to prove the main theorem of this chapter.

Theorem 3.7 (Riesz Representation Theorem)**.** *Let X be a locally compact Hausdorff space and let I be a positive linear functional on $C_c(X)$. Then there is a unique Radon measure μ on X such that $I(f) = \int f \, \mathrm{d}\mu$, for all $f \in C_c(X)$. Moreover, μ satisfies:*

$$\mu(U) = \sup\{I(f) \mid f \in C_c(X),\, f \prec U\}, \tag{3.1}$$

$$\mu(K) = \inf\{I(f) \mid f \in C_c(X), f \geq 1_K\}, \tag{3.2}$$

for all open $U \subseteq X$ and compact $K \subseteq X$.

Proof. We begin by proving uniqueness. Assume μ is a Radon measure such that $I(f) = \int f \,\mathrm{d}\mu$ for all $C_c(f)$. First, let $U \subseteq X$ be open. We want to show that U satisfies (3.1). Whenever $f \prec U$, clearly $I(f) = \int f \,\mathrm{d}\mu \leq \mu(U)$.

Now, using inner regularity of μ on U, for each $\varepsilon > 0$ we can choose a compact K contained in U such that $\mu(K) \geq \mu(U) - \varepsilon$. By Urysohn's lemma (Lemma 3.5), there exists $f \in C_c(X)$ such that $f \prec U$ and $f \geq 1_K$. By monotonicity of integrals, we get that $I(f) \geq \mu(K) \geq \mu(U) - \varepsilon$. Hence (3.1) holds, and $\mu(U)$ is uniquely determined by I. Since μ is outer regular, for each Borel set E, $\mu(E)$ is uniquely determined by $\mu(U)$ for open U and hence determined by I, and we conclude that μ is unique.

We now prove existence. Let $U \subseteq X$ be an open set and let $E \subseteq X$ be arbitrary. We first define

$$\rho(U) = \sup\{I(f) \mid f \in C_c(X),\, f \prec U\}.$$

Note that $\rho(\emptyset) = 0$. Furthermore, since I is positive and $f \prec U$ implies that $f \geq 0$, we get that $\rho(U) \geq 0$ for each open U. Then we define

$$\mu^*(E) = \inf\{\rho(U) \mid E \subseteq U,\, U \,\text{open}\}.$$

Note that if $U \subseteq V \subseteq X$ are open sets, then clearly $\rho(U) \leq \rho(V)$. Hence $\mu^*(U) = \rho(U)$ for each open $U \subseteq X$.

Last, define μ to be the restriction of μ^* to the Borel sets. In particular, $\mu(U) = \rho(U)$ for every open U.

The proof will proceed as follows:

1. μ^* is an outer measure.
2. Every open set in X is μ^*-measurable.
3. μ is a Borel measure.
4. μ is a Radon measure that satisfies (3.1) and (3.2).
5. $I(f) = \int f \,\mathrm{d}\mu$ for all $f \in C_c(X)$.

1. First, we show that for any sequence $\{U_i\}_{i\in\mathbb{N}}$ of open sets, $\rho\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \rho(U_i)$. Let $U := \bigcup_{i=1}^{\infty} U_i$ and let $f \in C_c(X)$ such that $f \prec U$. Let $K := \operatorname{supp}(f)$. Since $K \subseteq U$, we get that $\{U_i\}_{i\in\mathbb{N}}$ is an open cover of K and hence $K \subseteq \bigcup_{i=1}^{n} U_i$ for some $n \in \mathbb{N}$.

By Proposition 3.6, there exist functions $g_1, g_2, \ldots, g_n \in C_c(X)$ such that $g_i \prec U_i$ for each $1 \leq i \leq n$ and $\sum_{i=1}^{n} g_i(x) = 1$ for all $x \in K$. Clearly then $f = \sum_{i=1}^{n} f g_i$. Moreover, $f g_i \prec U$. By linearity of I and the definition of ρ on open sets, we get

$$I(f) = \sum_{i=1}^{n} I(f g_i) \leq \sum_{i=1}^{n} \rho(U_i) \leq \sum_{i=1}^{\infty} \rho(U_i).$$

Since f was chosen arbitrary such that $f \prec U$, we conclude that $\rho\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \rho(U_i)$.

Now it follows that for an arbitrary $E \subseteq X$

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \rho(U_i) \mid E \subseteq \bigcup_{i=1}^{\infty} U_i,\, U_i \,\text{open}\right\}.$$

Indeed, $\mu^*(E) \leq \inf\{\sum_{i=1}^{\infty} \rho(U_i) \mid E \subseteq \bigcup_{i=1}^{\infty} U_i,\, U_i \text{ open}\}$ by the preceeding, and the other inequality follows from the fact that $U = \bigcup_{i=1}^{\infty} U_i$ where $U_1 = U$ and $U_i = \emptyset$ for $i \geq 2$.

By Proposition 1.2, since ρ maps into the non-negative real numbers and $\rho(\emptyset) = 0$, we conclude from the equality just proven that μ^* is an outer measure.

2. Let $U \subseteq X$ be open. Recall from Chapter 1 that it is enough to show that

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \cap U^c),$$

for every $E \subseteq X$ with $\mu^*(E) < \infty$.

First, we consider a special case, namely the one where $E \subseteq X$ is open. Then $E \cap U$ is open. Let $\varepsilon > 0$. By the definition of ρ there is an $f \prec E \cap U$ such that $I(f) > \rho(E \cap U) - \varepsilon$. Since $\operatorname{supp}(f)$ is compact, hence closed, the set $V := E \backslash (\operatorname{supp}(f))$ is open, and there is $g \prec V$ with $I(g) > \rho(V) - \varepsilon$. Then clearly $f + g \prec E$, so by linearity of I and the fact that V covers $E \cap U^c$,

$$\begin{aligned}\mu^*(E) = \rho(E) &\geq I(f) + I(g) \\ &> \rho(E \cap U) + \rho(V) - 2\varepsilon \\ &\geq \mu^*(E \cap U) + \mu^*(E \cap U^c) - 2\varepsilon.\end{aligned}$$

Since ε was arbitrary, we conclude that $\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \cap U^c)$.

Now to the general case. Let $\varepsilon > 0$, $E \subseteq X$ and assume $\mu^*(E) < \infty$. By definition of μ^*, we can find an open V containing E such that $\rho(V) < \mu^*(E) + \varepsilon$. Using the special case discussed above, together with the facts that μ^* is an outer measure and that $E \subseteq V$, it follows that

$$\begin{aligned}\mu^*(E) + \varepsilon > \rho(V) = \mu^*(V) &\geq \mu^*(V \cap U) + \mu^*(V \cap U^c) \\ &\geq \mu^*(E \cap U) + \mu^*(E \cap U^c).\end{aligned}$$

This holds for every $\varepsilon > 0$ and hence $\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \cap U^c)$ holds.

3. By Carathéodory's theorem, we conclude that the restriction of μ^* to any σ-algebra consisting of μ^*-measurable sets is a measure. In particular, we get that μ is a Borel measure.

4. It is clear from the definition of ρ and μ^* that μ is outer regular and satisfies (3.1). Next we show that μ satisfies (3.2). Let $K \subseteq X$ be a compact set, and let $f \in C_c(X)$ satisfy $f \geq 1_K$. Let $\varepsilon > 0$ and set:

$$U := \{x \in X | f(x) > 1 - \varepsilon\}.$$

Since f is continuous, U is open, and clearly $K \subseteq U$.

For every $g \prec U$ we have that $g \leq (1-\varepsilon)^{-1} f$, and since I is positive and linear, we deduce that $I(g) \leq (1-\varepsilon)^{-1} I(f)$. It follows by monotonicity of μ together with the definition that $\mu(K) \leq \mu(U) = \rho(U) \leq (1-\varepsilon)^{-1} I(f)$. Since this holds for every $\varepsilon > 0$, we conclude that $\mu(K) \leq I(f)$. Hence

$$\mu(K) \leq \inf\{I(f) \mid f \in C_c(X), f \geq 1_K\}.$$

To show the other inequality, let $\varepsilon > 0$ and choose, using outer regularity of K, an open U such that $K \subseteq U$ and $\mu(U) \leq \mu(K) + \varepsilon$.

By Urysohn's lemma there is an $f \in C_c(X)$ such that $f \geq 1_K$ and $f \prec U$. Therefore, $I(f) \leq \mu(U) \leq \mu(K) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\mu(K) \geq \inf\{I(f) \mid f \in C_c(X), f \geq 1_K\}$, and hence equality (3.2) holds.

Now we show that $\mu(K) < \infty$ for every compact K. For this, choose an $f \in C_c(X)$ such that $f \geq 1_K$, which is clearly possible by Urysohn's lemma. Then $\mu(K) \leq I(f) < \infty$.

What is left to prove is inner regularity of μ on open sets. Let U be an open set. If $K \subseteq U$ then $\mu(K) \leq \mu(U)$, so $\mu(U) \geq \sup\{\mu(K) \mid K \text{ compact}, K \subseteq U\}$.

On the other hand, let $\varepsilon > 0$. By definition of μ, we can choose a $f \prec U$ such that $I(f) > \mu(U) - \varepsilon$. Set $K := \operatorname{supp}(f)$, which is compact. If $g \in C_c(X)$ such that $g \geq 1_K \geq f$, then $I(g) \geq I(f) > \mu(U) - \varepsilon$. But then $\mu(K) > \mu(U) - \varepsilon$ by (3.2), and since $\varepsilon > 0$ was arbitrary, we see that

$$\mu(U) \leq \sup\{\mu(K) \mid K \text{ compact}, K \subseteq U\}.$$

We therefore conclude that μ is inner regular on open sets, and hence μ is a Radon measure satisfying (3.1) and (3.2).

5. It is enough to prove that $I(f) = \int f \, \mathrm{d}\mu$ for every $f \in C_c(X)$ such that $0 \leq f \leq 1$, because every function in $C_c(X)$ can be written as a linear combination of such functions. Let $n \in \mathbb{N}$ and set $K_i := \{x \in X \mid f(x) \geq i/n\}$ for $1 \leq i \leq n$. Let $K_0 := \operatorname{supp}(f)$. We see that $K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_0$.

Furthermore, define $f_1, f_2, \ldots, f_n$ as

$$f_i = \min\left\{\max\left\{f - \frac{i-1}{n}, 0\right\}, \frac{1}{n}\right\},$$

where $1 \leq i \leq n$.

We note that $f_i(x) = 0$ if $x \notin K_{i-1}$, $0 \leq f_i(x) = f(x) - (i-1)/n \leq 1/n$ for $x \in K_{i-1} \backslash K_i$ and $f_i(x) = 1/n$ if $x \in K_i$.

We claim that $f = \sum_{i=1}^n f_i$. Indeed, if $x \notin K_0 = \bigcup_{i=1}^n K_i$, then $f(x) = 0 = \sum_{i=1}^n f_i(x)$. If $x \in K_{j-1} \backslash K_j$, then $0 \leq f(x) - (j-1)/n \leq 1/n$, so

$$\sum_{i=1}^n f_i(x) = f_j(x) + \sum_{i=1}^{j-1} \frac{1}{n} = f(x) - \frac{j-1}{n} + \frac{j-1}{n} = f(x).$$

Since this holds for all $1 \leq j \leq n$, we are done.

It is clear that the f_i's are continuous. Since $1/n \geq f \geq 0$ and $f_i(x) = 0$ for $x \notin K_{i-1}$, we know that $\operatorname{supp}(f_i) \subseteq K_{i-1} \subseteq \operatorname{supp}(f)$ so $f_i \in C_c(X)$, and it follows that $f_i \leq n^{-1} 1_{K_{i-1}}$. Moreover, since $f_i(x) = 1/n$ for $x \in K_i$ we conclude that $n^{-1} 1_{K_i} \leq f_i \leq n^{-1} 1_{K_{i-1}}$. Hence

$$\frac{1}{n}\mu(K_i) \leq \int f_i \, \mathrm{d}\mu \leq \frac{1}{n}\mu(K_{i-1}).$$

Furthermore, if U is an open set such that $K_{i-1} \subseteq U$ then $nf_i \prec U$ so $I(f_i) \leq n^{-1}\mu(U)$. By outer regularity of μ on K_{i-1} we conclude that $I(f_i) \leq n^{-1}\mu(K_{i-1})$. Moreover, from (3.2) we get that $I(f_i) \geq n^{-1}\mu(K_i)$, hence

$$\frac{1}{n}\mu(K_i) \leq I(f_i) \leq \frac{1}{n}\mu(K_{i-1}).$$

Since both I and the integral are linear and since $f = \sum_{i=1}^n f_i$, we get that

$$\frac{1}{n}\sum_{i=1}^{n} \mu(K_i) \leq \int f \, \mathrm{d}\mu \leq \frac{1}{n}\sum_{i=0}^{n-1} \mu(K_i)$$

and, respectively,

$$\frac{1}{n}\sum_{i=1}^{n}\mu(K_i) \le I(f) \le \frac{1}{n}\sum_{i=0}^{n-1}\mu(K_i).$$

In other words, we have that

$$\begin{aligned}\left|I(f) - \int f\,\mathrm{d}\mu\right| &\le \frac{1}{n}\left(\sum_{i=0}^{n-1}\mu(K_i) - \sum_{i=1}^{n}\mu(K_i)\right)\\ &= \frac{1}{n}(\mu(K_0) - \mu(K_n)) \le \frac{1}{n}\mu(K_0)\end{aligned}$$

Since K_0 is compact, we have already proved that $\mu(K_0) < \infty$ and since n was chosen arbitrarily, we get that

$$I(f) = \int f\,\mathrm{d}\mu.$$

This completes the proof. □

3.2 Properties of Radon measures

The Lebesgue measure is a well-known example of a Radon measure. Another example is the counting measure on a discrete group. We shall present a way of constructing new Radon measures once we already know some. In order to do this, we need to use the notion of lower semi-continuous functions, which we recall below.

Definition 3.8. *Let X be a topological space. A function $f : X \to \mathbb{R}$ is called* ***lower semi-continuous*** *if $f^{-1}((a,\infty))$ is open for every $a \in \mathbb{R}$.*

Note that every lower semi-continuous function is Borel measurable since the intervals (a,∞) generate the Borel σ-algebra on $\mathbb{R}$.

It follows straight from the definition that for every open set U, the characteristic function 1_U is lower semi-continuous. Furthermore, if the functions $f, g : X \to [0,\infty)$ are lower semi-continuous, then so is fg. This fact will be used in the proof of Proposition 3.10.

We will also need the following result:

Proposition 3.9. *Let X be a locally compact space with a Radon measure μ, and let $f : X \to [0,\infty)$ be a lower semi-continuous function. Then*

$$\int f\,\mathrm{d}\mu = \sup\left\{\int g\,\mathrm{d}\mu \mid g \in C_c(X), 0 \le g \le f\right\}.$$

Proof. It is clear that $\int f\,\mathrm{d}\mu \ge \sup\left\{\int g\,\mathrm{d}\mu \mid g \in C_c(X), 0 \le g \le f\right\}$. In order to show the other inequality, let $a \in \mathbb{R}$ such that $a < \int f\,\mathrm{d}\mu$. We want to find a $g \in C_c(X)$ such that $0 \le g \le f$ and $\int g\,\mathrm{d}\mu > a$.

For $j, n \in \mathbb{N}$ define $U_{n,j} = f^{-1}((j2^{-n},\infty))$. By assumption, $U_{n,j}$ is open for every $n, j \in \mathbb{N}$. Now define the function

$$s_n = 2^{-n}\sum_{j=1}^{2^{2n}} 1_{U_{n,j}}.$$

We immediately see that

$$s_n(x) = \begin{cases} j2^{-n} & x \in U_{n,j} \backslash U_{n,j+1},\ 1 \leq j < 2^{2n}, \\ 2^n & x \in U_{n,2^{2n}}, \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that $U_{n-1,j} = U_{n,2j}$ for $n > 1$ and $j \in \mathbb{N}$, we see that the sequence $(s_n(x))_{n\in\mathbb{N}}$ is increasing for every $x \in X$. Furthermore $(s_n)_{n\in\mathbb{N}}$ converges to f. Indeed, if $x \in f^{-1}((2^{-n}, \infty))$ then $0 \leq f - s_n \leq 2^{-n}$ and the result follows.

By the monotone convergence theorem, we conclude that $\int s_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$ as $n \to \infty$. Hence we can fix an $m \in \mathbb{N}$ such that

$$2^{-m} \sum_{j=1}^{2^{2m}} \mu(U_{m,j}) = \int s_m \,\mathrm{d}\mu > a.$$

Choose a constant $b \in \mathbb{R}$ such that $2^{-m} \sum_{j=1}^{2^{2m}} \mu(U_{m,j}) > b > a$. By inner regularity of μ on the open sets $U_{m,j}$, for each $1 \leq j \leq 2^{2m}$ we can find a compact set $K_j \subseteq U_{m,j}$ such that $\mu(K_j) > \mu(U_{m,j}) - (b-a)2^{-m}$. Then

$$2^{-m} \sum_{j=1}^{2^{2m}} \mu(K_j) > 2^{-m} \sum_{j=1}^{2^{2m}} \left(\mu(U_{m,j}) - \frac{b-a}{2^m} \right) > a.$$

Now, for $1 \leq j \leq 2^{2m}$, by Urysohn's lemma we can choose $g_j \in C_c(X)$ such that $g_j \geq 1_{K_j}$ and $g_j \prec U_{m,j}$. Then the function $g := 2^{-m} \sum_{j=1}^{2^{2m}} g_j$ belongs to $C_c(X)$, and clearly satisfies $g \leq s_m \leq f$ and

$$\int g \,\mathrm{d}\mu \geq 2^{-m} \sum_{j=1}^{2^{2m}} \mu(K_j) > a.$$

This completes the proof. □

Proposition 3.10. *Let μ be a Radon measure on a locally compact Hausdorff space X and let $\varphi : X \to (0, \infty)$ be a continuous function. Then the measure defined by $\nu(E) = \int_E \varphi \,\mathrm{d}\mu$, for all Borel sets $E \subseteq X$, is a Radon measure.*

Proof. First we show outer regularity of ν. Let $E \subseteq X$ be a Borel set. If $\nu(E) = \infty$, there is nothing to prove, so assume $\nu(E) < \infty$. Let $\varepsilon > 0$. We want to find an open $U \supseteq E$ such that $\nu(U \backslash E) < \varepsilon$.

For every $n \in \mathbb{N}$, define the set $W_n := \varphi^{-1}((1/n, n))$, which is open by continuity of φ. Using the definition of ν, we see that for a fixed $n \in \mathbb{N}$, whenever $F \subseteq W_n$ is a Borel set then

$$\frac{1}{n}\mu(F) \leq \nu(F) \leq n\mu(F).$$

By the first inequality, we get that $\mu(E \cap W_n) \leq n\nu(E \cap W_n) < \infty$. Hence, by outer regularity of μ, for every $n \in \mathbb{N}$ we can find an open $U_n \subseteq X$ such that $\mu(U_n \backslash (E \cap W_n)) < \varepsilon/(2^n \cdot n)$.

Set $U := \bigcup_{n=1}^{\infty} U_n \cap W_n$, which is an open set. Since $\bigcup_{n=1}^{\infty} W_n = X$, we have that $E = \bigcup_{n=1}^{\infty} (E \cap W_n)$, and since $E \cap W_n \subseteq U_n \cap W_n$ for all $n \in \mathbb{N}$, we conclude that $E \subseteq U$. Furthermore, since $E \cap W_n \subseteq E$ for all $n \in \mathbb{N}$, we obtain that

$$U \backslash E = \bigcup_{n=1}^{\infty} (U_n \cap W_n) \backslash E \subseteq \bigcup_{n=1}^{\infty} (U_n \cap W_n) \backslash (E \cap W_n).$$

Clearly $(U_n \cap W_n) \backslash (E \cap W_n) \subseteq W_n$ for all $n \in \mathbb{N}$ and hence

$$\begin{aligned}
\nu(U \backslash E) &\leq \sum_{n=1}^{\infty} \nu\left((U_n \cap W_n) \backslash (E \cap W_n)\right)\right) \\
&\leq \sum_{n=1}^{\infty} n\mu((U_n \cap W_n) \backslash (E \cap W_n)) \\
&\leq \sum_{n=1}^{\infty} n\mu(U_n \backslash (E \cap W_n)) \\
&< \sum_{n=1}^{\infty} n \frac{\varepsilon}{2^n \cdot n} = \varepsilon,
\end{aligned}$$

and we conclude that ν is outer regular.

The linear functional $f \mapsto \int f\varphi \,\mathrm{d}\mu$ is clearly positive and hence there is a Radon measure ν' such that $\int f\varphi \,\mathrm{d}\mu = \int f \,\mathrm{d}\nu'$ for all $f \in C_c(X)$. We will show that $\nu = \nu'$. For this, since both ν and ν' are outer regular, it is enough to show that $\nu(U) = \nu'(U)$ for every open $U \subseteq X$. Therefore, let $U \subseteq X$ be open. Using that 1_U is lower semi-continuous, by Proposition 3.9 we have that

$$\begin{aligned}
\nu'(U) = \int 1_U \,\mathrm{d}\nu' &= \sup\left\{\int g \,\mathrm{d}\nu' | g \in C_c(X), 0 \leq g \leq 1_U\right\} \\
&= \sup\left\{\int g\varphi \,\mathrm{d}\mu | g \in C_c(X), 0 \leq g \leq 1_U\right\}
\end{aligned}$$

Since $1_U\varphi$ is lower semi continuous, by the same Proposition, we get

$$\nu(U) = \int 1_U\varphi \,\mathrm{d}\mu = \sup\left\{\int h \,\mathrm{d}\mu | h \in C_c(X), 0 \leq h \leq 1_U\varphi\right\}.$$

We show that the two suprema are equal. Clearly

$$\sup\left\{\int g\varphi \,\mathrm{d}\mu | g \in C_c(X), 0 \leq g \leq 1_U\right\} \leq \sup\left\{\int h \,\mathrm{d}\mu | h \in C_c(X), 0 \leq h \leq 1_U\varphi\right\},$$

since $g\varphi \in C_c(X)$ and $0 \leq g\varphi \leq 1_U\varphi$ whenever $g \in C_c(X)$ and $0 \leq g \leq 1_U$.

To see the other inequality, let $h \in C_c(X)$ such that $0 \leq h \leq 1_U\varphi$. Define, for $n \in \mathbb{N}$, the function $h_n \in C_c(X)$ by

$$h_n(x) = \frac{h(x)}{1/n + \varphi(x)}.$$

This is well-defined and continuous, since the denominator is strictly positive and has compact support, since h has this property. Clearly $0 \leq h_n(x) \leq 1_U(x)$ and $h_n(x)\varphi(x)$ is an increasing sequence converging to $h(x)$ as n tends to infinity. Hence by the monotone convergence theorem

$$\int h \,\mathrm{d}\mu = \lim_{n\to\infty} \int h_n\varphi \,\mathrm{d}\mu \leq \sup\left\{\int g\varphi \,\mathrm{d}\mu | g \in C_c(X), 0 \leq g \leq 1_U\right\}.$$

Since h was arbitrary, the other inequality is proven.

This proves that $\nu(U) = \nu'(U)$ for every open set U, and we conclude that $\nu = \nu'$, so ν is a Radon measure. □

This way of constructing new Radon measure from already known ones will be important to us, especially in Chapters 6 and 7, where we look at examples of locally compact groups and their Haar measure.

Next, we shall prove a density theorem that states that if μ is a Radon measure on a locally compact Hausdorff space X, then $C_c(X)$ is p-norm dense in $L^p(G, \mu)$, the space of p-integrable functions on G, for every $1 \leq p < \infty$. This will be needed in Chapter 8.

Lemma 3.11. *If μ is a Radon measure on a locally compact Hausdorff space X, then μ is inner regular on all σ-finite Borel sets.*

Proof. First, let E be a Borel set and suppose $\mu(E) < \infty$. Let $\varepsilon > 0$. Using outer regularity of μ on E, choose and open set $U \supseteq E$ such that $\mu(U \backslash E) < \varepsilon$. Clearly U must have finite measure, and using inner regularity of μ on U, we can find a compact $F \subseteq U$ such that $\mu(F) > \mu(U) - \varepsilon$.

Since $\mu(U) < \mu(E) + \varepsilon$, we can choose an open set $V \supseteq U \backslash E$ such that $\mu(V) < \varepsilon$. Let $K = F \backslash V$. Then K is compact and $K \subseteq E$, and

$$\mu(K) = \mu(F) - \mu(V) > \mu(E) - \varepsilon - \mu(V) > \mu(E) - 2\varepsilon.$$

This entails that μ is inner regular on every E such that $\mu(E) < \infty$.

Now, assume that E is a σ-finite Borel set with $\mu(E) = \infty$. Then there is an increasing sequence $(E_i)_{i=1}^{\infty}$ of Borel sets of finite measure such that $E = \bigcup_{i=1}^{\infty} E_i$. Since $\mu(E_i) \to \infty$, as $i \to \infty$, we conclude that for every $n \in \mathbb{N}$ there is an $i \in \mathbb{N}$ such that $\mu(E_i) > n$. Since μ is inner regular on E_i by the preceeding, there is a compact $K \subseteq E_i \subseteq E$ such that $\mu(K) > n$, and we conclude that μ is inner regular on E. □

Theorem 3.12. *If μ is a Radon measure on a locally compact Hausdorff space X, then $C_c(X)$ is p-norm dense in $L^p(X, \mu)$.*

Proof. We know that the simple functions on X which are p-integrable are dense in $L^p(X, \mu)$. Therefore, since every simple function is a linear combination of indicator functions, we only need to show that for every Borel set E such that $\mu(E) < \infty$, the indicator function 1_E can be approximated in p-norm by functions from $C_c(X)$.

Let $E \subseteq X$ be a Borel set with $\mu(E) < \infty$ and let $\varepsilon > 0$. Then by Lemma 3.11 and outer regularity of μ on E, there is a compact set $K \subseteq E$ and an open set $U \supseteq E$ such that $\mu(U \backslash K) < \varepsilon$. By Urysohn's lemma, there is an $f \in C_c(X)$ such that $1_K \leq f \leq 1_U$. From this it follows that

$$\|1_E - f\|_p \leq \mu(U \backslash K)^{1/p} < \varepsilon^{1/p},$$

and the proof is complete. □

Chapter 4

Topological groups

In the preceeding chapters, we have considered topological spaces equipped with the Borel σ-algebra and have seen how sometimes a Borel measure interacts with the topology, e.g., by the regularity properties of the Radon measure. In this chapter, we shall discuss the notion of a topological group, which is a topological space carrying, in addition, a group structure compactible with the topology. More precisely,

Definition 4.1. *A **topological group** is a group G together with a topology on the set G such that the maps*

$$G \times G \to G \qquad (x, y) \mapsto xy \qquad \textit{(multiplication)}$$

$$G \to G \qquad x \mapsto x^{-1} \qquad \textit{(inversion)}$$

are both continuous.

4.1 Basic properties

The following notation will come in handy, when we consider some basic properties of topological groups.

Definition 4.2. *Let G be a group. We denote the neutral element of G by e_G or simply by e when no confusion can occur. For $A, B \subseteq G$ and $g \in G$ we define*

$$\begin{aligned} gA &:= \{ga | a \in A\}, \\ Ag &:= \{ag | a \in A\}, \\ A^{-1} &:= \{a^{-1} | a \in A\}, \\ AB &:= \{ab | a \in A, b \in B\}. \end{aligned}$$

*If $A = A^{-1}$ for a set $A \subseteq G$, we call A **symmetric**.*

Proposition 4.3 (Basic properties of topological groups)**.** *Let G be a topological group*

1. *For any $g \in G$, the maps $x \mapsto gx$, $x \mapsto xg$, and $x \mapsto x^{-1}$ are homeomorphisms of G.*

2. *If $U \subseteq G$ is open, respectively, closed, then the sets xU, Ux and U^{-1} are open, respectively, closed.*

3. *Every neighbourhood U of e contains a symmetric neighbourhood V of e.*

4. *For every neighbourhood U of e there exists a neighbourhood V of e such that $VV \subseteq U$.*

5. *For every neighbourhood U of e there is a symmetric neighbourhood V of e such that $VV \subseteq U$.*

6. *If H is a subgroup of G, then the closure $\overline{H}$ is a subgroup as well. If H is normal, then so is $\overline{H}$.*

7. *Every open subgroup of G is closed.*

8. *If $A, B \subseteq G$ are compact subsets of G, then AB is compact.*

Proof. 1. It follows directly from the continuity of multiplication and inversion in G that the maps are continuous with the continuous inverses: $y \mapsto g^{-1}y$, $y \mapsto yg^{-1}$ and $y \mapsto y^{-1}$.

2. Follows from 1.

3. Since U^{-1} is open and contains e, the set $U \cap U^{-1}$ is a clearly symmetric neighbourhood of e.

4. Let $U_1 \times U_2 \subseteq G \times G$ be the pre-image of U under the map $(x, y) \mapsto xy$. By continuity of multiplication, $U_1 \times U_2$ is open and clearly containing (e, e).

By definition of the product topology, there are open sets $V_1, V_2 \subseteq G$ such that $(e, e) \in V_1 \times V_2 \subseteq U_1 \times U_2$. Let $V := V_1 \cap V_2$, which is open. Then $VV \subseteq V_1V_2 \subseteq U_1U_2 = U$.

5. Choose, using part 4 of this proposition, a neighbourhood V of e such that $VV \subseteq U$. Then choose, using part 3, a symmetric neighbourhood W of e such that $W \subseteq V$. Then $WW \subseteq VV \subseteq U$ and we are done.

6. Obviously $e \in \bar{H}$. From continuity of multiplication and inversion, it follows that $\overline{H}$ is stable under the two operations. Now assume H is normal. Let $g \in G$. Then $H = gHg^{-1} \subseteq g\overline{H}g^{-1}$. Since $g\overline{H}g^{-1}$ is closed, $\overline{H} \subseteq g\overline{H}g^{-1}$. By this last inclusion, we also get that $g^{-1}\overline{H}g \subseteq \overline{H}$. Since g was arbitrary, it follows that $\overline{H} = g\overline{H}g^{-1}$ for all g and hence $\overline{H}$ is normal.

7. The sets gH are all open by 2. Hence $\bigcup_{g \in G \setminus H} gH$ is open. Since the cosets of H form a partition of G, we see that $H^c = \bigcup_{g \in G \setminus H} gH$, so H must be closed.

8. Since $A \times B$ is compact and multiplication is continuous, we conclude that the image AB is compact. □

4.2 Continuous functions on topological groups

Here we pay particular attention to continuous functions with compact support, which we have already encountered, because of properties that we need later.

Definition 4.4. *Let G be a topological group and let f be a function on G. Then we define the **left**, respectively, **right translate** L_y and R_y of f through y as follows:*

$$L_y f(x) = f(y^{-1}x),$$

$$R_y f(x) = f(xy).$$

Note that for all $y, z \in G$ and $f \in C(G)$, we have $L_{yz}f = L_y L_z f$ and $R_{yz}f = R_y R_z f$.

Definition 4.5. *Let G be a topological group. We say that $f \in C(G)$ is* ***left uniformly continuous*** *if for every $\varepsilon > 0$, there is a neighbourhood V of e such that $\|L_y f - f\| < \varepsilon$ whenever $y \in V$.*

Similarly, we define the notion of a ***right uniformly continous*** *function by replacing L_y with R_y above.*

Proposition 4.6. *Let G be a topological group and $f \in C_c(G)$. Then f is both left and right uniformly continuous.*

Proof. We shall only prove the left uniform continuity, as the right version is proved similarly. Let $\varepsilon > 0$. Let $K := \operatorname{supp}(f)$. For each $x \in K$ there is a neighbourhood U_x of e such that $|f(zx) - f(x)| < \varepsilon/2$ whenever $z \in U_x$. This follows directly from continuity of f. For every x, choose a symmetric neighbourhood V_x of e such that $V_x V_x \subseteq U_x$, which is possible by Proposition 4.3.5. Clearly $\{V_x x\}_{x\in K}$ cover K and since K is compact, there are $x_1, x_2, \ldots, x_n$ such that $K \subseteq \bigcup_{i=1}^n V_{x_i} x_i$. Set $V := \bigcap_{i=1}^n V_{x_i}$.

Now we will show that V is the desired neighbourhood. It is clear that V is a symmetric neighbourhood of e. First, assume $x \in K$ and $y \in V$. Then $xx_i^{-1} \in V_{x_i} \subseteq U_{x_i}$ for some $1 \le i \le n$, and hence $y^{-1}xx_i^{-1} \in VV_{x_i} \subseteq V_{x_i}V_{x_i} \subseteq U_{x_i}$. Therefore, we have that

$$\begin{aligned} |f(y^{-1}x) - f(x)| &= |f(y^{-1}xx_i^{-1}x_i) - f(xx_i^{-1}x_i)| \\ &\le |f(y^{-1}xx_i^{-1}x_i) - f(x_i)| + |f(x_i) - f(xx_i^{-1}x_i)| \\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Now, assume $x \notin K$. If $y^{-1}x \notin K$, then the ineqality is trivial.

Otherwise, if $y^{-1}x \in K$, then $y^{-1}xx_i^{-1} \in V_{x_i}$ for some i. Therefore $xx_i^{-1} = yy^{-1}xx_i^{-1} \in U_{x_i}$ and hence

$$\begin{aligned} |f(y^{-1}x) - f(x)| &\le |f(y^{-1}x) - f(x_i)| + |f(x_i) - f(x)| \\ &= |f(y^{-1}xx_i^{-1}x_i) - f(x_i)| + |f(x_i) - f(xx_i^{-1}x_i)| < \varepsilon. \end{aligned}$$

The proof is complete. □

Corollary 4.7. *For every $f \in C_c(G)$ and $\varepsilon > 0$, there is a neighbourhood V of e such that $|f(x) - f(y)| < \varepsilon$ whenever $y^{-1}x \in V$ or $yx^{-1} \in V$.*

Proof. Let $\varepsilon > 0$. Since f is left and right uniformly continuous, there is a neighbourhood U_1 of e such that for all $x \in G$ and $z \in U_1$, $|L_z f(x) - f(x)| < \varepsilon$ and a neighbourhood U_2 of e such that $z \in U_2$ implies $|R_z f(x) - f(x)| < \varepsilon$. Let $U := U_1 \cap U_2$. Then, if $y^{-1}x \in U$, that is, $x = yu$ for some $u \in U$, we deduce that $|f(x) - f(y)| = |f(yu) - f(y)| = |R_u f(y) - f(y)| < \varepsilon$. Similarly, we have that if $yx^{-1} = u \in U$, then $|f(x) - f(y)| = |L_u f(y) - f(y)| < \varepsilon$. □

Corollary 4.8. *For every $f \in C_c(G)$ and $\varepsilon > 0$ there is a symmetric neighbourhood V of e such that $\sup_{x\in G} |f(xy) - f(yx)| < \varepsilon$ for all $y \in V$.*

Proof. Since f is both left and right uniformly continuous, the conclusion follows from the fact that

$$\begin{aligned}|f(xy) - f(yx)| &\leq |f(xy) - f(x)| + |f(yx) - f(x)| \\ &= |R_y f(x) - f(x)| + |L_{y^{-1}} f(x) - f(x)|,\end{aligned}$$

for every $x, y \in G$, and that every neighbourhood of e contains a symmetric neighbourhood. □

4.3 The Hausdorff assumption

When dealing with the Haar measure, we always assume that the topological group in question is equipped with a Hausdorff topology. But in fact this is not much of a restriction due to the following proposition.

Proposition 4.9. *Let G be a topological group. Then we have the following:*

1. *If G is T_1, then G is Hausdorff.*
2. *Let $H := \overline{\{e\}}$. Then H is a normal subgroup and G/H equipped with the quotient topology, is a Hausdorff topological group.*

Proof. 1. Assume that G is T_1. Let $x \neq y$. By assumption $U = G \backslash \{xy^{-1}\}$ is a neighbourhood of e. Then by Proposition 4.3.5 there is a symmetric neighbourhood V of e such that $VV \subseteq U$ that is $xy^{-1} \notin VV$.

We claim that Vx and Vy are disjoint neighbourhoods of x and y. They are clearly open by Proposition 4.3.2. Furthermore, if $z = vx = wy$ for $v, w \in V$, then $xy^{-1} = v^{-1}zz^{-1}w = v^{-1}w \in V^{-1}V = VV$. Since this cannot be the case, Vx and Vy are disjoint. Hence G is Hausdorff.

2. By Proposition 4.3.6 H is a normal subgroup. It is easy to check that multiplication and inversion are continuous on G/H. We see that $\{e_{G/H}\}$ is closed in the quotient topology since H is closed. Hence every other one-point set is closed as well, using that gH is closed for every $g \in G$, and hence G/H is T_1 and by the first part of this proposition, we conclude that G/H is Hausdorff. □

In the following let G be a fixed topological group and let $H := \overline{\{e\}}$. As we shall see now, every Borel measurable function $f : G \to \mathbb{C}$ is constant on the cosets of the H, and as long as we consider measurable functions on G, we can restrict ourselves to the Hausdorff group G/H without losing much information.

Lemma 4.10. *For all $x \in H$, we have that $\overline{\{x\}} = H$.*

Proof. Clearly $e \in x^{-1}\overline{\{x\}}$, and since this set is closed by Proposition 4.3.1, we conclude that $H \subseteq x^{-1}\overline{\{x\}}$. On the other hand, since $x \in H$, we have that $\overline{\{x\}} \subseteq H$ and since H is a subgroup, $x^{-1} \in H$ and hence $x^{-1}\overline{\{x\}} \subseteq H$. Therefore, it follows that $\overline{\{x\}} = xH = H$. □

In the following, set $\mathcal{E} := \{\bigcup_{x \in A} xH \mid A \subseteq G\}$.

Lemma 4.11. *$\mathcal{E}$ is a σ-algebra.*

Proof. It is easy to see that $G \in \mathcal{E}$ and that $\mathcal{E}$ is stable under (countable) unions, and since the cosets xH for $x \in G$ form a partition of G, we have that $\left(\bigcup_{x \in A} xH\right)^c = \bigcup_{x \in A^c} xH \in \mathcal{E}$. □

Lemma 4.12. *The Borel sets are contained in $\mathcal{E}$.*

Proof. Let $\mathcal{B}(G)$ denote the σ-algebra consisting of the Borel sets on G. It is enough to show that every closed set is contained in $\mathcal{E}$, since the closed sets generate $\mathcal{B}(G)$. Therefore, let $A \subseteq G$ be closed. We now show that $A = \bigcup_{x\in A} xH$. It is trivial that $A \subseteq \bigcup_{x\in A} xH$. Hence, let $g \in \bigcup_{x\in A} xH$, that is $g \in xH$ for some $x \in A$, or equivalently, $x^{-1}g \in H$ and by Lemma 4.10 $H = \overline{\{x^{-1}g\}}$. Since $x^{-1}A$ is a closed set containing e, we have that $x^{-1}g \in \overline{\{x^{-1}g\}} = H \subseteq x^{-1}A$, so $g \in A$.

We thereby conclude that $A = \bigcup_{x\in A} xH \in \mathcal{E}$, and hence $\mathcal{B}(G) \subseteq \mathcal{E}$. □

Proposition 4.13. *Every Borel measurable function $f : G \to \mathbb{C}$ is constant on the cosets H*

Proof. Let $x \in G$ and let $z = f(x)$. By Lemma 4.12 f is also $\mathcal{E}$-measurable, so $f^{-1}(\{z\}) \in \mathcal{E}$, that is $f^{-1}(\{z\}) = \bigcup_{y\in A} yH$, for some $A \subseteq G$.

Since $x \in f^{-1}(\{z\})$ there is some $y \in A$ such that $x \in yH$. Hence $yH = xH$ and we get that $xH \subseteq f^{-1}(\{z\})$, that is, $f(xH) \subseteq \{z\}$. □

4.4 Locally compact groups

Definition 4.14. *A **locally compact group** is a topological group whose topology is locally compact and Hausdorff.*

We shall now define the Haar measure and discuss some important properties of it.

Definition 4.15. *Let G be a toplogical group. A Borel measure μ is called **left**, respectively, **right invariant** if*

$$\mu(xE) = \mu(E), \quad \textit{respectively}, \quad \mu(Ex) = \mu(E).$$

for all measurable sets E and $x \in G$.

Definition 4.16. *Let G be a locally compact group. Then a **left** respectively **right Haar measure** on G is a nonzero, left-, respectively, right-invariant Radon measure on G.*

Definition 4.17. *Let X be a locally compact Hausdorff space. We define*

$$C_c^+(X) := \{f \in C_c(X) | f \geq 0, \|f\| > 0\}.$$

Proposition 4.18. *Let G be a locally compact group.*

1. *A Radon measure μ on G is a left Haar measure if and only if the measure $\tilde{\mu}$ defined by $\tilde{\mu}(E) = \mu(E^{-1})$, for all Borel sets $E \subseteq G$, is a right Haar measure on G.*

2. *A non-zero Radon measure μ on G is a left Haar measure if and only if $\int f \,\mathrm{d}\mu = \int L_y f \,\mathrm{d}\mu$ for all $f \in C_c^+(G)$ and $y \in G$.*

3. *If μ is a left Haar measure on G then $\mu(U) > 0$ for all nonempty open $U \subseteq G$. Furthermore, $\int f \,\mathrm{d}\mu > 0$ for all $f \in C_c^+(G)$.*

4. *If μ is a left Haar measure on G, then $\mu(G) < \infty$ if and only if G is compact.*

Proof. 1. Assume that μ is a left Haar measure. Then it is easy to see that $\tilde{\mu}$ is a nonzero Radon measure. Let $E \subseteq G$ be a Borel set. That $\tilde{\mu}$ is right invariant simply follows from the fact that $(Ex)^{-1} = x^{-1}E^{-1}$ for all $x \in G$. Hence $\tilde{\mu}(Ex) = \mu(x^{-1}E^{-1}) = \mu(E^{-1}) = \tilde{\mu}(E)$, whenever $x \in G$. The other implication follows similarly.

2. First assume that μ is a left Haar measure and let $y \in G$. Note that $L_y 1_E = 1_{yE}$. Hence, for every simple function $s = \sum_{i=1}^n c_i 1_{E_i}$ with $c_i \geq 0$ we have that

$$\int s \, \mathrm{d}\mu = \sum_{i=1}^n c_i \mu(E_i) = \sum_{i=1}^n c_i \mu(yE_i) = \int L_y s \, \mathrm{d}\mu.$$

Since $\int f \, \mathrm{d}\mu = \sup\{\int s \, \mathrm{d}\mu \mid s \text{ simple}, f \geq s\}$, for all $f \in C_c^+(G)$, it follows that $\int f \, \mathrm{d}\mu = \int L_y f \, \mathrm{d}\mu$.

On the other hand, if $\int f \, \mathrm{d}\mu = \int L_y f \, \mathrm{d}\mu$ for all $f \in C_c^+(G)$ and $y \in G$ then the equation must hold for all $f \in C_c(G)$, beacuse such functions are linear combinations of functions in $C_c^+(G)$. Since the integral is a positive linear functional on $C_c(G)$, by applying the Riesz representation theorem, we deduce from the equation (3.1) that

$$\begin{aligned} \mu(U) &= \sup\left\{\int f \, \mathrm{d}\mu | f \prec U\right\} = \sup\left\{\int L_y f \, \mathrm{d}\mu | f \prec U\right\} \\ &= \sup\left\{\int f \, \mathrm{d}\mu | f \prec yU\right\} = \mu(yU), \end{aligned}$$

by using that $f \prec U$ precisely when $L_y f \prec yU$. Hence μ is a left Haar measure.

3. Since μ is nonzero, G must have nonzero measure. By inner regularity of μ on the open set G, there must be a compact set K such that $\mu(K) > 0$. Now, let U be an arbitrary nonempty, open set. Then there exist $x_1, x_2, \ldots, x_n \in G$ such that $K \subseteq \bigcup_{i=1}^n x_i U$. Since $\mu(x_i U) = \mu(U)$ for all $1 \leq i \leq n$, and $\mu(K) \leq \sum_{i=1}^n \mu(x_i U)$, we conclude that $\mu(U) > 0$.

For $f \in C_c^+(G)$, let $U := \{x \in G \mid f(x) > \|f\|/2\}$. Clearly, U is open and nonempty, since $\|f\| > 0$, so $\int f \, \mathrm{d}\mu \geq \|f\|\mu(U)/2 > 0$.

4. Assume G is compact with Haar measure μ. Since μ is a Radon measure, $\mu(G) < \infty$. Now, assume that μ is a Haar measure on G and $\mu(G) < \infty$. Since G is locally compact, there is a compact neighbourhood N of e. Since N contains an open set, $\mu(xN) = \mu(N) > 0$ for all $x \in G$. If there were a sequence $(x_n)_{n \in \mathbb{N}} \subseteq G$ such that $x_n N \cap x_m N = \emptyset$ for $n \neq m$, then $\mu\left(\bigcup_{n=1}^\infty x_n N\right) = \sum_{n=1}^\infty \mu(x_n N) = \sum_{n=1}^\infty \mu(N) = \infty$.

Hence there must exist finitely many $x_1, x_2, \ldots, x_n \in G$ such that $x_i N \cap x_j N = \emptyset$ for $i \neq j$ and $x_i N \cap xN \neq \emptyset$ for all $x \neq x_1, x_2, \ldots, x_n$. Since $x_i N$ is compact for every i, the union $K = \bigcup_{i=1}^n x_i N$ is compact, as well.

We claim that $G = KK^{-1}$. Indeed, for every $x \in G$ we have that $K \cap xK \neq \emptyset$. That is, for every $x \in G$, there are $h, k \in K$ such that $k = xh$, which is the same as $x = kh^{-1} \in KK^{-1}$.

Since K is compact, so is KK^{-1} by the continuity of inversion and Proposition 4.3 8. Hence G is compact. □

Note that because of Proposition 4.18.1, it follows easily that there are right Haar measure analogues of Proposition 4.18.2-4.

Chapter 5

The Haar measure

We are now ready to prove what we consider as our main goal, namely the existence and uniqueness of the left, respectively, right Haar measure on every locally compact group G. The existence will be proven by constructing a positive linear functional I on $C_c(G)$, which is also left translation invariant in the sense that $I(L_y f) = I(f)$, for every $f \in C_c(G)$ and $y \in G$. Using the Riesz representation theorem and Proposition 4.18.2, we can prove the existence of a left Haar measure, and by the same proposition (4.18.1), we immediately get the existence of a right Haar measure. The construction of this functional is most of all technical, but we need some good ideas, especially when we want to show that our functional is additive.

5.1 The Existence of Haar measure

In the following, we let G be a locally compact group.

Definition 5.1. *For $f, \varphi \in C_c^+(G)$ we define*

$$C_{f,\varphi} := \left\{ \sum_{j=1}^{n} c_j \,\middle|\, n \in \mathbb{N}, c_1, c_2, \ldots, c_n > 0, x_1, x_2, \ldots, x_n \in G, f \leq \sum_{j=1}^{n} c_j L_{x_j}\varphi \right\},$$

and

$$(f : \varphi) = \inf C_{f,\varphi}.$$

Lemma 5.2. *Let $f, \varphi \in C_c^+(G)$. There exist $x_1, x_2, \ldots, x_n \in G$ such that $f \leq 2\|f\| \cdot \|\varphi\|^{-1} \sum_{j=1}^{n} L_{x_j}\varphi$.*

Proof. $U := \{x \in G \mid \varphi(x) > \|\varphi\|/2\}$ is open and nonempty since $\|\varphi\| > 0$. Hence $\{xU\}_{x\in G}$ is an open covering of the support of f and hence there exist $x_1, x_2, \ldots, x_n \in G$ such that $\{x_j U\}_{j=1}^{n}$ covers $\operatorname{supp} f$.

For $y \in U$ and $x \in G$, we have that $f(x) \leq 2\|f\| \cdot \|\varphi\|^{-1}\varphi(y)$ by using that $f(x), \varphi(y) \geq 0$. For $x \in x_j U$ we have that that $x_j^{-1}x \in U$ so

$$f(x) \leq 2\|f\| \cdot \|\varphi\|^{-1} L_{x_j}\varphi(x).$$

Hence $f \leq 2\|f\| \cdot \|\varphi\|^{-1} \sum_{j=1}^{n} L_{x_j}\varphi$. □

It follows from this lemma that $C_{f,\varphi}$ is nonempty for every $f, \varphi \in C_c^+(G)$, and hence $(f : \varphi) \in \mathbb{R}$ is well-defined. Furthermore, it is clear that $(f : \varphi) \geq 0$.

The map $f \mapsto (f : \varphi)$ will be a very rough estimate of the linear functional that we are looking for. Of course, it is only defined for functions in $C_c^+(G)$ but since every function in $C_c(G)$ is a linear combination of positive functions, this will suffice.

We next show that the map has the following properties.

Proposition 5.3. *Let $f, g, \varphi \in C_c^+(G)$ and $c > 0$. Then*

1. $(f : \varphi) = (L_x f : \varphi)$ *for all* $x \in G$,
2. $(f + g : \varphi) \leq (f : \varphi) + (g : \varphi)$,
3. $(cf : \varphi) = c(f : \varphi)$,
4. *If* $f \leq g$, *then* $(f : \varphi) \leq (g : \varphi)$,
5. $(f : \varphi) \geq ||f|| \cdot ||\varphi||^{-1}$,
6. $(f : \varphi) \leq (f : g)(g : \varphi)$.

Proof. 1. Using that $L_x L_y = L_{xy}$ for all $x, y \in G$, it follows that if

$$f \leq \sum_{j=1}^{n} c_j L_{x_j} \varphi, \qquad \text{then} \qquad L_x f \leq \sum_{j=1} c_j L_{x_j x} \varphi.$$

We conclude that $C_{f,\varphi} \subseteq C_{L_x f,\varphi}$. To show the other inclusion, note that if $L_x f \leq \sum_{j=1}^n c_j L_{x_j} \varphi$, then $f \leq \sum_{j=1} c_j L_{x_j x^{-1}} \varphi$, for every $x \in G$.

2. To prove this, note that $f \leq \sum_{j=1}^n c_j L_{x_j} \varphi$ and $g \leq \sum_{j=1}^m d_j L_{y_j} \varphi$ implies that $(f + g : \varphi) \leq \sum_{j=1}^n c_j + \sum_{j=1}^m d_j$, which entails the desired conclusion.

3. Here we have that $f \leq \sum_{j=1}^n c_j L_{x_j} \varphi$ if and only if $cf \leq \sum_{j=1}^n c c_j L_{x_j} \varphi$, which gives the conclusion.

4. Assume $f \leq g$. Let $\sum_{j=1}^n c_j \in C_{g,\varphi}$. Then $f \leq g \leq \sum_{j=1}^n c_j L_{x_j} \varphi$ for some x_j's. Hence $C_{g,\varphi} \subseteq C_{f,\varphi}$, and we are done.

5. Let $\sum_{j=1}^n c_j \in C_{f,\varphi}$. Then there are $x_1, \ldots, x_n \in G$ such that $f(x) \leq \sum_{j=1}^n c_j L_{x_j} \varphi(x) \leq \sum_{j=1}^n c_j \|\varphi\|$, for all $x \in G$, using that all the c_j's are positive. Hence, by definition of the supremum norm, we conclude that $\|f\| \leq \sum_{j=1}^n c_j \|\varphi\|$, and we obtain the desired conclusion.

6. Assume that $f \leq \sum_{i=1}^n c_i L_{x_i} g$ and $g \leq \sum_{j=1}^m d_j L_{y_j} \varphi$. Then

$$f \leq \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j L_{x_i y_j} \varphi,$$

so $\sum_{i=1}^n \sum_{j=1}^m c_i d_j \in C_{f,\varphi}$. Since all the c_i's and d_j's are positive, we have that $(f : \varphi) \leq \sum_{i=1}^n \sum_{j=1}^m c_i d_j \leq \left(\sum_{i=1}^n c_i\right) \left(\sum_{j=1}^m d_j\right)$, and since this holds for any set of c_i's and d_j's, the conclusion follows. □

Definition 5.4. *Let $f_0 \in C_c^+(G)$ be fixed. For $\varphi \in C_c^+(G)$ define the map $I_\varphi : C_c^+(G) \to (0, \infty)$ by*

$$I_\varphi(f) = \frac{(f : \varphi)}{(f_0 : \varphi)}.$$

From now on we will consider I_φ instead of the map $f \mapsto (f : \varphi)$. The reason is the following inequalities, which we will need.

Lemma 5.5. *For every $f, \varphi \in C_c(G)$, we have that*

$$(f_0 : f)^{-1} \le I_\varphi(f) \le (f : f_0).$$

Proof. Follows from Proposition 5.3.6. □

Proposition 5.6. *The functional I_φ is sublinear and left translation-invariant.*

Proof. Follows from Proposition 5.3. □

This is almost what we are looking for. The functionals are not linear, but the following proposition will show that if the support of φ is small enough, then I_φ is approximately linear.

Lemma 5.7. *For $f, g \in C_c^+(G)$ and $\varepsilon > 0$ there is a neighborhood V of e such that if $\operatorname{supp}(\varphi) \subseteq V$ then $I_\varphi(f) + I_\varphi(g) \le I_\varphi(f+g) + \varepsilon$.*

Proof. By Urysohn's lemma (Lemma 3.5) there exists an $h_0 \in C_c^+(G)$ such that $h_0(x) = 1$ for all $x \in \operatorname{supp}(f+g)$.

Let $\delta > 0$ and set $h := f + g + \delta h_0$. Set

$$h_1(x) := \begin{cases} f/h & \text{if } x \in \operatorname{supp}(f), \\ 0 & \text{otherwise.} \end{cases}$$

$$h_2(x) := \begin{cases} g/h & \text{if } x \in \operatorname{supp}(g), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that these functions are well-defined and that $h_1, h_2 \in C_c^+(G)$. By Corollary 4.7, there is some neighbourhood V of e such that $|h_1(x) - h_1(y)| < \delta$ and $|h_2(x) - h_2(y)| < \delta$ for all $y^{-1}x \in V$.

Let $\varphi \in C_c^+(G)$ satisfying $\operatorname{supp}(\varphi) \subseteq V$. Choose $\sum_{j=1}^n c_j \in C_{h,\varphi}$ and corresponding x_j's. Then for $x \in G$ such that $x_j^{-1}x \in \operatorname{supp}(\varphi)$ we have that $|h_1(x) - h_1(x_j)| < \delta$.

Therefore, for all $x \in G$,

$$f(x) = h(x)h_1(x) \le \sum_{j=1}^n c_j L_{x_j}\varphi(x) h_1(x) \le \sum_{j=1}^n c_j L_{x_j}\varphi(x) \left(h_1(x_j) + \delta\right),$$

and, respectively,

$$g(x) = h(x)h_2(x) \le \sum_{j=1}^n c_j L_{x_j}\varphi(x) h_2(x) \le \sum_{j=1}^n c_j L_{x_j}\varphi(x) \left(h_2(x_j) + \delta\right).$$

Hence

$$(f : \varphi) \le \sum_{j=1}^n c_j (h_1(x) + \delta), \qquad \text{and} \qquad (g : \varphi) \le \sum_{j=1}^n c_j (h_2(x) + \delta).$$

From the definition we see that $h_1 + h_2 \leq 1$, and hence

$$(f : \varphi) + (g : \varphi) \leq \sum_{j=1}^{n} c_j(h_1(x_j) + h_2(x_j) + 2\delta) \leq \sum_{j=1}^{n} c_j(1 + 2\delta).$$

Again, since $\sum_{j=1}^{n} c_j \in C_{h,\varphi}$ is arbitrary, we conclude that

$$(f : \varphi) + (g : \varphi) \leq (h : \varphi)(1 + 2\delta)$$

This entails that

$$I_\varphi(f) + I_\varphi(g) \leq I_\varphi(h)(1 + 2\delta) \leq (1 + 2\delta)(I_\varphi(f + g) + \delta I_\varphi(h_0)),$$

using that I is sublinear.

Hence, for a given $\varepsilon > 0$, since $\delta > 0$ was arbitrary, we can choose δ small enough to ensure that

$$2\delta(f + g : f_0) + \delta(1 + 2\delta(h_0 : f_0)) < \varepsilon,$$

then by Lemma 5.5 we conclude that

$$\begin{aligned} I_\varphi(f) + I_\varphi(g) &\leq (1 + 2\delta)(I_\varphi(f + g) + \delta I_\varphi(h_0)) \leq (1 + 2\delta)(I_\varphi(f + g) + \delta(h_0 : f_0)) \\ &\leq I_\varphi(f + g) + 2\delta(f + g : f_0) + \delta(1 + 2\delta)(h_0 : f_0) < I_\varphi(f + g) + \varepsilon. \end{aligned}$$

This completes the proof. □

We are now ready to take the last step.

Theorem 5.8. *Every locally compact group G possesses a left Haar measure.*

Proof. Define the interval $X_f := [(f_0 : f)^{-1}, (f : f_0)]$, for each $f \in C_c^+(G)$. Define $X := \prod_{f \in C_c^+(G)} X_f$. By Lemma 5.5 $I_\varphi \in X$, for all $\varphi \in C_c^+(G)$. By Tychonoff's theorem (Theorem 2.14), X is compact.

For each neighbourhood V of e, let

$$K_V := \overline{\{I_\varphi \in X | \varphi \in C_c^+(G), \operatorname{supp}\varphi \subseteq V\}}.$$

We show that $\{K_V\}_V$ has the finite intersection property. Let $V_1, V_2, \ldots, V_n$ be given neighbourhoods of e. Clearly $e \in \bigcap_{i=1}^n V_i$, so the open set $\bigcap_{i=1}^n V_i$ is non-empty. Hence, by Urysohn's lemma there is a function $\varphi \in C_c^+(G)$, such that $\operatorname{supp}\varphi \subseteq \bigcap_{i=1}^n V_i$. We see that $\varphi \in K_{\bigcap_{i=1}^n V_i}$. Clearly $K_{\bigcap_{i=1}^n V_i} \subseteq \bigcap_{i=1}^n K_{V_i}$ and this entails that the latter set is non-empty.

Since $\{K_V\}_V$ is a family of closed subsets that has the finite intersection property, we conclude that $\bigcap_V K_V$ is non-empty. Choose an $I \in \bigcap_V K_V$.

For each neighbourhood V of e we have that $I \in K_V$. Using the definition of K_V together with the definition of the product topology, we deduce that for each $f_1, f_2, \ldots, f_n \in C_c^+(G)$ and $\varepsilon > 0$, there exists $\varphi \in C_c^+(G)$ such that $\operatorname{supp}\varphi \subseteq V$ and $|I(f_i) - I_\varphi(f_i)| < \varepsilon$, for $i = 1, 2, \ldots, n$.

In the following, let $\varepsilon > 0$, and $f, g \in C_c^+(G)$ be fixed. For each $x \in G$ there is $\varphi \in C_c^+(G)$ such that $|I(f) - I_\varphi(f)| < \varepsilon/2$ and $|I(L_x f) - I_\varphi(L_x f)| < \varepsilon/2$. Using Lemma 5.6 we get that

$$|I(f) - I(L_x f)| \leq |I(f) - I_\varphi(f)| + |I_\varphi(L_x f) - I(L_x f)| < \varepsilon.$$

From this. it follows that $I(f) = I(L_x f)$, for all $x \in G$.

Easily, in a similar way, we get that $I(cf) = cI(f)$, for each constant $c > 0$.

By Lemma 5.7 there is a neighbourhood V of e such that

$$|I_\varphi(f+g) - I_\varphi(f) - I_\varphi(g)| < \varepsilon/3,$$

whenever $\operatorname{supp}\varphi \subseteq V$. Moreover, we can choose a $\varphi \in C_c^+(G)$ such that $\operatorname{supp}\varphi \subseteq V$ and $|I(f+g)-I_\varphi(f+g)| < \varepsilon/3$ and $|I_\varphi(f)+I_\varphi(g)-I(f)-I(g)| < \varepsilon/3$.

By using the triangle inequality, it now easily follows that

$$|I(f+g) - I(f) - I(g)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $I(f+g) = I(f) + I(g)$.

Since $f, g \in C_c^+(G)$ were arbitrarily chosen, we conclude that I is left translation invariant and linear.

We now extend I to $C_c(G)$. First of all, we set $I(0) := 0$. For $f \in C_c(G)$ real-valued, define $I(f) = I(f^+) - I(f^-)$ and for $f \in C_c(G)$ arbitrary, define $I(f) = I(\operatorname{Re} f) + iI(\operatorname{Im} f)$. Then I is clearly a positive linear functional on $C_c(G)$ and hence, by the Riesz representation theorem, we conclude that there is a unique Radon measure μ such that $I(f) = \int f \, \mathrm{d}\mu$, for all $f \in C_c(G)$.

Definition of I, we have that $\int f \, \mathrm{d}\mu = I(f) \geq (f_0 : f)^{-1} > 0$ for all $f \in C_c^+(G)$ and hence μ must be non-zero. Since $\int f \, \mathrm{d}\mu = I(f) = I(L_x f) = \int L_x f \, \mathrm{d}\mu$ for all $x \in G$ and $f \in C_c^+(G)$, we conclude by Proposition 4.18.2 that μ is a left Haar measure. □

Corollary 5.9. *Every locally compact group possesses a right Haar measure.*

Proof. It follows from Proposition 4.18.1 and what we have just proven. □

5.2 Uniqueness of the Haar measure

The following theorem shows that, in a certain sense, the (left, respectively, right) Haar measure on any locally compact group is unique.

Theorem 5.10. *Let μ and ν be left Haar measures on G. Then there is a constant $c > 0$ such that $\mu = c\nu$.*

Proof. Let $f, g \in C_c^+(G)$ be given. First, we prove that

$$\frac{\int f \, \mathrm{d}\mu}{\int f \, \mathrm{d}\nu} = \frac{\int g \, \mathrm{d}\mu}{\int g \, \mathrm{d}\nu}.$$

Let V_0 be a fixed symmetric compact neighbourhood of e and let

$$A = \operatorname{supp}(f)V_0 \cup V_0 \operatorname{supp}(f),$$

$$B = \operatorname{supp}(g)V_0 \cup V_0 \operatorname{supp}(g).$$

For $y \in V_0$, consider the functions $f_y : x \mapsto f(xy) - f(yx)$ and $g_y : x \mapsto g(xy) - g(yx)$. It is clear that $f_y, g_y \in C_c(G)$ with $\operatorname{supp}(f_y) \subseteq A$ and $\operatorname{supp}(g_y) \subseteq B$ for every $y \in V_0$.

Let $\varepsilon > 0$. By Corollary 4.8 there is a symmetric neighbourhood V_1 of e such that $|f_y(x)| < \varepsilon$ and $|g_y(x)| < \varepsilon$ for all $y \in V_1$. Let $V := V_0 \cap V_1$ which is a symmetric neighbourhood of e.

Now, pick $\tilde{h} \in C_c^+(G)$ with $\operatorname{supp}(\tilde{h}) \subseteq V$ and let $h(x) = \tilde{h}(x) + \tilde{h}(x^{-1})$, for all $x \in G$. Then $h \in C_c(G)$, $h(x) = h(x^{-1})$ for all $x \in G$, and since V is symmetric, $\operatorname{supp}(h) \subseteq V$.

Since f, g and h are all compactly supported, hence supported on sets of finite measure, Tonelli's theorem applies in the following calculations. Therefore,

$$\begin{aligned}\left(\int h \, \mathrm{d}\nu\right)\left(\int f \, \mathrm{d}\mu\right) &= \int\int h(y)f(x) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \\ &= \int\int h(y)f(yx) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y).\end{aligned}$$

Furthermore, using properties of h together with Tonelli's theorem, we get

$$\begin{aligned}\left(\int h \, \mathrm{d}\mu\right)\left(\int f \, \mathrm{d}\nu\right) &= \int\int h(x)f(y) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \\ &= \int\int h(y^{-1}x)f(y) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \\ &= \int h(x^{-1}y)f(y) \, \mathrm{d}\nu(y) \, \mathrm{d}\mu(x) \\ &= \int\int h(y)f(xy) \, \mathrm{d}\nu(y)\mu(x) \\ &= \int\int h(y)f(xy) \, \mathrm{d}\mu(x)\nu(y).\end{aligned}$$

Combining these equalities, we deduce that

$$\begin{aligned}&\left|\left(\int h \, \mathrm{d}\mu\right)\left(\int f \, \mathrm{d}\nu\right) - \left(\int h \, \mathrm{d}\nu\right)\left(\int f \, \mathrm{d}\mu\right)\right| \\ &= \left|\int\int h(y)f(xy) \, \mathrm{d}\mu(x)\nu(y) - \int\int h(y)f(yx) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y)\right| \\ &= \left|\int\int h(y)(f(xy) - f(yx)) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y)\right| \leq \varepsilon\mu(A)\int h \, \mathrm{d}\nu.\end{aligned}$$

The last inequality is due to the fact that $|h(y)(f(xy) - f(yx))| < \varepsilon h(y)$ for all $x, y \in G$, since h is supported in V.

Dividing by the strictly positive number $\left(\int h \, \mathrm{d}\nu\right)\left(\int f \, \mathrm{d}\nu\right)$, we get that

$$\left|\frac{\int h \, \mathrm{d}\mu}{\int h \, \mathrm{d}\nu} - \frac{\int f \, \mathrm{d}\mu}{\int f \, \mathrm{d}\nu}\right| \leq \frac{\varepsilon\mu(A)}{\int f \, \mathrm{d}\nu},$$

Upon replacing f by g, and doing exactly the same calculations, we see that

$$\left|\left(\int h \, \mathrm{d}\mu\right)\left(\int g \, \mathrm{d}\nu\right) - \left(\int h \, \mathrm{d}\nu\right)\left(\int g \, \mathrm{d}\mu\right)\right| \leq \varepsilon\mu(B)\int h \, \mathrm{d}\nu.$$

Dividing by $\left(\int h \, \mathrm{d}\nu\right)\left(\int g \, \mathrm{d}\nu\right)$ we obtain

$$\left|\frac{\int h \, \mathrm{d}\mu}{\int h \, \mathrm{d}\nu} - \frac{\int g \, \mathrm{d}\mu}{\int g \, \mathrm{d}\nu}\right| \leq \frac{\varepsilon\mu(B)}{\int g \, \mathrm{d}\nu}.$$

Now we can combine these inequalities to obtain

$$\begin{aligned}\left|\frac{\int f\,\mathrm{d}\mu}{\int f\,\mathrm{d}\nu}-\frac{\int g\,\mathrm{d}\mu}{\int g\,\mathrm{d}\nu}\right| &\leq \left|\frac{\int f\,\mathrm{d}\mu}{\int f\,\mathrm{d}\nu}-\frac{\int h\,\mathrm{d}\mu}{\int h\,\mathrm{d}\nu}\right|+\left|\frac{\int h\,\mathrm{d}\mu}{\int h\,\mathrm{d}\nu}-\frac{\int g\,\mathrm{d}\mu}{\int g\,\mathrm{d}\nu}\right| \\ &\leq \frac{\varepsilon\mu(A)}{\int f\,\mathrm{d}\nu}+\frac{\varepsilon\mu(B)}{\int g\,\mathrm{d}\nu}.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\frac{\int f\,\mathrm{d}\mu}{\int f\,\mathrm{d}\nu}=\frac{\int g\,\mathrm{d}\mu}{\int g\,\mathrm{d}\nu}.$$

Since this holds for any $f, g \in C_c^+(G)$, we conclude that

$$\int f\,\mathrm{d}\mu = c\int f\,\mathrm{d}\nu$$

for any $f \in C_c^+(G)$, where $c = \int g\,\mathrm{d}\mu / \int g\,\mathrm{d}\nu > 0$ independent of the choice of $g \in C_c^+(G)$. It now clearly follows that $\int f\,\mathrm{d}\mu = c\int f\,\mathrm{d}\nu$ for every $C_c(G)$, and by the uniqueness part of the Riesz representation theorem we conclude that $\mu = c\nu$. □

Note that uniqueness of the left Haar measures also implies uniqueness of the right Haar measures, because of Proposition 4.18.1.

5.3 An alternative proof of the existence of Haar measure on a compact abelian group

In this section we shall present a proof of the existence of the Haar measure in the special case where G is compact and abelian. In this case, the Haar measure is, of course, both left and right translation invariant.

This proof is based on a fixed point theorem by Markov and Kakutani. To understand this proof properly, one needs to know something about topological vector spaces, in particular locally convex spaces and the weak*-topology (see, e.g., [2]).

In order to prove the Markov-Kakutani theorem, we need a lemma involving affine maps.

Definition 5.11. *Let X and Y be vector spaces and let C be a convex subset of X. A map $T : C \to Y$ is called **affine** if*

$$T(tx + (1-t)y) = tTx + (1-t)Ty$$

for all $x, y \in C$ and $0 \leq t \leq 1$.

Lemma 5.12. *Let $T : C \to C$ be a continuous affine map on a compact, convex subset C of a locally convex space X. Then T has a fixed point.*

Proof. We define the sequence $(x_n)_{n=0}^{\infty}$ recursively by

$$x_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i x_0,$$

where $x_0 \in C$ is arbitrarily chosen. Since C is convex and T maps into C, it follows that $x_n \in C$, for all $n \geq 0$. Further, since C is compact, there is a subnet $(x_{n_\alpha})_{\alpha \in A}$ of our sequence that converges to some point, say $x \in C$. We will show that $Tx = x$.

Since the continuous linear functionals seperate points in X (see [5], corollary to Theorem 3.4), it is enough to show that $\varphi(Tx) = \varphi(x)$ for each continuous linear functional φ on X.

We see, that for a continuous linear functional φ, for all $n \geq 0$,

$$|\varphi(Tx_n - x_n)| = \left|\varphi\left(\frac{1}{n}T^n x_0 - \frac{1}{n}x_0\right)\right| \leq \frac{2}{n}\sup_{y \in C}|\varphi(y)|.$$

The last number is finite, since C is compact, and hence $|\varphi(Tx_n - x_n)| \to 0$ as $n \to \infty$.

In particular, $\varphi(Tx - x) = \lim_\alpha \varphi(Tx_{n_\alpha} - x_{n_\alpha}) = 0$, and we are done. $\square$

Theorem 5.13 (Markov-Kakutani)**.** *Let C be a compact convex subset of a locally convex space. Let $\mathcal{F}$ be a family of continuous affine maps of C into itself. Assume furthermore that $TSx = STx$, for all $T, S \in \mathcal{F}$ and $x \in C$. Then $\mathcal{F}$ has a common fixed point, i.e., there is an $x \in C$ such that $Tx = x$, for all $T \in \mathcal{F}$.*

Proof. For each finite subset $F \subseteq \mathcal{F}$, define $f_F = \{x \in C \mid Tx = x, \forall T \in F\}$. We immediately observe that f_F is closed, hence compact. Indeed, if $(x_\alpha)_{\alpha \in A}$ is a net in f_F such that $x_\alpha \to x \in C$, then for each $T \in F$, since T is continuous, we have that $\lim_\alpha Tx_\alpha = \lim_\alpha x_\alpha = x$, hence $Tx = x$.

Next, we show that f_F is non-empty by induction after $|F|$. First, we note that each of the maps $T \in \mathcal{F}$ has a fixed point by Lemma 5.12. Hence $f_{\{T\}} \neq \emptyset$ for all $T \in \mathcal{F}$.

Now assume that f_F is non-empty whenever $|F| = n$. Let $S \in \mathcal{F}$. We want to show that $f_{F \cup \{S\}}$ is non-empty. Since all the T's are affine, we have that if $x, y \in f_F$ then

$$T(tx + (1-t)y) = tTx + (1-t)Ty = tx + (1-t)y,$$

for all $T \in F$, and hence $tx + (1-t)y \in f_F$ and we conclude that f_F is convex.

Furthermore, if $x \in F$, then for each $S \in \mathcal{F}$ we have that $TSx = STx = Sx$ for every $T \in F$, so $Sx \in f_F$. In other words the restriction of S to f_F can be viewed as a map $S\restriction_{f_F} : f_F \to f_F$. Since f_F is non-empty by assumption and, moreover, convex and compact, by Lemma 5.12 the map $S\restriction_{f_F} : f_F \to f_F$ has a fixed point, that is, $Sx = x$ for some $x \in f_F$. Hence $x \in f_{F \cup \{S\}}$ and therefore the latter set is non-empty.

Clearly, we have that $f_{F_1} \cap f_{F_2} = f_{F_1 \cup F_2} \neq \emptyset$, and from this it follows that the family of closed sets f_F has the finite intersection property. Since C is compact, there is an $x \in \bigcap_F f_F$. Clearly, x has the property that $Tx = x$ for all $T \in \mathcal{F}$, and hence x is a common fixed point. $\square$

Theorem 5.14. *Every compact abelian group G possesses a Haar measure.*

Proof. Our goal is to find a probability measure on G, which is also a Haar measure.

Note that if I is a positive linear functional on $C_c(G)$ then by the Riesz representation theorem there is a Radon measure μ such that $I(f) = \int f \, \mathrm{d}\mu$ for all $f \in C_c(G)$. It is easy to see that $\|I\| = \int 1_G \, \mathrm{d}\mu = \mu(G) < \infty$. In particular, $I \in C_c(G)^*$, where $C_c(G)^*$ denotes the set of bounded linear functionals on $C_c(G)$.

Let $\mathcal{M}$ be the set of positive linear functionals on $C_c(G)$ of norm 1. We see that $\mathcal{M} \subseteq C_c(G)^*$ is closed in the weak*-topology. Indeed, if $(I_\alpha)_{\alpha \in A}$ is a net in $\mathcal{M}$ converging to $I \in C_c(G)^*$ then clearly I is positive and hence there is a Radon measure μ such that $I(f) = \int f \, \mathrm{d}\mu$ for all $f \in C_c(G)$. Then

$$\|I\| = \mu(G) = \lim_\alpha I_\alpha(1_G) = \lim_\alpha 1 = 1,$$

and we conclude that $I \in \mathcal{M}$.

Hence, by the Banach-Alaoglu theorem (see [1], Theorem 5.18), $\mathcal{M}$ is compact. Furthermore, $\mathcal{M}$ is convex. Indeed, if $I_1, I_2 \in \mathcal{M}$ are given by $I_1(f) = \int f \, \mathrm{d}\mu_1$ and $I_2(f) = \int f \, \mathrm{d}\mu_2$, then $\|tI_1 + (1-t)I_2\| = t\mu_1(G) + (1-t)\mu_2(G) = 1$.

Now, for $y \in G$, consider the operator T_y on $\mathcal{M}$ defined by

$$T_y I(f) = I(L_y f),$$

for all $f \in C_c(G)$. It is clear that T_y is weak*-continuous. Moreover, it is easy to see that $T_y I \in \mathcal{M}$ for all $I \in \mathcal{M}$ and $y \in G$.

We see that $T_y T_z = T_z T_y$ for all $y, z \in G$, since G is abelian. And clearly T_y is affine, so by the Markov-Kakutani theorem, we conclude that there is a functional I given by $I(f) = \int f \, \mathrm{d}\mu$, $f \in C_c(G)$, for some Radon probability measure μ such that for $T_y(I) = I$, for all $y \in G$. In other words, $\int f \, \mathrm{d}\mu = \int L_y f \, \mathrm{d}\mu$ for all $y \in G$. Hence by 4.18.2, μ is a left Haar measure and therefore also a right Haar measure on G. $\square$

Chapter 6

The modular function

Having established the existence and uniqueness of the left, respectively, right Haar measure, we shall next examine the relationship between the them.

If μ is a left Haar measure on some locally compact group G, then for every $x \in G$, the measure μ_x defined by $\mu_x(E) = \mu(Ex)$, for every Borel set E, is clearly also a left Haar measure. Therefore, by the uniqueness (Theorem 5.10) there is a positive number $\Delta(x)$ such that $\mu_x = \Delta(x)\mu$.

We see that the map $\Delta : G \to (0, \infty)$ is independent of the choice of μ. Indeed, if ν is another left Haar measure, the uniqueness theorem gives us a constant $c > 0$ such that $\nu = c\mu$, and hence we get that for all Borel sets $E \subseteq X$,

$$\nu_x(E) = \nu(Ex) = c\mu(Ex) = \Delta(x)c\mu(E) = \Delta(x)\nu(E).$$

The map Δ will be called the **modular function** on G. To study some basic properties of Δ, we need a lemma.

Lemma 6.1. *Let μ be a Radon measure on a locally compact group G and let $f \in C_c(G)$. Then the maps $x \mapsto \int L_x f \, \mathrm{d}\mu$ and, respectively, $x \mapsto \int R_x f \, \mathrm{d}\mu$ are continuous.*

Proof. First, define $K := \operatorname{supp} f$ and fix a compact neighbourhood V of e. By Proposition 4.3.8, the set VK is compact and since μ is a Radon measure, $\mu(VK) < \infty$.

Let $(x_\alpha)_{\alpha \in A}$ be a net in G converging to x. We want to show that

$$\lim_\alpha \int L_{x_\alpha} f \, \mathrm{d}\mu = \int L_x f \, \mathrm{d}\mu.$$

First, assume that $x = e$. Then, given $\varepsilon > 0$, since f is left uniformly continuous, there is a neighbourhood U of e such that $\|L_y f - f\| < \varepsilon / \mu(VK)$ for all $y \in U$. By replacing U with $U \cap V$, we may even assume that $U \subseteq V$. Then clearly $\operatorname{supp}(L_y f - f) \subseteq VK$ for all $y \in U$. This ensures that there is an α_0 such that if $\alpha \geq \alpha_0$, then $x_\alpha \in U$, that is

$$\left| \int L_{x_\alpha} f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu \right| \leq \|L_{x_\alpha} f - f\| \cdot \mu(VK) < \varepsilon.$$

Hence $\lim_\alpha \int L_{x_\alpha} f \, \mathrm{d}\mu = \int L_e f \, \mathrm{d}\mu$.

In general, if $\lim_\alpha x_\alpha = x \in G$ then $\lim_\alpha x_\alpha x^{-1} = e$. Combining this with the fact that $L_x f \in C_c(G)$, we conclude by the previous arguments that

$$\lim_\alpha \int L_{x_\alpha} f \,\mathrm{d}\mu = \lim_\alpha \int L_{x_\alpha x^{-1}} L_x f \,\mathrm{d}\mu = \int L_x f \,\mathrm{d}\mu,$$

and we are done.

The continuity of the map $x \mapsto \int R_x f \,\mathrm{d}\mu$ is proven in exactly the same way. ☐

Proposition 6.2. *The modular function on a locally compact group G is a continuous group homomorphism into the multiplicative group $((0,\infty),\cdot)$.*

Proof. Let $x, y \in G$. It is clear that $\mu_{xy} = (\mu_x)_y$ and hence $\Delta(xy) = \Delta(x)\Delta(y)$. To establish continuity, we observe that

$$\int R_x f \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu_{x^{-1}} = \Delta(x^{-1}) \int f \,\mathrm{d}\mu.$$

By Lemma 6.1, we get that the map $x \mapsto \Delta(x^{-1}) \int f \,\mathrm{d}\mu$ is continuous. Since the inversion map $x \mapsto x^{-1}$ is continuous and $\int f \,\mathrm{d}\mu$ is independent of x, we conclude that $x \mapsto \Delta(x)$ is continuous, as wanted. ☐

An important feature of the modular function is that it can determine when the Haar measure is both left and right translation invariant, using the following definition:

Definition 6.3. *A locally compact group G is called* ***unimodular*** *if Δ is the constant function* 1.

It is clear that G is unimodular if and only if the left Haar measure μ on G is also right translation invariant, because (in the "only if"-case) $\mu(Ex) = \mu_x(E) = \mu(E)$, for all Borel sets E.

Note that abelian groups are unimodular. Now let us look at some not entirely trivial examples:

Proposition 6.4. *If G is compact, then G is unimodular*

Proof. By Proposition 4.18.4, we infer that $0 < \mu(G) < \infty$. If μ is a left Haar measure on μ, then $\mu(G) = \mu(Gx) = \Delta(x)\mu(G)$ for all $x \in G$, and hence clearly $\Delta(x) = 1$ for all $x \in G$. ☐

Proposition 6.5. *Let $[G, G]$ denote the commutater subgroup of G, that is, the subgroup consisting of elements on the form $[x, y] = xyx^{-1}y^{-1}$, where $x, y \in G$. If $G/[G, G]$ is finite, then G is unimodular*

Proof. It is easy to see that $[G, G]$ is a normal subgroup of G, and furthermore, it is clear that $\Delta(xyx^{-1}y^{-1}) = 1$ for all $x, y \in G$. Hence $[G, G] \subseteq \ker \Delta$. From this, we conclude that $\Delta(G) \simeq G/\ker \Delta \subseteq G/[G, G]$, but the latter set is finite by assumption, and hence $\Delta(G)$ is finite. Since the only finite subgroup of $(0, \infty)$ is $\{1\}$, we deduce that $\Delta(G) = \{1\}$. ☐

We will conclude this chapter with a proposition which shows a connection between a left Haar measure μ and its right translation-invariant version $\tilde{\mu}$ from Proposition 4.18.1.

Proposition 6.6. *Let G be a locally compact group. For every $x \in G$ we have that* $\mathrm{d}\tilde{\mu}(x) = \Delta(x^{-1})\,\mathrm{d}\mu(x)$.

Proof. Define $\delta : x \mapsto \Delta(x^{-1})$ for all $x \in G$, which is of course a continuous homomorphism from G to $(0, \infty)$.

It is enough to prove that $\int f\,\mathrm{d}\tilde{\mu} = \int f(x)\Delta(x^{-1})\,\mathrm{d}\mu(x)$ for every $f \in C_c(G)$, because by Proposition 3.10, the measure $\delta\,\mathrm{d}\mu$ is a Radon measure and by the uniqueness part in Riesz representation theorem, we can conclude that $\mathrm{d}\tilde{\mu}(x) = \Delta(x^{-1})\,\mathrm{d}\mu(x)$.

Let $f \in C_c(G)$ and $y \in G$. Then $f\delta \in C_c(G)$, and as in the proof of Proposition 6.2, we have that

$$\int R_y(f\delta)\,\mathrm{d}\mu = \delta(y)\int f\,\mathrm{d}\mu,$$

and hence

$$\begin{aligned}\int R_y f(x)\delta(x)\,\mathrm{d}\mu(x) &= \delta(y^{-1})\int f(xy)\delta(xy)\,\mathrm{d}\mu(x)\\ &= \delta(y^{-1})\int R_y(f\delta)\,\mathrm{d}\mu = \int f\delta\,\mathrm{d}\mu.\end{aligned}$$

By Proposition 4.18.2, we obtain that $\delta(x)\,\mathrm{d}\mu(x)$ is a right Haar measure, and by uniqueness, we conclude that there is a positive c such that $\int f\delta\,\mathrm{d}\mu = c\int f\,\mathrm{d}\tilde{\mu}$, for all $f \in C_c(G)$.

Now we show that $c = 1$. Let $\varepsilon > 0$. Since δ is continuous and $\delta(e) = 1$, we can find a neighbourhood V of e such that $|\delta(x) - 1| < \varepsilon$ for $x \in V$. We may even assume that V is symmetric.

As we have seen in the proof of the uniqueness of the Haar measure, we can find $h \in C_c^+(G)$ such that $h(x) = h(x^{-1})$ and $\operatorname{supp} h \subseteq V$. Then, by defition of $\tilde{\mu}$, we get that

$$\int h(x)\,\mathrm{d}\tilde{\mu}(x) = \int h(x^{-1})\,\mathrm{d}\mu(x) = \int h(x)\,\mathrm{d}\mu(x),$$

and hence, we obtain

$$\left|(c-1)\int h\,\mathrm{d}\mu\right| = \left|c\int h\,\mathrm{d}\tilde{\mu} - \int h\,\mathrm{d}\tilde{\mu}\right| = \left|\int h\delta\,\mathrm{d}\mu - \int h\,\mathrm{d}\mu\right| \leq \varepsilon\int h\,\mathrm{d}\mu.$$

Since ε was arbitrary, we conclude that $c = 1$.

Hence, for every $f \in C_c(G)$ we get that

$$\int f\delta\,\mathrm{d}\mu = \int f\,\mathrm{d}\tilde{\mu}.$$

This completes the proof. □

Chapter 7

Examples

In this chapter we shall consider examples of locally compact groups and their Haar measures.

Example. The Lebesgue measure on the additive group $\mathbb{R}^n$ is a standard example of a left and right Haar measure. It is well-known that this measure is translation invariant, and we shall not discuss this any further.

Example. A group equipped with the discrete topology is clearly a locally compact Hausdorff group.

On such a discrete group G, the counting measure τ is a left and right Haar measure and hence every Haar measure on G is on the form $c\tau$ for some $c > 0$. In this case, clearly every singleton $\{x\}$ has strictly positive measure, namely $\mu(\{x\}) = c$. What is more curious, is the following fact:

Proposition 7.1. *Let G be a locally compact group and let μ be a left or right Haar measure on G. If $\mu(\{e\}) > 0$ then $\mu(\{x\}) > 0$ for all $x \in G$ and G is a discrete group.*

Proof. The first statement follows from translation-invariance of μ.

Now let $x \in G$ and let U be a precompact open neighbourhood of x, which exists by Proposition 2.12. Note that by additivity of μ, every infinite set has infinite measure. Since every compact set has finite measure, the open set U must have finite measure and hence be finite. For an $x \in U$, since G is assumed to be Hausdorff, there is a neighbourhood V of x such that $V \cap U = \{x\}$. Hence $\{x\}$ is open, so all the singletons are open in G and hence G is discrete. □

In order to provide more examples, we need to refresh a theorem about the Lebesgue integral on $\mathbb{R}^n$. This theorem is an elementary measure theoretical result, and we will not spend time on it here. The proof can be found in [1], Theorem 2.47 a.

Theorem 7.2. *Let U be an open subset of $\mathbb{R}^n$ and $h : U \to \mathbb{R}^n$ be a C^1-diffeomorphism with Jacobi-matrix $Dh(x)$. If $f \in L^1(h(U), m)$ then:*

$$\int_{h(U)} f(x)\,\mathrm{d}x = \int_U f(h(x))|\det Dh(x)|\,\mathrm{d}x$$

where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^n$.

Using this result, we are able to prove a proposition that enables us to provide the aforementioned examples. In the following, by $GL_n(\mathbb{R})$ we denote the invertible $n \times n$-matrices with real entries.

Proposition 7.3. *Let G be a locally compact group which is homeomorphic to an open subset $U \subseteq \mathbb{R}^n$ in such a way that if we identify G with U, left translation is an affine map in the sense that $xy = A_x y + b_x$, where $A_x \in GL_n(\mathbb{R})$ and $b_x \in \mathbb{R}^n$. Then $|\det A_x|^{-1}\,\mathrm{d}x$ is a left Haar measure on G.*

Proof. First note that $A_{xy} = A_x A_y$, for all $x, y \in G$. Indeed, for $x, y, z \in G$ we have that

$$A_x A_y z + A_x b_y + b_x = A_x(A_y z + b_y) + b_x = x(yz) = (xy)z = A_{xy} z + b_{xy},$$

and hence

$$(A_x A_y - A_{xy})z = b_{xy} - A_x b_y - b_x.$$

This means that the linear map $A_x A_y - A_{xy}$ is constant on the open set $U \subseteq \mathbb{R}^n$ and hence clearly equal to the zero matrix. In other words, $A_x A_y = A_{xy}$ and the claim is proven.

For $x \in G$, let $h_x : U \to \mathbb{R}^n$ be the function defined by

$$h_x(y) = xy = A_x y + b_x,$$

for $y \in G$. Clearly, $h_x(U) = U$, for all $x \in G$. Furthermore, we see that h is a diffeomorphism with Jacobi-matrix $Dh_x(y) = A_x$.

Now to the proof of the statement. By Proposition 3.10 and 4.18.2, it is enough to show that for all $y \in G$ and $f \in C_c(G)$,

$$\int_U f(yx)|\det A_x|^{-1}\,\mathrm{d}x = \int_U f(x)|\det A_x|^{-1}\,\mathrm{d}x.$$

By Theorem 7.2 and the fact that $h_{y^{-1}}(U) = U$, it follows that

$$\begin{aligned}\int_U f(yx)|\det A_x|^{-1}\,\mathrm{d}x &= \int_U f(yh_{y^{-1}}(x))|\det A_{h_{y^{-1}}(x)}|^{-1}|\det Dh_{y^{-1}}(x)|\,\mathrm{d}x \\ &= \int_U f(yy^{-1}x)|\det A_{y^{-1}x}|^{-1}|\det A_{y^{-1}}|\,\mathrm{d}x \\ &= \int_U f(x)|\det A_{y^{-1}}\det A_x|^{-1}|\det A_{y^{-1}}|\,\mathrm{d}x \\ &= \int_U f(x)|\det A_x|^{-1}\,\mathrm{d}x.\end{aligned}$$

This completes the proof. □

There is, of course, a similar statement, concerning right translation being an affine map, that is, $yx = A_x y + b_x$. Then $|\det A_x|^{-1}\,\mathrm{d}x$ is a right Haar measure, and we shall use this in the following examples. The proof is similar.

Now we are finally ready to provide those examples:

Example. A very simple example is the multiplicative group $((0,\infty),\cdot)$. Since $(0,\infty)$ is an open subset of $\mathbb{R}$, we can easily apply Proposition 7.3 to find the Haar measure. Since the group is abelian, the measure is both left and right invariant. We see that for all $x, y \in \mathbb{R}$, $xy = A_x y$ where $A_x := (x) \in GL_1(\mathbb{R})$. Since $\det A_x = x > 0$ for all x, we conclude that $x^{-1}\,\mathrm{d}x$ is the Haar measure on $(0,\infty)$.

Example. Now we turn our attention to another example, namely the topological group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R}^n \right\},$$

equipped with matrix multiplication.

It is clear, that G is actually a group, since every matrix in G is invertible and the product of two such matrices are given by:

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+u & xw+y+v \\ 0 & 1 & z+w \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

Using the topology inherited from $\mathbb{R}^9$ we get a locally compact group. Furthermore, we can identify G with $\mathbb{R}^3$ by the homeomorphism

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Using this identification, we see that left translation is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} * \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x+u \\ xw+y+v \\ z+w \end{pmatrix} = A_{(x,y,z)} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + b_{(x,y,z)}$$

where $b_{(x,y,z)} := (x, y, z)$ and

$$A_{(x,y,z)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

By Proposition 7.3, we get that $1 \cdot \mathrm{d}(x, y, z) = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ is a left Haar measure on G.

This time, the group is not abelian, so this measure may not necessarily be a right Haar measure. But as a matter of fact, it turns out to be. Using the fact that right translation is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} * \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ w & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

we get, by the right translation version of Proposition 7.3 that $1 \cdot \mathrm{d}(u, v, w) = \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w$ is also a right Haar measure. In particular, G is unimodular.

Example. Now, consider the group

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x > 0, y \in \mathbb{R} \right\}.$$

Again, G is clearly a group and multiplication is given by

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xu & xv+y \\ 0 & 1 \end{pmatrix}.$$

Again, the group is not abelian, and this time the left Haar measure is different from the right invariant one.

By identifying G with $(0,\infty)\times\mathbb{R}$ via the homeomorphism

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix},$$

we see that

$$\begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} xu \\ xv+y \end{pmatrix}.$$

This means that left translation is given as in Proposition 7.3 with

$$A_{(x,y)} := \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

and $b_{(x,y)} := (0,y)$. Hence $x^{-2}\,\mathrm{d}x\,\mathrm{d}y$ is a left Haar measure.

Similarly, right translation is given by the matrix

$$A_{(u,v)} := \begin{pmatrix} u & 0 \\ v & 1 \end{pmatrix}$$

and $b_{(u,v)} := (0,0)$, so $u^{-1}\,\mathrm{d}u\,\mathrm{d}v$ is a right Haar measure on G, which is different from the left invariant one above. Hence the group is not unimodular.

Example. Now we want to use Proposition 7.3 on the group of invertible matrices, $\mathrm{GL}_n(\mathbb{R})$.

For a real-valued matrix $X \in M_n(\mathbb{R})$ we write X_i for the i'th column in X, that is, $X = (X_1\, X_2\, \ldots\, X_n)$, and we identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ by the map

$$X \mapsto \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \in \mathbb{R}^{n^2}.$$

Using this identification, the group $G = \mathrm{GL}_n(\mathbb{R})$ is a locally compact group identified with an open subset of $\mathbb{R}^{n^2}$, by the continuity of the determinant map.

We see that left translation is given as

$$XY = (XY_1\, XY_2\, \ldots\, XY_n),$$

for $X, Y \in \mathrm{GL}_n(\mathbb{R})$, where $Y = (Y_1, Y_2, \ldots, Y_n)$. By our identification map of $M_n(\mathbb{R})$ and $\mathbb{R}^{n^2}$ we see that left translation is a linear map given by $XY = A_X Y$ where

$$A_X := \begin{pmatrix} X & & & \\ & X & & \\ & & \ddots & \\ & & & X \end{pmatrix}$$

is a $n^2\times n^2$-matrix. Proposition 7.3 states that $|\det A_X|^{-1}\,\mathrm{d}X$ is a left Haar measure where $\mathrm{d}X$ is the lebesgue measure on $\mathbb{R}^{n^2}$. It is easy to see that $\det A_X = (\det X)^n$ using the definition of the determinant, and hence $|\det X|^{-n}\,\mathrm{d}X$ is a left Haar measure on $\mathrm{GL}_n(\mathbb{R})$.

In a similar way, we see that the measure is also a right Haar measure.

Chapter 8

Unitary representations of compact groups

This last chapter is an application of the results we have established in the preceeding ones. We shall discuss unitary representations of compact groups and use the existence of the Haar measure on such groups to prove certain results, concerning these representations.

First, we will need some notation. For a Hilbert space H, we denote the inner product on H by $\langle \cdot, \cdot \rangle_H$ or just $\langle \cdot, \cdot \rangle$, if no confusion can occur. By $\mathcal{B}(H)$ we denote the set of bounded linear operators on H. The identity operator on H, will be called I_H. For a subset $\mathcal{A} \subseteq \mathcal{B}(H)$ we denote the commutant of $\mathcal{A}$ by $\mathcal{A}'$. By $\mathcal{U}(H)$ we denote the set of unitary operators. Recall that an operator U on H is unitary if by definition $\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle$, for all $\xi, \eta \in H$.

Definition 8.1. *Let G be a topological group. A **unitary representation** of G on a Hilbert space H is a group homomorphism $\pi : G \to \mathcal{U}(H)$ which is continuous with respect to the strong operator topology on $\mathcal{U}(H)$.*

Definition 8.2. *Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation. A subspace $V \subseteq H$ is called **invariant** under π if $\pi(G)V \subseteq V$.*

Definition 8.3. *Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation. A unitary representation $\pi_0 : G \to \mathcal{U}(V)$ is called a **subrepresentation** of π on V, if $\pi_0(g) = \pi(g)\restriction_V$, for all $g \in G$.*

It follows directly from the definitions that V is invariant under π if and only if there is a subrepresentation of π on the Hilbert space V. It is clear that a subrepresentation of a unitary representation π is uniquely determined by the invariant subspace V, and hence we speak of *the* subrepresentation of π on V.

Definition 8.4. *A unitary representation π of a topological group G on a Hilbert space H is said to be **irreducible** if the commutant $\pi(G)' = \mathbb{C}I_H$.*

We shall later see why this is called irreducible, but first we will state the main result that we want to prove in this chapter. It is a generalization of the Peter-Weyl theorem.

Theorem 8.5. *Let π be a unitary representation of a compact group G on a Hilbert space H. Then we can write*

$$H = \bigoplus_{i \in I} V_i,$$

where $\{V_i\}_{i \in I}$ is a family of finite dimensional, invariant subspaces for π such that the subrepresentation of π on V_i is irreducible for every $i \in I$.

Now we shall discuss the notion of unitary representations a little further. The first question one should ask before proving anything about them is, if such things even exist! Of course, for any topological group G and Hilbert space H we can construct a trivial example, namely $\pi : G \to \mathcal{U}(H)$ defined by $\pi(g) = I_H$, $g \in G$. But, as a matter of fact, we can use existence of the Haar measure on locally compact groups to construct non-trivial examples of unitary representations.

Definition 8.6. *Let G be a locally compact group with left Haar measure μ. The* ***left regular representation*** *on $H = L^2(G, \mu)$ is the map $\pi : G \to \mathcal{B}(H)$ defined by $\pi(g)(\xi) = L_g\xi$ for all $g \in G$ and $\xi \in L^2(G, \mu)$ (see Definition 4.4).*

It would be adequate to show that π is actually a unitary representation of G on $L^2(G, \mu)$. This will be settled in the following proposition:

Proposition 8.7. *With G, H and π as above, we have the following*

1. *$\pi(g) \in \mathcal{U}(H)$ for all $g \in G$,*
2. *The map π is a homomorphism, and*
3. *π is continuous with respect to the strong operator topology.*

Proof. In the following we denote the 2-norm on $L^2(G, \mu)$ by $\|\cdot\|_2$.

1. Let $\xi, \eta \in H$. Then by definition of the inner product on $L^2(G, \mu)$ and by 4.18.2 we have

$$\begin{aligned}\langle L_g\xi, L_g\eta\rangle &= \int L_g\xi(x)\overline{L_g\eta(x)}\,\mathrm{d}\mu(x) \\ &= \int L_g(\xi \cdot \bar{\eta})(x)\,\mathrm{d}\mu = \int \xi(x)\eta(x)\,\mathrm{d}\mu(x) \\ &= \langle \xi, \eta\rangle .\end{aligned}$$

2. It is clear that $\pi(gh) = L_{gh} = L_gL_h = \pi(g)\pi(h)$ for all $g, h \in G$.

3.We want to prove that the map $g \mapsto L_g\xi$ is continuous, for every $\xi \in H$. We begin by assuming that $\xi \in C_c(G)$. This part of the proof will be somewhat similar to the proof of Lemma 6.1. Let $K := \mathrm{supp}(\xi)$ and fix a compact neighbourhood V of e. Note that VK is compact and hence $\mu(VK) < \infty$. The function ξ is left uniformly continuous, so we can choose a neighbourhood U og e in G such that $|L_g\xi(x) - \xi(x)| < \sqrt{\varepsilon/\mu(VK)}$ whenever $g \in U$ and $x \in G$ (cf. Proposition 4.6). We can assume that $U \subseteq V$. Then, for all $g \in U$,

$$\|L_g\xi - \xi\|_2^2 = \int |L_g\xi(x) - \xi(x)|^2\,\mathrm{d}\mu(x) < \varepsilon\frac{\mu(VK)}{\mu(VK)} = \varepsilon,$$

where we used that the map $G \ni x \mapsto |L_g\xi(x) - \xi(x)|^2$ is supported in VK for all $g \in U$.

By replacing ξ with $L_h\xi$ for $h \in G$, it now follows that for every $g, h \in G$ there is a neighbourhood U of e such that if $gh^{-1} \in U$, then

$$\|L_g\xi - L_h\xi\|_2 = \|L_{gh^{-1}}(L_h\xi) - L_h\xi\|_2 < \varepsilon.$$

This entails that the map $g \mapsto L_g\xi$ is continous.

Now, let $\xi \in L^2(G, \mu)$, and let $(g_\alpha)_{\alpha\in A}$ be a net in G that converges to $g \in G$. Let $\varepsilon > 0$. Since $C_c(G)$ is 2-norm dense in $L^2(G, \mu)$ by Theorem 3.12, we can choose a $\eta \in C_c(G)$ such that $\|\xi - \eta\| < \varepsilon/3$. Obviously, then $\|L_h\xi - L_h\eta\| < \varepsilon/3$ for every $h \in G$. By continuity of the map $g \mapsto L_g\eta$, we can choose an α_0 such that if $\alpha \geq \alpha_0$, then $\|L_{g_\alpha}\eta - L_g\eta\| < \varepsilon/3$. By this, we get

$$\begin{aligned}\|L_{g_\alpha}\xi - L_g\xi\|_2 &= \|L_{g_\alpha}\xi - L_{g_\alpha}\eta + L_{g_\alpha}\eta - L_g\eta + L_g\eta - L_g\xi\|_2 \\ &\leq \|L_{g_\alpha}\xi - L_{g_\alpha}\eta\|_2 + \|L_{g_\alpha}\eta - L_g\eta\|_2 + \|L_g\eta - L_g\xi\|_2 < \varepsilon,\end{aligned}$$

which completes the proof. □

After this example of a unitary representation of a compact group, we will begin to examine properties of unitary representations in general.

Proposition 8.8. *A unitary representation π is irreducible if and only if the only closed invariant subspaces under π are the trivial subspaces $\{0\}$ and V.*

Proof. Let π be a unitary representation of G on H. For a closed subspace $V \subseteq H$, let P_V denote the projection onto V. First note that V is invariant under $\pi(G)$ if and only if P_V commutes with $\pi(g)$ for all $g \in G$. Indeed, if V is invariant, then for $g \in G$, we have $\pi(g)P_V(H) = \pi(g)V \subseteq V$. Hence $P_V\pi(g)P_V = \pi(g)P_V$. Similarly, $P_V\pi(g^{-1})P_V = \pi(g^{-1})P_V$ and it follows that for $\xi, \eta \in H$

$$\begin{aligned}\langle \pi(g)P_V\xi, \eta\rangle &= \langle P_V\pi(g)P_V\xi, \eta\rangle = \langle \xi, (P_V\pi(g)P_V)^*\eta\rangle \\ &= \left\langle \xi, P_V\pi(g^{-1})P_V\eta\right\rangle = \left\langle \xi, \pi(g^{-1})P_V\eta\right\rangle \\ &= \langle P_V\pi(g)\xi, \eta\rangle .\end{aligned}$$

We thereby conclude that $\pi(g)P_V = P_V\pi(g)$, for all $g \in G$.

On the other hand, if $P_V\pi(g) = \pi(g)P_V$ for all $g \in G$, then $\pi(g)V = \pi(g)P_V(H) = P_V\pi(g)(H) \subseteq V$ for all $g \in G$, and the assertion is proven.

Now we turn to the proof of the proposition. Assume that π is irreducible. Then by definition $\pi(G)' = \{A \in \mathcal{B}(H) | \pi(g)A = A\pi(g), \forall g \in G\} = \mathbb{C}I_H$, so, in particular, the only projections that commute with $\pi(g)$ for all $g \in G$ are the trivial ones, namely the projection 0 onto $\{0\}$ and the identity I_H onto H. From the argument above, it follows that the only closed invariant subspaces under π are the trivial ones.

Assume now that if $\pi(G)V \subseteq V$ then either $V = \{0\}$ or $V = H$. Then again by the above assertion, we get that the only projections in $\pi(G)'$ are 0 and I_H. It is well-known that the commutant of a self-adjoint set is a von Neumann-algebra and that a von Neuman algebra is the closed span of the projections in it. Hence

$$\pi(G)' = \overline{\operatorname{span}\{0, I_H\}} = \mathbb{C}I_H.$$

This completes the proof. □

Proposition 8.9. *Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation. Furthermore, let $E \in \mathcal{B}(H)$ be a finite dimensional projection. Then the map $G \to \mathcal{B}(H)$ given by*

$$g \mapsto \pi(g)E\pi(g^{-1}),$$

for $g \in G$, is continuous, when $\mathcal{B}(H)$ is equipped with the norm topology.

Proof. Since $\pi(g)E\pi(g^{-1}) = \pi(g)E(\pi(g)E)^*$ for all $g \in G$, it suffices to show that the map defined by $g \mapsto \pi(g)E$ for $g \in G$ is norm continuous.

Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis for $E(H)$. Let $g, h \in G$ and $\xi \in H$ with $\|\xi\| \leq 1$. Then $E\xi = \sum_{i=1}^{n} \lambda_i e_i$ for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. We get that

$$\begin{aligned}
\|(\pi(g)E - \pi(h))\xi\| &= \left\| \sum_{i=1}^{n} \lambda_i(\pi(g) - \pi(h))e_i \right\| \\
&\leq \sum_{i=1}^{n} |\lambda_i| \cdot \|(\pi(g) - \pi(h))e_i\| \\
&\leq \left(\sum_{i=1}^{n} |\lambda_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \|(\pi(g) - \pi(h))e_i\|^2 \right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n} \|(\pi(g) - \pi(h))e_i\|^2 \right)^{1/2},
\end{aligned}$$

where we used that $\left(\sum_{i=1}^{n} |\lambda_i|\right)^{1/2} = \|E\xi\| \leq \|\xi\| \leq 1$. Hence

$$\|(\pi(g)E - \pi(h))\| \leq \left(\sum_{i=1}^{n} \|(\pi(g) - \pi(h))e_i\|^2 \right)^{1/2}.$$

Since the map $g \mapsto \pi(g)$ is continous in the strong operator topology, we conclude that this last expression can be made arbitrarily small when g and h are close, and hence $\pi(g)E$ is norm continuous. □

Now we will discuss the theory of vector-valued integration, which we will need in order to prove Theorem 8.5.

Definition 8.10. *Let (X, μ) be a measure space and let H be a Hilbert space. By $L^1(X, \mathcal{B}(H))$ we denote the set of functions $f : X \to \mathcal{B}(H)$ satisfying the following two contitions:*

1. *For all $\xi, \eta \in H$, the function defined by*

$$x \mapsto \langle f(x)\xi \,, \eta \rangle \,,$$

   for $x \in X$, is measurable.

2. *The function defined by $x \mapsto \|f(x)\|$, for $x \in X$, belongs to $L^1(X, \mu)$.*

Note that it follows straight from the second condition and the Cauchy-Schwarz inequality that the functions $x \mapsto \langle f(x)\xi, \eta\rangle$ belong to $L^1(X, \mu)$ for all $\xi, \eta \in H$.

The next proposition shows that whenever $f \in L^1(X, \mathcal{B}(H))$, we can give a meaningful definition of the integral $\int f(x)\, \mathrm{d}\mu(x)$.

Proposition 8.11. *For each $f \in L^1(X, \mathcal{B}(H))$, there is a unique element $\int f(x)\,\mathrm{d}\mu(x) \in \mathcal{B}(H)$ which satisfies that*

$$\left\langle \left(\int f(x)\,\mathrm{d}\mu(x) \right) \xi, \eta \right\rangle = \int \langle f(x)\xi, \eta \rangle \,\mathrm{d}\mu(x),$$

for all $\xi, \eta \in H$. Moreover,

$$\left\| \int f(x)\,\mathrm{d}\mu(x) \right\| \leq \int \|f(x)\|\,\mathrm{d}\mu(x).$$

Proof. Consider the map $F : H \times H \to \mathbb{C}$ defined by $F(\xi, \eta) = \int \langle f(x)\xi, \eta \rangle \,\mathrm{d}\mu(x)$, $\xi, \eta \in H$. This is well-defined since the function integrated belongs to $L^1(X, \mu)$ for every $\xi, \eta \in H$ by assumption. Furthermore, F is clearly linear in the first variable and conjugate linear in the second. Moreover, for all $\xi, \eta \in H$,

$$|F(\xi, \eta)| \leq \int |\langle f(x)\xi, \eta \rangle| \,\mathrm{d}\mu(x) \leq \|\xi\|\|\eta\| \int \|f(x)\|\,\mathrm{d}\mu(x).$$

Hence, by assumption again, F is bounded with $\|F\| \leq \int \|f(x)\|\,\mathrm{d}\mu(x)$. By a theorem due to Riesz (also called the Riesz representation theorem (see [4], Theorem 1.2)), there is a unique operator $T \in \mathcal{B}(H)$ such that $F(\xi, \eta) = \langle T\xi, \eta \rangle$, for all $\xi, \eta \in H$. Obviously, the operator $\int f\,\mathrm{d}\mu(x) := T$ satisfies the desired properties. ☐

Having defined the vector-valued integral properly, we shall examine some of its basic properties.

Proposition 8.12. *Consider $L^1(X, \mathcal{B}(H))$ defined as above. Then*

1. *The vector-valued integral is linear in the sense that if $f, g \in L^1(X, \mathcal{B}(H))$ and $A, B \in \mathcal{B}(H)$, then $(AfB + g) \in L^1(X, \mathcal{B}(H))$ and*

$$\int Af(x)B + g(x)\,\mathrm{d}\mu(x) = A \left(\int f(x)\,\mathrm{d}\mu(x) \right) B + \int g(x)\mu(x).$$

2. *If $E \subseteq X$ is measurable of finite measure, then $1_E \cdot I_H \in L^1(X, \mathcal{B}(H))$ and*

$$\int 1_E(x)\,\mathrm{d}\mu(x) = \mu(E) I_H.$$

3. *If $f \in L^1(X, \mathcal{B}(H))$ and $f(x) \geq 0$ for all $x \in X$, then $\int f(x)\,\mathrm{d}\mu(x) \geq 0$.*

Proof. 1. Let $f, g \in L^1(X, \mathcal{B}(H))$ and $A, B \in \mathcal{B}(H)$ be given. Since

$$\|Af(x)B + g(x)\| \leq \|A\|\|f(x)\|\|B\| + \|g(x)\|,$$

for $x \in X$, it follows that $(AfB + g) \in L^1(X, \mathcal{B}(H))$, and for $\xi, \eta \in H$,

$$\begin{aligned} \left\langle \left(\int Af(x)B + g(x)\,\mathrm{d}\mu(x) \right) \xi, \eta \right\rangle &= \int \langle (Af(x)B + g(x))\xi, \eta \rangle \,\mu(x) \\ &= \int \langle f(x)(B\xi), A^*\eta \rangle \,\mathrm{d}\mu(x) + \int \langle g(x)\xi, \eta \rangle \,\mathrm{d}\mu(x) \\ &= \left\langle \left(\int f(x)\,\mathrm{d}\mu(x) \right) B\xi, A^*\eta \right\rangle + \left\langle \left(\int g(x)\,\mathrm{d}\mu(x) \right) \xi, \eta \right\rangle \\ &= \left\langle \left(A \int f(x)\,\mathrm{d}\mu(x) B + \int g(x)\,\mathrm{d}\mu(x) \right) \xi, \eta \right\rangle, \end{aligned}$$

and the desired conclusion follows.

2. It is clear that $\langle 1_E(x) I_H \xi, \eta \rangle = 1_E(x) \langle \xi, \eta \rangle$ for all $\xi, \eta \in H$ and the right hand side is clearly measurable.

Furthermore, $\|1_E(x) I_H\| = 1_E(x) \|I_H\|$, so

$$\int \|1_E(x) I_H\| \, \mathrm{d}\mu(x) = \mu(E) < \infty,$$

and hence $1_E I_H \in L^1(X, \mathcal{B}(H))$. Since

$$\int \langle 1_E(x) I_H \xi, \eta \rangle \, \mathrm{d}\mu(x) = \mu(E) \langle \xi, \eta \rangle ,$$

for all $\xi, \eta \in H$, we conclude that $\int 1_E(x) I_H \, \mathrm{d}\mu(x) = \mu(E) I_H$.

3. Assume that $f(x) \geq 0$ for all $x \in X$. By definition of positive operators, this means that $\langle f(x)\xi, \eta \rangle \geq 0$ for all $\xi, \eta \in H$ and $x \in X$. Hence $\left\langle \left(\int f(x) \, \mathrm{d}\mu(x) \right) \xi, \eta \right\rangle = \int \langle f(x)\xi, \eta \rangle \, \mathrm{d}\mu(x) \geq 0$ for all $\xi, \eta \in H$ and therefore $\int f(x) \, \mathrm{d}\mu(x) \geq 0$. □

For a subset $\mathcal{A} \subseteq \mathcal{B}(H)$, we let $L^1(X, \mathcal{A})$ denote the set of functions $f \in L^1(X, \mathcal{B}(H))$ such that $f(X) \in \mathcal{A}$. The following proposition is a nice application of the double commutant theorem, and even though we shall not use it, it is proven here for completeness.

Proposition 8.13. *If $\mathcal{M} \subseteq \mathcal{B}(H)$ is a von Neumann algebra, then the operator $\int f(x) \, \mathrm{d}\mu(x)$ belongs to $\mathcal{M}$ whenever $f \in L^1(X, \mathcal{M})$.*

Proof. Let $f \in L^1(X, \mathcal{M})$ be given and let $A \in \mathcal{M}'$. Then by Proposition 8.12.1 we have that

$$A \int f(x) \, \mathrm{d}\mu(x) = \int A f(x) \, \mathrm{d}\mu(x) = \int f(x) A \, \mathrm{d}\mu(x) = \int f(x) \, \mathrm{d}\mu(x) A,$$

so $\int f(x) \, \mathrm{d}\mu(x) \in \mathcal{M}'' = \mathcal{M}$ by the double commutant theorem (see [4], Theorem 18.6). □

We will now consider certain topological spaces and examine vector-valued integration with respect to a Borel measure.

Lemma 8.14. *Let X be a compact Hausdorff space and let μ be a finite Borel measure on X. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C^*-algebra. If $f : X \to \mathcal{A}$ is a continuous function when $\mathcal{A}$ is equipped with the norm topology, then $f \in L^1(X, \mathcal{A})$ and $\int f(x) \, \mathrm{d}\mu(x) \in \mathcal{A}$.*

Moreover, if f is positive and nonzero, that is, $f(x) \geq 0$ for all $x \in X$ and there is an $x_0 \in X$ such that $f(x_0) \neq 0$, and if $\mu(U) > 0$ for all non-empty open sets U, then $\int f(x) \, \mathrm{d}\mu(x)$ is positive and nonzero.

Proof. The functions $X \ni x \mapsto \langle f(x)\xi, \eta \rangle$ are continuous for all $\xi, \eta \in H$, since f is continuous. The function $X \ni x \mapsto \|f(x)\|$ is continuous as well. Since X is compact with $\mu(X) < \infty$, every continuous function from X to $\mathbb{C}$ is bounded hence integrable and therefore $f \in L^1(X, \mathcal{A})$.

We now prove that $\int f \, \mathrm{d}\mu \in \mathcal{A}$. Let $\varepsilon > 0$. Since $f(X)$ is compact in the norm-topology, we can find a finite number of operators $(T_i)_{i=1}^n \subseteq f(X)$, such that the balls

$$B(T_i, \varepsilon) := \{S \in f(X) | \, \|T_i - S\| < \varepsilon\},$$

cover $f(X)$. Hence the sets $(f^{-1}(B(T_i,\varepsilon)))_{i=1}^n$ cover X, and we can find a finite partition $(E_i)_{i=1}^n$ of X, consisting of measurable sets, such that $E_i \subseteq f^{-1}(B(T_i,\varepsilon))$, for $i = 1, 2, \ldots, n$.

Choose $A_i \in f(E_i) \subseteq B(T_i,\varepsilon)$ for each $i = 1, 2, \ldots, n$, and set $g = \sum_{i=1}^n 1_{E_i} A_i$. We see that for every $x \in E_i$ we have that $\|f(x) - g(x)\| = \|f(x) - A_i\| < \varepsilon$.

Since the sets E_i have finite measure for every i, by Proposition 8.12 it follows that $\int g \,\mathrm{d}\mu = \sum_{i=1}^n \mu(E_i)A_i \in \mathcal{A}$. Furthermore, by Proposition 8.11,

$$\left\| \int f \,\mathrm{d}\mu - \int g \,\mathrm{d}\mu \right\| \leq \int \|f - g\| \,\mathrm{d}\mu < \varepsilon\mu(X).$$

Since $\mathcal{A}$ is norm-closed, $\mu(X) < \infty$, and $\varepsilon > 0$ was arbitrary, we conclude that $\int f \,\mathrm{d}\mu \in \mathcal{A}$.

Now assume that f is positive and non-zero and that $\mu(U) > 0$ for all open sets U. Then by Proposition 8.12.3, the integral $\int f \,\mathrm{d}\mu \geq 0$.

Let $x_0 \in X$ such that $f(x_0) \neq 0$. Since $f(x_0) \geq 0$, there must be a $\xi \in H$ such that $\langle f(x_0)\xi, \xi\rangle \neq 0$. Since $\langle f(x)\xi, \xi\rangle$ is a nonzero, nonnegative continuous function, the integral $\int \langle f(x)\xi, \xi\rangle \,\mathrm{d}\mu > 0$. Hence

$$\left\langle \left(\int f \,\mathrm{d}\mu \right) \xi, \xi \right\rangle = \int \langle f(x)\xi, \xi\rangle \,\mathrm{d}\mu > 0,$$

and we conclude that $\int f \,\mathrm{d}\mu$ is nonzero. □

The requirement that $\mu(U) > 0$, for all open U, in the last proposition is in particular satisfied for the Haar measure of a compact group. We shall now begin to combine the theory we developed in this chapter with our knowledge of the Haar measure.

Proposition 8.15. *Let G be a compact group with Haar measure μ. Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation. Let $E \in \mathcal{B}(H)$ be a finite dimensional nonzero projection. Set*

$$A := \int \pi(g)E\pi(g^{-1}) \,\mathrm{d}\mu(g).$$

Then we have the following:

1. *The operator A is compact.*
2. *Furthermore, $A \geq 0$ and $A \neq 0$.*
3. *Lastly, $A \in \pi(G)'$.*

Proof. Let $\mathcal{K}(H)$ denote the compact operators on H, and let $f : G \to \mathcal{B}(H)$ be defined by $g \mapsto \pi(g)E\pi(g^{-1})$, for $g \in G$. We see that $\pi(g)E\pi(g^{-1})$ is a finite rank operator for every $g \in G$ since E is finite dimensional. Hence $f(g) \in \mathcal{K}(H)$ for every $g \in G$. By Proposition 8.9 the map f is norm-continuous and hence by Lemma 8.14 we conclude that $f \in L^1(G, \mathcal{K}(H))$ and that $A = \int f \,\mathrm{d}\mu \in \mathcal{K}(H)$.

We note that for all $g \in G$, we have that $f(g)^2 = f(g)^* = f(g)$, so $f(g)$ is a projection and in particular $f(g) \geq 0$ for all $g \in G$. Furthermore, $f(e) = E \neq 0$ and using that μ is a Haar measure, by the last part of Lemma 8.14, we get that $A \geq 0$ and $A \neq 0$.

To show that $A \in \pi(G)'$, we prove that for all $g \in G$,

$$\pi(g)A\pi(g^{-1}) = A.$$

Using Proposition 8.12.1, it follows that

$$\pi(g)A\pi(g^{-1}) = \int \pi(g)\pi(h)E\pi(h^{-1})\pi(g^{-1})\,\mathrm{d}\mu(h) = \int \pi(gh)E\pi((gh)^{-1})\,\mathrm{d}\mu(h).$$

Let $\xi, \eta \in H$. Using Proposition 4.18.2 on the Haar measure μ, we get that for all $g \in G$

$$\begin{aligned}
\left\langle \pi(g)A\pi(g^{-1})\xi, \eta \right\rangle &= \left\langle \left(\int \pi(gh)E\pi((gh)^{-1})\,\mathrm{d}\mu(h) \right) \xi, \eta \right\rangle \\
&= \int \left\langle \pi(gh)E\pi((gh)^{-1})\xi, \eta \right\rangle \mathrm{d}\mu(h) \\
&= \int \left\langle \pi(h)E\pi(h^{-1})\xi, \eta \right\rangle \mathrm{d}\mu(h) \\
&= \left\langle \left(\int \pi(h)E\pi(h^{-1})\,\mathrm{d}\mu(h) \right) \xi, \eta \right\rangle = \langle A\xi, \eta \rangle .
\end{aligned}$$

We conclude that $A = \pi(g)A\pi(g^{-1})$ for all $g \in G$ and hence that $A \in \pi(G)'$. □

Proposition 8.16. *Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation on $H \neq \{0\}$. Furthermore, assume that G is compact. Then there is a nonzero, finite-dimensional subspace $V \subseteq H$ which is invariant under π.*

Proof. Let μ be the Haar measure on G, and let $E \in \mathcal{B}(H)$ be an arbitrary, 1-dimensional projection. Set

$$A := \int \pi(g)E\pi(g^{-1})\,\mathrm{d}\mu(g).$$

Then, by Proposition 8.15, A is a positive, compact operator, which is nonzero, and $A \in \pi(G)'$. Pick a nonzero eigenvector λ for A and let V be the eigenspace corresponding to λ. This is possible by the spectral theorem for compact operators (see [3] Theorem 5.2.2). The same theorem gives that V is finite dimensional and, of course, $V \neq \{0\}$.

We see that if $\xi \in V$ and $g \in G$, then $A\xi = \lambda\xi$ and hence

$$A\pi(g)\xi = \pi(g)A\xi = \pi(g)\lambda\xi = \lambda\pi(g)\xi,$$

which implies that $\pi(g)\xi$ is an eigenvector for A with eigenvalue λ and we conclude that $\pi(g)\xi \in V$. In other words, $\pi(g)V \subseteq V$ for all $g \in G$, and V is invariant under π. The proof is complete. □

As a corollary, we immediately obtain that if $\pi : G \to \mathcal{U}(H)$ is an irreducible unitary representation of a compact group G, then H is finite dimensional.

Lemma 8.17. *Let G be a compact group and let H be a nonzero, finite dimensional Hilbert space. Assume that $\pi : G \to \mathcal{U}(H)$ is a unitary representation. Then there is a nonzero subspace $V \subseteq H$ such that the subrepresentation of π on V is irreducible.*

Proof. We prove this by strong induction after $\dim(H)$. Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation and assume that $\dim(H) = 1$. Then π itself is irreducible. Indeed, all the subspaces of H are $\{0\}$ and H and hence by Proposition 8.8 the desired follows.

Let $n > 1$ and assume that all unitary representations $\pi' : G \mapsto \mathcal{U}(H')$, where $\dim(H') < n$, have an irreducible subrepresentation on a subspace $V \subseteq H'$ such that $V \neq \{0\}$. Consider a Hilbert space H with $\dim(H) = n$ and a unitary representation $\pi : G \to \mathcal{U}(H)$. If π is irreducible, we are done. If not, by Proposition 8.8 there is a nonzero subspace $H' \subseteq H$ such that $H' \neq H$ and H' is invariant under π. Then the map $\pi_0 : G \to \mathcal{U}(H')$ defined as $\pi_0(g) = \pi(g)\restriction_V$ is a subrepresentation of π. Since $0 < \dim(H') < \dim(H) = n$, by our assumption, there is a irreducible subrepresentation π_1 of π_0 on a subspace $V \subseteq H' \subseteq H$ such that $V \neq \{0\}$. It is clear that π_1 is also a subrepresentation of π, and the proof is complete. □

Corollary 8.18. *Let G be a compact group and let $\pi : G \to \mathcal{U}(H)$ be a unitary representation on a Hilbert space $H \neq \{0\}$. Then there is an irreducible subrepresentation of π on a nonzero, finite dimensional subspace $V \subseteq H$.*

Proof. It follows directly from Proposition 8.16 and Lemma 8.17. □

It turns out that this corollary will be the key to prove the theorem stated in the beginning of this chapter. We will conclude this chapter and thus the whole thesis with a presentation of this proof.

Proof of Theorem 8.5. Let G be a compact group and let π be a unitary representation of G on a Hilbert space H. Let $\mathcal{P}$ be the collection of all families $\{V_i\}_{i\in I}$, where $V_i \subseteq H$ is a nonzero finite-dimensional subspace, such that $V_i \perp V_j$ whenever $i \neq j$ and the subrepresentation of π on V_i is irreducible for every $i \in I$.

We note that it follows from Corollary 8.18 that $\mathcal{P}$ is non-empty. Indeed, there is a finite dimensional subspace $V \subseteq H$ such that the subrepresentation of π on V is irreducible. Hence $\{V\} \in \mathcal{P}$.

The collection $\mathcal{P}$ is partially ordered under the inclusion order "$\subseteq$". We want to use Zorn's lemma in order to conclude that $\mathcal{P}$ contains a maximal element. Therefore, let $\mathcal{T}$ be a totally ordered subset of $\mathcal{P}$. We want to show that $T_0 = \bigcup_{T\in\mathcal{T}} T$ is an upper bound for $\mathcal{T}$ in $\mathcal{P}$, that is, $T_0 \in \mathcal{P}$ and $T \subseteq T_0$ for all $T \in \mathcal{T}$. The last assertion is clear. To see that $T_0 \in \mathcal{P}$, we note that T_0 clearly consists of nonzero, finite dimensional subspaces of H. What remains is to show that the subspaces are pairwise orthogonal. Let $V_1, V_2 \in T_0$. Since $\mathcal{T}$ is totally ordered, there is a $T \in \mathcal{T}$ such that $V_1, V_2 \in T$, and since $T \in \mathcal{P}$, we conclude that $V_1 \perp V_2$.

Hence every totally ordered subset of $\mathcal{P}$ has an upper bound, and, by Zorn's lemma, there exists a maximal element $\{V_i\}_{i\in I}$ in $\mathcal{P}$.

We now claim that

$$H = \bigoplus_{i\in I} V_i.$$

Assume by contradiction that this is not the case. Then the orthogonal complement

$$F := \left(\bigoplus_{i\in I} V_i\right)^{\perp},$$

is a nonzero subspace of H.

We check that F is invariant under π. Let $g \in G$ and $\xi \in F$. We have to show that $\langle \pi(g)\xi, \eta \rangle = 0$ for every $\eta \in \bigoplus_{i \in I} V_i$. Since V_i is invariant under π for every $i \in I$, we conclude that $\pi(g^{-1})\eta \in \bigoplus_{i \in I} V_i$. Hence

$$\langle \pi(g)\xi, \eta \rangle = \left\langle \xi, \pi(g^{-1})\eta \right\rangle = 0.$$

The last equality follows from the definition of ξ being in F.

Since F is invariant under π, we can consider the subrepresentation π_0 of π on F. By Corollary 8.18, there is a nonzero, finite-dimensional subspace $V \subseteq F$ such that the subrepresentation π_1 of π_0 to V is irreducible. Note that the subrepresentation of π on V is π_1 as well. Since $V \subseteq F$, we conclude that $V \perp V_i$ for every $i \in I$. Hence $\{V_i\}_{i \in I} \cup \{V\}$ satisfies every requirement for being an element of $\mathcal{P}$, but since $\{V_i\}_{i \in I} \subsetneq \{V_i\}_{i \in I} \cup \{V\}$ this contradicts the fact that $\{V_i\}_{i \in I}$ is a maximal element of $\mathcal{P}$. $\square$

Litterature

The material for this thesis is mainly collected from [1]. Almost all the major proofs in Chapters 1-7 are from this book, in particular the main theorem about existence and uniqueness of the Haar measure can be found in [1]. The ideas of some of the details, though, stem from [3], e.g. the proof of Proposition 7.1. Moreover, the proof of uniqueness of the Haar measure in the compact abelian case is from [2].

Chapter 8 is based on notes by the supervisor of this thesis. The main result discussed in this chapter can be found in [3] as well, but the approach is different therein.

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