Real Division Algebras

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It is well known that division is possible in the field $\mathbb{R}$. By division I mean that there is a multiplication "$*$" on $\mathbb{R}$ and for all elements $x \neq 0 \in \mathbb{R}$ there exists an element $x^{-1} \in \mathbb{R}$ such that $x * x^{-1} = 1$, where 1 is the unit element. x^{-1} is called the multiplicative inverse. It is also well known that the complex numbers $\mathbb{C}$ satisfy the same condition and since $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic the same holds for $\mathbb{R}^2$. So divison is also possible in the vector field $\mathbb{R}^2$. The obvious question is now: Is division possible in $\mathbb{R}^3$ or in $\mathbb{R}^n$ for some arbitrary $n \in \mathbb{N}$?

The Irish mathematician Hamilton (1805-1865) did a great effort during his career to answer this question but did not succeed. Though while he was working on constructing a multiplication on $\mathbb{R}^3$ he actually came up with one on $\mathbb{R}^4$. This made it possible to show that division is possible in the vector field $\mathbb{R}^4$ or rather in the isomorphic vector field now named the "Quaternions". Later on it was discovered independently by John T. Graves and Arthur Cayley that division is also possible in the so called "Octonions"that are isomorphic to $\mathbb{R}^8$.

So now we know that the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quarternions $\mathbb{H}$ and the octonions $\mathbb{O}$ are all vector fields over $\mathbb{R}$ where division is possible. Since these vector fields are all algebras they are called "Real division algebras". One very important property of these are that they all have a norm. This norm actually implies the existence of a multiplicative inverse. So the real division algebras are also called "Normed division algebras".

Despite the fact that they are all real division algebras they are quite different from each other. While $\mathbb{R}$ and $\mathbb{C}$ and $\mathbb{H}$ are all associative $\mathbb{H}$ and $\mathbb{O}$ are not commutative in fact $\mathbb{O}$ is not even associative.

Well so far we know that if $n = 1, 2, 4, 8$ then $\mathbb{R}^n$ is a real division algebra, but what about for an arbitrary $n \in \mathbb{N}$? how should we proceed? Some clever guys came up with the following idea: Why not, instead of trying to find these rather complicated multiplications in some $\mathbb{R}^n$ for some n just prove that they can not be found because they don't exists unless n is 1,2,4 or 8. In other words they proved that there are only four real division algebras namely $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{R}^8$.I will now try to sketch out how this was done.

The proof is very demanding and takes a lot of pages so I am not going to try to give a thorough description but just try to outline the basic ideas. The fundamental part of the proof is

Proposition If $\mathbb{R}^n$ is a real division algebra then S^{n-1} is parallelizable.

Well well what does that mean? It means that if S^{n-1} is not parallelizable then $\mathbb{R}^n$ is not a real division algebra. But does that make it easier? I don't know but thats the way the problem has been solved, cause it is actually possible to show that S^{n-1} is only parallelizable if $n = 1, 2, 4, 8$ and hence there exists only four real division algebras (up to isomorphism) namely the four already mentioned.

Now I guess it would be appropriate to explain a few notions before I proceed. First, S^{n-1} is the unit sphere in $\mathbb{R}^n$. S^{n-1} is parallelizable if there exists an (n-1)-frame on S^{n-1}. An (n-1)-frame on S^{n-1} is a set of vector fields such that

$$r_1 s_1(x) + r_2 s_2(x) + ... + r_{n-1} s_{n-1}(x) = \mathbf{0}$$

if and only if $r_1, r_2, ..., r_{n-1} = 0$

where $r_i \in \mathbb{R}$ and s_i is a function $S^{n-1} \to \mathbb{R}^n$ such that $s_i(x)$ is a tangent vector to S^{n-1} for all $x \in S^{n-1}$. So the equation just states that this set of vector fields is linearly independent for each $x \in \mathbb{R}$.

So how do we prove the preceding proposition? The idea is to assume that $\mathbb{R}^n$ is a real division algebra. Then we can construct a set of vector fields, that are tangent to S^{n-1}, using an orthogonal basis for $\mathbb{R}^n$. Then we use the fact that a real division algebra possesses a norm , which makes it possible to show that this set of vector fields is linearly independent and therefor it is a (n-1)-frame to S^{n-1}.

So the next step of the proof is to show that S^{n-1} is only parallelizable if $n = 1, 2, 4, 8$. This was first done independently by Milner and Bott in 1958. The proof was rather complicated but in 1962 a guy named J.F. Adams came up with an easier way to prove it.

The trick here is to recognize that for S^{n-1} to be parallelizable the span of S^{n-1} must be $n - 1$. Then we use that some clever people, Hurwitz and Radon came up with a subtle function ρ that has the following property: $span\{S^{n-1}\} = \rho(n) - 1$. $\rho(n)$ is called the Hurwitz-Radon number.

$$\rho(n) = 8a + 2b \ and \ n = 2^{4a+b}k$$

k is an odd number and a and b are integers such that $a \geq 0$ and $0 \leq b \leq 3$

Now the problem is reduced to finding the n's that gives $\rho(n) = n$. Cause it follows that S^{n-1} is parallelizable if and only if $\rho(n) = n$.

So now we just need to solve the equation $\rho(n) = n$. By a bit of calculation it can be shown that this equation is solved for $n = 1, 2, 4, 8$.

These were the headlines of the proof and by these steps it is shown that S^{n-1} is parallelizable if $n = 1, 2, 4, 8$ hence $\mathbb{R}^n$ is a real division algebra if and only if $n = 1, 2, 4, 8$. So $\mathbb{R} \simeq \mathbb{R}$, $\mathbb{R}^2 \simeq \mathbb{C}$, $\mathbb{R}^4 \simeq \mathbb{H}$, $\mathbb{R}^8 \simeq \mathbb{O}$ are the only real division algebras.