The Banach-Tarski paradox
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Formulating the "Paradox"

There are many ways of formulating The Banach-Tarski paradox. The most famous is probably this one:

"Any solid ball in three dimensions can be split up into finitely many pieces and reassembled by rotating and translating these pieces into two exact copies of the original".

This is not actually a paradox. It is is a mathematical theorem that can be proven. The reason why it is called a paradox is because it is rather counterintuitive.Rotating and translating the pieces will not change the shape or volume of each piece and therefor it is surprising that we can get two copies of the original sphere by just rotating and translating. I consider it a pure mathematical result, not a physical property.

Definition

Let G be a group acting on a set X. If there is a subset $E \subseteq X$ and pairwise disjoint subsets $A_1, ...A_m, B_1, ..., B_n$ of E and elements $g_1, ...g_m, h_1, ...h_n \in G$ such that $E = \bigcup_{i=1}^{m} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j)$ Then X is paradoxical with respect to G or X is G-paradoxical.

This means that in the set X there is a subset (it can be all X) that can be split up into to two disjoint sets each of which can be split up into finitely many disjoint subsets and reassembled into a copy of the original. It can be shown that this can be done with the two sets $\bigcup_{i=1}^{m} g_i(A_i)$ and $\bigcup_{j=1}^{m} h_j(B_j)$ together forming a partition of X. So that we use all of X to create two copies of X.

Formulating the paradox

Now we can formulate the paradox by the use of the preceding definition:

"The solid ball in $\mathbb{R}^3$ is G_3-paradoxical. Where G_3 is the group of isometries in $\mathbb{R}^3$"

The group of isometries is the group of bijections from $\mathbb{R}^3$ to $\mathbb{R}^3$ that preserves distance.

Free groups of rank 2

To get started we look at the groups called the free groups of rank 2. The elements in such a group are the words that can be made out of two generators and their inverses. The composition in these groups is concatenation. The empty word is the neutral element $\{e\}$. If we have a generating set $M = \{\sigma, \tau\}$ we can define a free group F of rank 2 like this

$$F = \{\sigma, \sigma^{-1}, \tau, \tau^{-1} | \sigma, \tau \in M\}$$

If we let ρ be one of $\sigma, \sigma^{-1}, \tau, \tau^{-1}$ we can define a word in F by $w(\rho)$. So for examble $w(\sigma)$ are all the possible words starting with σ. If there are no pairs of an element and its inverse in a word we say that the word is reduced. It is easy to check that F is actually a group.

Theorem

"A free group F is paradoxical when it acts in itself by left translation"

Proof:

We have that

$$F = \{e\} \cup w(\sigma) \cup w(\sigma^{-1}) \cup w(\tau) \cup w(\tau^{-1})$$

Where each set is disjoint. But we also have that

$$F = w(\sigma) \cup \sigma w(\sigma^{-1}) \qquad F = w(\tau) \cup \tau w(\tau^{-1})$$

Since if $h \in F \backslash w(\sigma)$ then $\sigma^{-1}h \in w(\sigma^{-1})$ but $h = \sigma(\sigma^{-1}h) \in \sigma w(\sigma^{-1})$

This argument is the same for the other paradoxical decomposition of F.

In this proof we cut F up into five pieces and uses only four to make the paradoxical decomposition. It can be shown that is is possible to do it with only four pieces together forming a disjoint partition of F.

Proposition

"If G is a paradoxical group acting on the set X with no nontrivial fixed point ($g.x = x$ and $g \neq e$) then X is G-paradoxical. In particular if a free group F of rank to acts on X with no non trivial fixed points then X is paradoxical with respect to F".

We are not going to prove this but just give a strategy for the proof. First we construct a set M by using the axiom of choice to pick one element from each G-orbit in X. Then it can be shown that the collection of sets $((g(M))_{g \in G}$ is a disjoint partition of X. Now we construct two disjoint subsets of X by letting G act on the set M and then by using that G is paradoxical it can be shown that these two disjoint subsets of X each can be split up in finitely many pieces and reassembled into X.

Corollary

"If a group G has a paradoxical subgroup F then G itself is paradoxical. Hence any group that has a free subgroup of rank 2 is paradoxical"

This is a direct consequence of the preceding proposition. We can simply let F act on G by left translation.

Theorem

"The group SO_3 of rotations in $\mathbb{R}^3$ has a free subgroup of rank 2, hence SO_3 is paradoxical"

We are not going to prove this result but the idea is to show that there are rotations θ and ϕ in SO_3 that are generators for a free group of rank 2. These two rotations could be around the x-axis and the z-axis respectively through the same angle. But it is important that it is around two different orthogonal axis. Otherwise the resulting group would not be non Abelian which a free group of two generators is. These two rotations can be represented by matrices. Maybe it is not so difficult to see that such two rotations and concatenations of them are equivalent to the words that can be formed of two generators in a free group of rank 2.

Hausdorff's paradox

"$S^2 \backslash D$ where D is a countable set is SO_3-paradoxical"

We let D denote the set of nontrivial fixed points of F as a subgroup of SO_3 when it acts on S^2. This set is countable because F is countable. Each element in F has two non trivial fixed points when it acts on S^2 namely the intersection between the axis of rotation and S^2. We can count this set like we would count $\mathbb{Z}$. So D is countable. Now that this set is removed it follows from the preceding propostion that $S^2 \backslash D$ is SO_3-paradoxical.

Equidecomposeability

Now we are getting closer to proving the Banach-Tarski paradox and by introducing a new definition we can soon show the paradox for the unit sphere S^2.

Definition

If we have a group G acting on a set X and there exists subsets $A, B \subseteq X$ and elements $g_1, ..., g_n \in G$ such that

$$A = \bigcup A_i \qquad B = \bigcup B_i$$

Where $A_i \cap A_j = B_i \cap B_j = \emptyset$ for $i < j \leq n$ and $A_i = g_i(B_i)$ Then we say that A and B are G-equidecomposable.

This means that X contain two disjoint set that can be split up in the same number of pieces and put together by G to form each other.

Theorem

"If D is a countable subset of S^2 then $S^2 \backslash D$ and S^2 are SO_3-equidecomposable."

Proposition

If G is a group acting on X and E is a G-paradoxical subset of X and E is decomposable with a subset $\tilde{E}$ of X. Then $\tilde{E}$ is paradoxical.

The Banach-Tarski paradox 1

"S^2 is SO_3-Paradoxical."

This follows from the fact that $S^2 \setminus D$ is SO_3-paradoxical and S^2 is SO_3-equidecomposable with $S^2 \setminus D$ then by the preceding S^2 is SO_3-paradoxical.

The Banach-Tarski paradox 2

"The unit ball B in $\mathbb{R}^3$ is G_3-paradoxical.

The paradoxical decomposition of S^2 leads to a paradoxical decomposition of the unit ball $B \setminus \{0\}$ by using the rays from each point in the decomposition of S^2 into the origo. Then it can be shown that $B \setminus \{0\}$ is G_3-equidecomposable with B and hence the unit ball is G_3-paradoxical.

Short discussion

A paradoxical decomposition of S^2 can be achieved using a minimum of four pieces.

The decomposition of B can be done using a minimum of five pieces.

I claim that when we decompose S^2 by using SO_3 we will not be able to see the two copies as two individual spheres. They are situated on top of each other since I haven't used any translations. But since the decomposition can be done using 4 pieces a can color each piece in a distinct color and then the two copies will be distinguishable since they each are made of pieces of color, that the other does not contain.