Euler characteristics of *p*-subgroup categories

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Outline of talk

G is a finite group and p is a prime number

- $\mathcal{S}_G^*$ the poset of nonidentity p-subgroups of G
- $\mathcal{T}_G^*(H,K) = N_G(H,K)$ the transporter category of G
- $\mathcal{L}_G^*(H,K) = O^pC_G(H)\backslash N_G(H,K)$ the linking category of G
- $\mathcal{F}_G^*(H,K) = C_G(H)\backslash N_G(H,K)$ the fusion category of G
- $\mathcal{O}_G^*(H,K) = N_G(H,K)/K$ the orbit category of G
- $\widetilde{\mathcal{F}}_G(H,K) = C_G(H)\backslash N_G(H,K)/K$ the exterior quotient of the fusion category of G

What are the Euler characteristics of these finite categories?

Outline of talk

1. Euler characteristics of matrices
   - Euler characteristic of an invertible matrix
   - Euler characteristic of the poset $\mathcal{S}_G^*$
   - Euler characteristic of square matrix
2. Euler characteristics of fusion categories
3. Euler characteristics of p-subgroup categories
   - Categories of nonidentity subgroups
   - Categories of centric subgroups

Typical result

$$\chi(\mathcal{F}_G^*) = \sum_{[K]} \frac{-\mu(K)}{|\mathcal{F}_G^*(K)|}$$

Definition (The Euler characteristic of an invertible matrix)

The Euler characteristic of $\zeta = \big(\zeta(a,b)\big)$ is the sum

$$\chi(\zeta) = \sum_{a,b} \mu(a,b)$$

of the entries in the inverse $\mu = \zeta^{-1} = \sum_{k=0}^{\infty}(-1)^k(\zeta - E)^k$.

Example

$$\zeta = (g) \qquad \mu = (g^{-1}) \qquad \chi(\zeta) = g^{-1}$$

$$\zeta = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mu = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \chi(\zeta) = 1$$

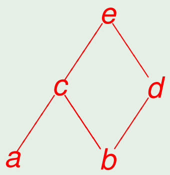
Definition (The incidence matrix of a poset $\mathcal{S}$)

$$\zeta(\mathcal{S}) = \big(\zeta(a,b)\big)_{a,b}, \qquad \zeta(a,b) = \begin{cases} 1 & a \le b \\ 0 & \text{otherwise} \end{cases}$$

Definition (The Euler characteristic of a finite poset $\mathcal{S}$)

$$\chi(\mathcal{S}) = \chi(\zeta(\mathcal{S}))$$

Example (A poset with a terminal element)



$$\zeta(\mathcal{S}) = \begin{pmatrix} 1&0&1&0&1 \\ 0&1&1&1&1 \\ 0&0&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{pmatrix}$$

$$\chi(\mathcal{S}) = \chi(\zeta(\mathcal{S})) = 1$$

Definition (Simplices in a poset)

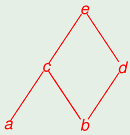
A k-simplex, $k \geq 0$, (from a to b) is a totally ordered subset of $k+1$ points (with a as smallest and b as greatest element).

Example ($(\zeta - E)^k$ counts k-simplices)

0-simplices $\begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} = E$

1-simplices $\begin{pmatrix} 0&0&1&0&1 \\ 0&0&1&1&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \end{pmatrix} = \zeta(\mathcal{S}) - E$

2-simplices $\begin{pmatrix} 0&0&0&0&1 \\ 0&0&0&0&2 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} = (\zeta(\mathcal{S}) - E)^2$



Lemma (Counting simplices in poset $\mathcal{S}$)

$(\zeta - E)^k(a,b) = \#\{k\text{-simplices from } a \text{ to } b\}$ $(k \geq 0)$
$\sum_{a,b}(\zeta - E)^k(a,b) = \#\{k\text{-simplices in } \mathcal{S}\}$ $(k \geq 0)$

Topological Euler characteristic of the realization $|\mathcal{S}|$

$$\begin{aligned}
\chi(|\mathcal{S}|) &= \sum_{k=0}^{\infty}(-1)^k \#\{k\text{-simplices in } \mathcal{S}\} \\
&= \sum_{k=0}^{\infty}(-1)^k \sum_{a,b\in\mathcal{S}} (\zeta - E)^k(a,b) \\
&= \sum_{a,b}\sum_{k=0}^{\infty}(-1)^k(\zeta - E)^k(a,b) \\
&= \sum_{a,b}\zeta^{-1}(a,b) = \sum_{a,b}\mu(a,b) = \zeta(\mathcal{S})
\end{aligned}$$

$x^{-1} = (1 + (x-1))^{-1} = \sum_{k=0}^{\infty}(-1)^k(x-1)^k$

Assumptions

G is a finite group and p is a prime number

Definition (The posets $\mathcal{S}_G$ and $\mathcal{S}_G^*$)

- $\mathcal{S}_G$ is the poset of all p-subgroups of G
- $\mathcal{S}_G^*$ is the poset of nonidentity p-subgroups of G

Proposition (Euler characteristic of the poset $\mathcal{S}_G^*$ I)

$$\chi(\mathcal{S}_G^*) = \sum_{K \in \mathrm{Ob}(\mathcal{S}_G^*)} -\mu(K) = \sum_{[K]\neq 1} -\mu(K)|G \colon N_G(K)|$$

Proof

$$1 = \chi(\mathcal{S}_G) = \sum_{H,K} \mu(H,K) = \sum_{1 \leq K} \mu(1,K) + \sum_{1 \lneqq H \leq K \leq G} \mu(H,K)$$

$$= \mu(1,1) + \sum_{1 \lneqq K} \mu(1,K) + \chi(\mathcal{S}_G^*)$$

Lemma (The Möbius function on $\mathcal{S}_G$ [1, Lemme 4.1])

Let H and K be p-subgroups of G. Then $\mu(H,K)=0$ unless $H \lhd K$ with elementary abelian factor group where

$$\mu(H,K)=(-1)^n p^{\binom{n}{2}}, \qquad |K : H| = p^n$$

In particular, $\mu(K)=\mu(1,K)=0$ unless K is elementary abelian where

$$\mu(K)=(-1)^n p^{\binom{n}{2}}, \quad p^n = |K|$$

Example (Alternating groups at $p=2$)

n	4	5	6	7	8	9	10	11
$\chi(\mathcal{S}_{A_n}^*)$	1	5	-15	-175	65	5121	15105	55935

Proposition (Euler characteristic of the poset $\mathcal{S}_G^*$ II)

$$\chi(\mathcal{S}_G^*) = \sum_H \left(1 - \chi(\mathcal{S}^*_{N_G(H)/H})\right)$$

Recursive calculations?

Proof

$$\begin{aligned}\chi(\mathcal{S}_G^*) &= \sum_{H,K} \mu(H,K) = \sum_H \sum_K \mu(H,K) = \sum_H \sum_{K\in[H,N_G(H)]} \mu(H,K) \\ &= \sum_H \sum_{K\in[H,N_G(H)]} \mu(K/H) = \sum_H \sum_{\bar{K}\in N_G(H)/H} \mu(\bar{K}) \\ &= \sum_H \left(1 - \chi(\mathcal{S}^*_{N_G(H)/H})\right)\end{aligned}$$

What is known about $\chi(\mathcal{S}_G^*)$?

Theorem (Product formula)

$$1 - \chi(\mathcal{S}^*_{\prod_{i=1}^n G_i}) = \prod_{i=1}^{n} 1 - \chi(\mathcal{S}^*_{G_i})$$

Proposition

If G has a nonidentity normal p-subgroup then $\chi(\mathcal{S}_G^*) = 1$

Theorem (Brown 1975, Quillen 1978)

$1 - \chi(\mathcal{S}_G^*)$ *is divisible by* $|G|_p$

Example

$\chi(\mathcal{S}^*_{\mathrm{SL}_n(\mathbf{F}_q)}) = 1 + (-1)^n q^{\binom{n}{2}}$ when q is a power of p

What is unknown about $\chi(\mathcal{S}_G^*)$!

- (Euler characteristics of Chevalley groups) Is $1-\chi(\mathcal{S}^*_{G_n(q)}) = (-1)^n q^{\#\{\text{positive roots}\}}$ for $G = A, B, C, D, E$?
- (Euler characteristics of alternating groups) Describe the sequence $n \to \chi(\mathcal{S}^*_{A_n})$
- (Quillen conjecture 1978) $O_p(G) > 1 \iff \mathcal{S}^*_G \simeq *$

The Euler characteristic of a matrix $\zeta = (\zeta(a,b))$

How do we define $\chi(\zeta)$ when ζ is not invertible?

Definition (Weightings and coweightings)

A weighting for ζ is a column vector $(k^\bullet)$ such that

$$(\zeta(a,b))(k^b) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

A coweighting for ζ is a row vector $(k_\bullet)$ such that

$$(k_a)(\zeta(a,b)) = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$$

- A matrix may have none or many (co)weightings
- If $\big(\mu(a,b)\big)$ is an inverse to $\big(\zeta(a,b)\big)$ then

$$(k^a) = (\mu(a,b)) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \Big(\sum_{b \in S} \mu(a,b)\Big) \quad \text{(column sums)}$$

$$(k_b) = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} (\mu(a,b)) = \Big(\sum_{a \in S} \mu(a,b)\Big) \quad \text{(row sums)}$$

  are the unique weighting and coweighting for ζ
- $\chi(\zeta) = 1$ for $\zeta = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ (and ζ is not invertible)
- $\zeta = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ has no Euler characteristic

If ζ admits both a weighting $k^\bullet$ and a coweighting $k_\bullet$ then the sum of the values of the weighting

$$\sum_b k^b = \sum_b \Big(\sum_a k_a \zeta(a,b)\Big) k^b = \sum_a k_a \Big(\sum_b \zeta(a,b) k^b\Big) = \sum_a k_a$$

Definition (The Euler characteristic of a matrix (Leinster 2008))

$$\chi(\zeta) = \sum_b k^b = \sum_a k_a$$

If ζ is invertible then

$$\chi(\zeta) = \sum_a k^a = \sum_{a,b} \mu(a,b)$$

as before.

Definition (The incidence matrix of a finite category $\mathcal{C}$)

$\zeta(\mathcal{C}) = \big(\zeta(a,b)\big)_{a,b} \qquad \zeta(a,b) = |\mathcal{C}(a,b)|$

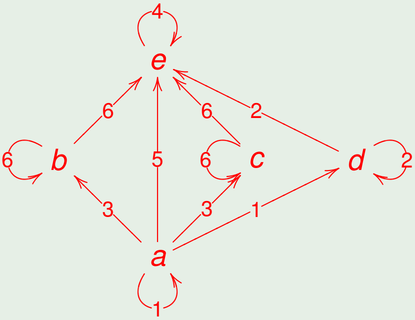
Definition (Euler characteristic of a category (Leinster 2008))

$$\chi(\mathcal{C}) = \chi(\zeta(\mathcal{C}))$$

Proposition (Invarians under equivalence (Leinster 2008))

- *If there is an adjunction* $\mathcal{C} \rightleftarrows \mathcal{D}$ *then* $\chi(\mathcal{C}) = \chi(\mathcal{D})$
- *If* $\mathcal{C}$ *has an initial or terminal element then* $\chi(\mathcal{C}) = 1$
- *If* $\mathcal{C}$ *and* $\mathcal{D}$ *are equivalent then* $\chi(\mathcal{C}) = \chi(\mathcal{D})$

Example (Skeletal subcategory of $\mathcal{F}^*_{A_6}$, $p = 2$)



$$\zeta(\mathcal{C}) = \begin{pmatrix} 1 & 3 & 3 & 1 & 5 \\ 0 & 6 & 0 & 0 & 6 \\ 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\mu(\mathcal{C}) = \begin{pmatrix} 1 & -1/2 & -1/2 & -1/2 & 1/2 \\ 0 & 1/6 & 0 & 0 & -1/4 \\ 0 & 0 & 1/6 & 0 & -1/4 \\ 0 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 0 & 0 & 1/4 \end{pmatrix}$$

$$\chi(\mathcal{C}) = \sum \mu(i,j) = 1/3$$

Theorem (A weighting, a coweighting, and the Euler characteristic of $\mathcal{F}_G^*$)

$$k_K^{\mathcal{F}} = \frac{-\mu(K)}{|G \colon C_G(K)|}$$

$$k_{\mathcal{F}}^H = \sum_K \frac{\mu(H,K)}{|G \colon C_G(K)|} = \frac{1}{|G|} \sum_{x \in C_G(H)} \left(1 - \chi(\mathcal{S}^*_{C_{N_G(H)}(x)/H})\right)$$

$$\chi(\mathcal{F}_G^*) = \sum_{K \in \mathrm{Ob}(\mathcal{F}_G^*)} \frac{-\mu(K)}{|G \colon C_G(K)|} = \sum_{[K] \in \pi_0(\mathcal{F}_G^*)} \frac{-\mu(K)}{|\mathcal{F}_G^*(K)|}$$

In all cases

$$k_K^{\mathcal{F}} \neq 0 \iff K \text{ is elementary abelian}$$

If the Sylow p-subgroup P is normal in G then

$$k_{\mathcal{F}}^H \neq 0 \iff H = C_P(g) \text{ for some } g \in G$$

What is known about $\chi(\mathcal{F}_G^*)$?

Theorem (Product formula)

$$1 - \chi(\mathcal{F}^*_{\prod_{i=1}^n G_i}) = \prod_{i=1}^n 1 - \chi(\mathcal{F}^*_{G_i})$$

Proposition

- *If G has a nonidentity central p-subgroup then* $\chi(\mathcal{F}_G^*) = 1$
- $|G|_{p'} \cdot \chi(\mathcal{F}_G^*) \in \mathbf{Z}$
- $\chi(\mathcal{F}_G^*) = \dfrac{|\{\varphi \in \mathcal{F}_G^*(P) \mid C_P(\varphi) > 1\}|}{|\mathcal{F}_G^*(P)|}$ *when P, the Sylow p-subgroup, is abelian.*
- $\chi(\mathcal{F}_G^*) = \chi(\mathcal{F}_G^{\mathrm{a}})$ *and* $\chi(\mathcal{F}_G^*) = \chi(\widetilde{\mathcal{F}}_G^*)$

Example (Alternating groups A_n at $p = 2$)

n	$\chi(\mathcal{S}^*_{A_n})$	$\chi(\mathcal{F}^*_{A_n})$
4	1	$1/3$
5	5	$1/3$
6	-15	$1/3$
7	-175	$1/3$
8	68	$41/63$
9	5121	$41/63$

n	$\chi(\mathcal{S}^*_{A_n})$	$\chi(\mathcal{F}^*_{A_n})$
10	55105	$18/35$
11	55935	$18/35$
12	-288255	$389/567$
13	1626625	$389/567$
14	23664641	$233/405$
15	150554625	$233/405$

Example (The smallest group with $\chi(\mathcal{F}^*_G) > 1$)

There is a group $G = C_2^4 \rtimes H$, where $H = (C_3 \times C_3) \rtimes C_2$, of order $|G| = 288$ with Euler characteristic $\chi(\mathcal{F}^*_G) = 10/9$ at $p = 2$.

What is unknown about $\chi(\mathcal{F}_G^*)$

- Are $\mathcal{F}_G^*$ and $\mathcal{F}_G^a$ homotopy equivalent?
- Are $\mathcal{F}_G^*$ and $\widetilde{\mathcal{F}}_G^*$ homotopy equivalent?
- Is $\chi(\mathcal{F}_G^*)$ always positive when p divides the order of G?
- Can $\chi(\mathcal{F}_G^*)$ get arbitrarily large?
- What is $\chi(\mathcal{F}_{A_n}^*)$? Does it converge for $n \to \infty$?
- What is $\chi(\mathcal{F}_{\mathrm{SL}_n(\mathbf{F}_q)}^*)$?
- Is there a $|G|_{p'}$-fold covering map $E \to B\mathcal{F}_G^*$ where E is (homotopy) finite and $\chi(E) = |G|_{p'}\chi(\mathcal{F}_G^*)$?

Theorem (Euler characteristics of nonidentity p-subgroup categories)

$\mathcal{C}$	$\chi(\mathcal{C})$
$\mathcal{T}_G^*$	$\displaystyle\sum_{[H]} \frac{-\mu(H)}{\lvert\mathcal{T}_G^*(H)\rvert}$
$\mathcal{L}_G^*$	$\displaystyle\sum_{[H]} \frac{-\mu(H)}{\lvert\mathcal{L}_G^*(H)\rvert}$
$\mathcal{F}_G^*$	$\displaystyle\sum_{[H]} \frac{-\mu(H)}{\lvert\mathcal{F}_G^*(H)\rvert}$
$\mathcal{O}_G^*$	$\displaystyle\chi(\mathcal{T}_G^*) + \frac{p-1}{p}\sum_{[C]} \frac{1}{\lvert\mathcal{O}_G^*(C)\rvert}$

Combinatorial identities

Corollary

For any finite group G and any prime p,

$$\sum_{H} \left(1 - \chi(\mathcal{S}^*_{N_G(H)/H}) + \mu(H)\right) = 0$$

$$\sum_{H} \sum_{x \in C_G(H)} \left(1 - \chi(\mathcal{S}^*_{C_{N_G(H)}(x)/H}) + \mu(H)\right) = 0$$

$$\sum_{H} \left(|H| - \chi(\mathcal{S}^*_{N_G(H)/H})|H| + \mu(H)\right) = \frac{p-1}{p} \sum_{C} |C|$$

where H runs over the set of nonidentity p-subgroups of G and C over the set of nonidentity cyclic p-subgroups of G.

Definition

The p-subgroup $H \leq G$ is p-centric if $p \nmid |C_G(H) \colon C_H(H)|$

Example (Euler characteristics of centric subgroup categories for alternating groups at $p = 2$)

n	4	5	6	7	8	9	10	11
$\lvert A_n\rvert\chi(\mathcal{L}^c_{A_n})$	1	5	-15	-105	65	585	11745	129195
$\chi(\mathcal{S}^c_{A_n})$	1	5	-15	-175	65	585	11745	107745
$\chi(\mathcal{L}^c_{A_n})$	$1/12$		$-1/24$		$13/4032$		$29/4480$	
$\chi(\mathcal{F}^c_{A_n})$	$1/3$		$1/3$		$13/63$		$19/105$	
$\chi(\widetilde{\mathcal{F}}^c_{A_n})$	$1/3$		$1/3$		$13/63$		$19/105$	

Conjecture

$$\chi(\mathcal{F}^c_G) = \chi(\widetilde{\mathcal{F}}^c_G)$$

References

Charles Kratzer and Jacques Thévenaz, *Type d'homotopie des treillis et treillis des sous-groupes d'un groupe fini*, Comment. Math. Helv. **60** (1985), no. 1, 85–106. MR 787663 (87b:06017)