0811 Localization functors & genus sets

Joint w. J. Scherer ①
Preprint on arxiv

1. Localization funct
2. Genus sets - classical and new : Classification thm.
3. COMP1 : Rationalization genus of S^n
4. COMP2 : Postnikov genus of S^n

} two cases cellularization localization

$CX \to X \to LX$

1. Localization functor

$L : TOP \to TOP$ $\quad X \to LX \quad LLX \simeq LX$

Example 1 ~~A any space~~ Fix space A. X is A-local if $map_*(A,X) \simeq *$.
There is a functorial way to $X \to P_A X$ to produce the maximal A-local quotient of X:
least destructive way $\quad map(X,Y) = map(P_A X, Y)$

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ P_A X \arrow[rr, dashed, "\exists !"] & & Y \end{tikzcd}$$

Y A-local $\qquad P_A A \simeq *$

$A = S^{n+1}$. X is A-local if $\Omega^{n+1} X \simeq *$, $\pi_{>n}(X) = 0$.
$P_A X = P_n X$ (Postnikov), $\overline{P}_n X = X\langle n \rangle$, $\quad P_n \overline{P}_n X = *$ (general rule)

Example 2 Fix a map $f : A \to B$. X is f-local if $map(A,X) \simeq map(B,X)$.
$X \to L_f X$ makes X f-local in ~~the most~~ less drastic way:

$$map(X,Y) = map(L_f X, Y)$$

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ L_f X \arrow[rr, dashed] & & Y \end{tikzcd}$$

Y f-local

f monster map. $L_f X = X_0$, $\mathcal{C}_f X = X_\tau$, $L_f \mathcal{C}_f X = *$ (not general)

f —— : $L_f X = X^\wedge = \prod X^\wedge_p$

X is f-local iff it thinks that every $H_*\mathbb{Q}$-iso is a h.e.

2. Genus sets.

2A. Classical

Sullivan square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^\wedge \arrow[d] \\ X_0 \arrow[r] & (X^\wedge)_0 \end{tikzcd} \qquad \begin{array}{ccc} X & \Rightarrow & X^\wedge \\ X_0 & \rightarrow & (X^\wedge)_0 \end{array}$$

$$G(X) = \{ Y \mid X_0 \simeq Y_0,\ X^\wedge = Y^\wedge \}$$

$$\Rightarrow X_\tau \to X^\wedge \to X_0$$

<u>Obs</u> $X^\wedge = Y^\wedge \Leftrightarrow X_\tau = Y_\tau \Leftarrow (X_\tau \to X \to X_0) \to (X_\tau \to X^\wedge \to *)$

$$G(X) = \{ Y \mid X_0 \simeq Y_0,\ X_\tau \simeq Y_\tau \} = \{ Y \mid L_f X = L_f Y,\ \bar{L}_f Y = \bar{L}_f X \}$$

<u>2B</u>. New genus set

same localization and colocalization as X

L localization functor $\quad G_L(X) = \{ Y \mid LX = LY,\ \bar{L}X = \bar{L}Y \}$

$\boxed{\text{If } X \text{ is } L\text{-local, } LX = X,\ \bar{L}X = * \text{ and } G_L(X) = \{X\}.}$

<u>Example</u> $LX = X_0 \qquad G_L(X) = G_0(X) = G(X)$

measuring how far X is from being local

$LX = P_n X \qquad G_L(X) = G_n(X) = \{ Y \mid P_n X = P_n Y,\ X\langle n \rangle = Y\langle n \rangle \}$

Classification Theorem Assume $L\bar{L}X = *$. Then

$$G_L(X) = {}_{\mathrm{Aut}(X)}\backslash [LX, \mathrm{Baut}(\bar{L}X)]$$

$$= \mathrm{Aut}(LX) \backslash [LX, \mathrm{Baut}(\bar{L}X)]_* / \mathrm{Aut}(\bar{L}X)$$

<u>Example</u> $\quad G_0(X) = \mathrm{Aut}(X_0) \backslash [X_0, \mathrm{Baut}(X_\tau)]_* / \mathrm{Aut}(X_\tau)$

$\boxed{\mathrm{Baut}(X_\tau)\ X_0\text{-local}}$?

3. Rationalization genus of S^n, n odd

$$LX = X_0 \qquad M(\mathbb{Q}/\mathbb{Z}, n-1) = S^n_\tau \to S^n \to K(\mathbb{Q},n) = M(\mathbb{Q},n)$$

$$G_0(S^n) = \{Y \mid K(\mathbb{Q},n) = Y_0,\ S^n_\tau = Y_\tau\} = \{Y \mid Y_0 = M(\mathbb{Q},n),\ Y_\tau = M(\mathbb{Q}/\mathbb{Z},n-1)\}$$

<u>Thm</u> $\quad G_0(S^n) \xrightarrow[\cong]{\pi_n} \{H \mid \mathbb{Z} \le H \le \mathbb{Q}\}$ abelian torsion-free groups of rank 1

$\rightarrow$ Why? $\quad Y \in G_0(S^n) \quad \exists \quad M(\mathbb{Q}/\mathbb{Z}, n-1) \to Y \to M(\mathbb{Q},n)$

$$0 \to \pi_n Y \to \mathbb{Q} \xrightarrow{\partial} \mathbb{Q}/\mathbb{Z} \to \pi_{n-1}(Y) \to 0$$

Can have $\pi_{n-1}(Y) \neq 0$

$\leftarrow$ Any such H fits into $\quad 0 \to H \to \mathbb{Q} \xrightarrow{\partial} \mathbb{Q}/\mathbb{Z}$.

$$M(\mathbb{Q}, n-1) \xrightarrow{\partial} \underbrace{M(\mathbb{Q}/\mathbb{Z}, n-1) \to Y(\partial) \to M(\mathbb{Q},n)}_{\text{cofibration and fibration !}}$$

cofibre

so $Y(\partial) \in G_0(S^n)$.

Alternative: $\quad G_0(S^n) = \mathrm{Aut}(\mathbb{Q}) \backslash \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) / \mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$

<u>Corollary</u> $\quad \{Y \in G_0(S^n) \mid \pi_{n-1} Y = 0\} \xrightarrow[\cong]{\pi_n} \{H \mid \mathbb{Z} \le H \le \mathbb{Q},\ \forall p:\ H \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}\}$ pseudo-integers

<u>Problem</u> n even?

4. Postnikov genus of S^n, n odd

(4)

Is it fin. $G_2(X)$

$LX = P_nX \qquad S^n\langle n\rangle \to S^n \to K(\mathbb{Z},n)$

$G_n(S^n) = \{Y \mid P_nY = K(\mathbb{Z},n),\ Y\langle n\rangle = S^n\langle n\rangle\}$

<u>Thm</u> $\mathbb{N}_+ = \{-\infty\} \cup \{0,1,2,\dots\}$

$$\prod_p \mathbb{N}_+ \xrightarrow{\cong} G_n(S^n)$$

Given a sequence $(k_p)_p$, $k_p \in \mathbb{N}_+$, let $Y((k_p)_p)$ be the pullback

$$\begin{tikzcd}
Y((k_p)) \arrow[rrrr] \arrow[d] & & & & \prod_p \mathrm{Baut}_*(\ ) \arrow[d] \arrow[r, phantom, "="] & \mathrm{Baut}_*(S^n\langle n\rangle) \arrow[d] \\
K(\mathbb{Z},n) \arrow[r, "\Delta"] & \prod K(\mathbb{Z},n) \arrow[r, "\prod_p p^{k_p}"] & \prod K(\mathbb{Z},n) \arrow[rr, "\prod s^n"] & & \prod_p \mathrm{Baut}(\ ) \arrow[r] & \mathrm{Baut}(S^n\langle n\rangle)
\end{tikzcd}$$

<u>Rmk</u> 1) The only finite cplx in $G_n(S^n)$ is S^n

2) $H^*\mathbb{Z}$ with stable operations distinguish members of $G_n(S^n)$.

Indication of proof: $\quad G_n(S^n_p) = \{\pm 1\}\backslash[K(\mathbb{Z},n), \mathrm{Baut}(S^n\langle n\rangle)]$

$$\begin{tikzcd}
K(\hat{\mathbb{Z}},n-1) \arrow[r] \arrow[d] & S^n\langle n\rangle \arrow[r] \arrow[d] & (S^n)^\wedge \arrow[r] & K(\hat{\mathbb{Z}},n) \\
* \arrow[r] \arrow[d] & \mathrm{Baut}_*(S^n\langle n\rangle) \arrow[r, shift left, "p"] \arrow[d] & \mathrm{Baut}_*((S^n)^\wedge) \arrow[l, shift left, "CW_{S^{n+1}}"] \arrow[d] & \\
K(\hat{\mathbb{Z}},n) \arrow[r] & \mathrm{Baut}(S^n\langle n\rangle) \arrow[r, "p"'] & \underbrace{\mathrm{Baut}((S^n)^\wedge)}_{\text{should be } K(\mathbb{Z},n)\text{-local}} &
\end{tikzcd}$$

$CW_{S^{n+1}}$ is a pointed functor

interested in this space

<u>Neisendorfer localization</u> $\quad X$ finite complex, $\pi_{\leq 2}(X)=0$, $\pi_{>n}(X)\otimes\mathbb{Q}=0$

$$P_{VB\mathbb{Z}/p}(X\langle n\rangle) = X_\tau \qquad P_{VB\mathbb{Z}/p}(X\langle n\rangle)^\wedge = X^\wedge$$

As $\mathrm{Baut}((S^n)^\wedge)$ turns out to be $K(\mathbb{Z},n)$-local we get

$$[K(\mathbb{Z},n), \mathrm{Baut}(S^n\langle n\rangle)] = [K(\mathbb{Z},n), K(\hat{\mathbb{Z}},n)]/(\hat{\mathbb{Z}})^\times$$

$$= \mathrm{Hom}(\mathbb{Z},\hat{\mathbb{Z}})/(\hat{\mathbb{Z}})^\times$$

We conclude that

$$G_n(S^n) = \mathbb{Z}^\times \backslash \mathrm{Hom}(\mathbb{Z},\hat{\mathbb{Z}})/(\hat{\mathbb{Z}})^\times = \hat{\mathbb{Z}}/(\hat{\mathbb{Z}})^\times$$

$\mathrm{Hom}(\mathbb{Z},\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}$, $\quad \mathbb{Z}^\times \leq (\hat{\mathbb{Z}})^\times$