

Localization functors & genus sets

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 Preprint on arXiv

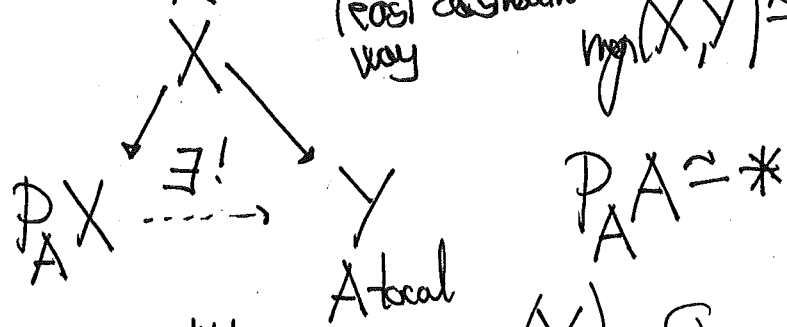
1. Localization functors
2. Genus sets - demand all new: Classification Thm.
3. COMP1: Rationalization genus of S^n
4. COMP2: Postnikov genus of S^n

two cases localization localization
 $\mathbb{Z} \rightarrow X \rightarrow LX$

1. Localization functor $L: TOP \rightarrow TOP$ $X \rightarrow LX$ $LLX = LX$

Example 1

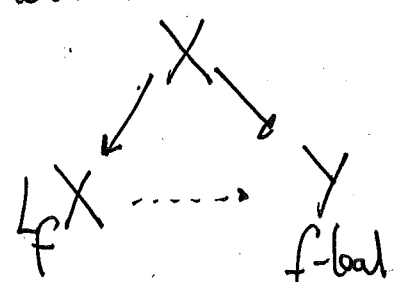
Fix A . X is A -local if $\text{map}_*(A, X) \simeq *$
 There is a functorial way to $X \rightarrow P_A X$ to produce the maximal A -local part of X :
 (most drastic way) $\text{map}_*(X, Y) \simeq \text{map}_*(P_A X, Y)$



$A = S^{n+1}$. X is A -local if $\pi_{n+1} X \simeq *$, $\pi_n(X) = 0$.
 $P_A X = P_n X$ (Postnikov), $\bar{P}_n X = X \langle n \rangle$, $P_n \bar{P}_n X = *$ (general rule)

Example 2

Fix a map $f: A \rightarrow B$. X is f -local if $\text{map}_*(A, X) \simeq \text{map}_*(B, X)$.
 $X \rightarrow L_f X$ makes X f -local in the most (less drastic) way:
 $\text{map}_*(X, Y) \simeq \text{map}_*(L_f X, Y)$



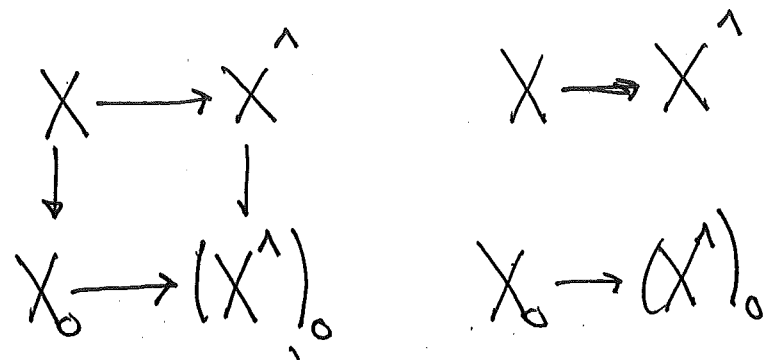
f monic map. $L_f X = X_0$, $L_f X = X_\tau$, $L L_f X = *$ (not general)
 $f \rightarrow$: $L_f X = X^\wedge = \prod X_\rho^\wedge$

X is f -local iff it thinks that every $H\mathbb{Q}$ -iso is a.e.

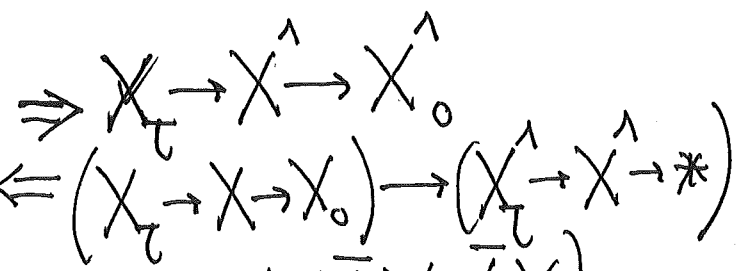
2. Genus sets.

2A. Classical

Sullivan square:



$$G(X) = \{Y \mid X_0 \simeq Y_0, X^\wedge = Y^\wedge\}$$



Obs

$$X^\wedge = Y^\wedge \Leftrightarrow X_\tau = Y_\tau$$

$$G(X) = \{Y \mid X_0 \simeq Y_0, X_\tau \simeq Y_\tau\} = \{Y \mid L_f X = \Delta_f, \bar{L}_f Y = \bar{L}_f X\}$$

2B. New genus set

L localization functor

$$G_L(X) = \{Y \mid LX = \Delta, \bar{L}X = \bar{L}Y\}$$

Some localization and completion of X in local
 $LX = X, X \neq *$
 and
 $G_L(X) = \{X\}$

Example $LX = X_0$

$$G_L(X) = G_0(X) = G(X)$$

however how
 for X is from
 being local

$LX = P_n X$

$$G_L(X) = G_n(X) = \{Y \mid P_n X = P_n Y, X\langle n \rangle = Y\langle n \rangle\}$$

Classification Theorem

Assume $LX = *$. Then

$$G_L(X) = \text{Aut}(X) \backslash [LX, \text{Baut}(LX)]$$

$$= \text{Aut}(X) \backslash [X, \text{Baut}(X)]_* / \text{Aut}(X)$$

Example

$$G_0(X) = \text{Aut}(X_0) \backslash [X_0, \text{Baut}(X_\tau)]_* / \text{Aut}(X_\tau)$$

$\text{Baut}(X_\tau)$
 X_0 -local

3. Rationalization gens of S^n , n odd

(3)

$$LX = X_0 \underbrace{M(\mathbb{Q}/\mathbb{Z})}_{\mathbb{Z}} S^n \rightarrow S^n \rightarrow K(\mathbb{Q}, n) = M(\mathbb{Q}, n)$$

$$G_0(S^n) = \{Y \mid K(\mathbb{Q}, n) = Y_0, S^n = Y_1\} = \{Y \mid Y_0 = M(\mathbb{Q}, n), Y_1 = M(\mathbb{Z}, n-1)\}$$

Thm $G_0(S^n) \xrightarrow{\cong} \{H \mid \mathbb{Z} \cong H \cong \mathbb{Q}\}$ abelian torsion-free groups of rank 1

Why? $Y \in G_0(S^n) \quad \mathbb{S}^n \quad M(\mathbb{Q}/\mathbb{Z}, n-1) \rightarrow Y \rightarrow M(\mathbb{Q}, n)$

$$0 \rightarrow \pi_n Y \rightarrow \mathbb{Q} \xrightarrow{\partial} \mathbb{Q}/\mathbb{Z} \rightarrow \pi_{n-1}(Y) \rightarrow 0$$

(can have $\pi_{n-1}(Y) \neq 0$)

Any such fib fits into $0 \rightarrow H \rightarrow \mathbb{Q} \xrightarrow{\partial} \mathbb{Q}/\mathbb{Z}$.

$$M(\mathbb{Q}, n-1) \xrightarrow{\partial} M(\mathbb{Q}/\mathbb{Z}, n-1) \rightarrow Y(\partial) \rightarrow M(\mathbb{Q}, n)$$

cube

commutative and fibration!

so $Y(\partial) \in G_0(S^n)$.

Alternative: $G_0(S^n) = \text{Aut}(\mathbb{Q}) \backslash \text{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) / \text{Aut}(\mathbb{Q}/\mathbb{Z})$

Corollary $\{Y \in G_0(S^n) \mid \pi_{n-1} Y = 0\} \xrightarrow{\cong} \{H \mid \mathbb{Z} \cong H \cong \mathbb{Q}, \forall \varphi: H \otimes \mathbb{Z}_{(\varphi)} = \mathbb{Z}_{(\varphi)}\}$ pseudo-integers

Problem n even?

4. Postnikov genes of S^n , n odd

Ⓢ $\left. \begin{array}{l} \text{Is } \downarrow \\ \text{for} \\ G(X) \end{array} \right\}$

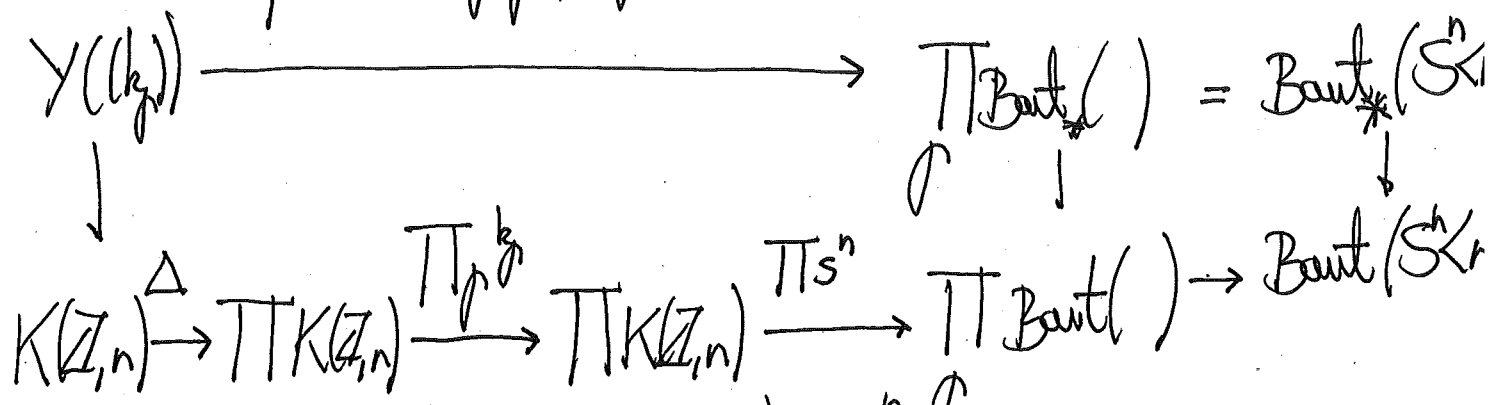
$$LX = P_n X \quad S^n \langle n \rangle \rightarrow S^n \rightarrow K(\mathbb{Z}, n)$$

$$G_n(S^n) = \{ Y \mid P_n Y = K(\mathbb{Z}, n), Y \langle n \rangle = S^n \langle n \rangle \}$$

Thm $\mathbb{N}_+ = \{-\infty\} \cup \{0, 1, 2, \dots\}$

$$\prod \mathbb{N}_+ \xrightarrow{\cong} G_n(S^n)$$

Given a square $\begin{matrix} \uparrow \\ (k_{p,p}) \\ \downarrow \end{matrix}$, $k_p \in \mathbb{N}_+$, let $Y((k_{p,p}))$ be the pull-back



Remark The only finite complex in $G_n(S^n)$ is S^n

2) $H^*\mathbb{Z}$ with stable operations distinguishes members of $G_n(S^n)$.

Indication of proof:

$$G_n(S^h) = \{\pm 1\} \setminus [K(\mathbb{Z}, n), \text{Baut}(S^h)] \quad (5)$$

$$K(\mathbb{Z}^{\wedge}_{n-1}) \longrightarrow S^h \longrightarrow (S^h)^{\wedge} \longrightarrow K(\mathbb{Z}^{\wedge}_n)$$

$$\begin{array}{ccccc}
 \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & \text{Baut}_*(S^h) & \xrightarrow{p} & \text{Baut}_*(S^h)^{\wedge} \\
 \downarrow & & \downarrow & \swarrow \text{CW}_{S^h} & \downarrow \\
 K(\mathbb{Z}^{\wedge}_n) & \longrightarrow & \text{Baut}(S^h) & \xrightarrow{p} & \text{Baut}(S^h)^{\wedge} \\
 & \nearrow & & & \text{shall be } K(\mathbb{Z}, n)\text{-local} \\
 & \text{integral in this case} & & &
 \end{array}$$

CW_{S^h} is a pointed functor

Nerve of localization X finite complex, $\pi_{\leq 2}(X) = 0$, $\pi_{> n}(X) \otimes \mathbb{Q} = 0$

$$P_{V\mathbb{B}\mathbb{Z}/n}(X\langle n \rangle) = X_{\mathbb{Z}} \quad P_{V\mathbb{B}\mathbb{Z}/n}(X\langle n \rangle)^{\wedge} = X^{\wedge}$$

As $\text{Baut}(S^h)^{\wedge}$ turns out to be $K(\mathbb{Z}, n)$ -local we set

$$\begin{aligned}
 [K(\mathbb{Z}, n), \text{Baut}(S^h)] &= [K(\mathbb{Z}, n), K(\mathbb{Z}^{\wedge}_n)] / (\mathbb{Z}^{\wedge})^{\times} \\
 &= \text{Hom}(\mathbb{Z}, \mathbb{Z}^{\wedge}) / (\mathbb{Z}^{\wedge})^{\times}
 \end{aligned}$$

We conclude that

$$G_n(S^h) = \mathbb{Z}^{\times} \setminus \text{Hom}(\mathbb{Z}, \mathbb{Z}^{\wedge}) / (\mathbb{Z}^{\wedge})^{\times} = \mathbb{Z}^{\wedge} / (\mathbb{Z}^{\wedge})^{\times}$$

$$\text{Hom}(\mathbb{Z}, \mathbb{Z}^{\wedge}) = \mathbb{Z}^{\wedge}, \quad \mathbb{Z}^{\times} \cong (\mathbb{Z}^{\wedge})^{\times}$$