Relaxed vertex colorings of simplicial complexes

Natalia Dobrinskaya[1] Jesper M. Møller[2] Dietrich Notbohm[1]

[1]Department of Mathematics
Vrije Universiteit, Amsterdam

[2]Department of Mathematics
University of Copenhagen

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  - Stanley–Reisner rings
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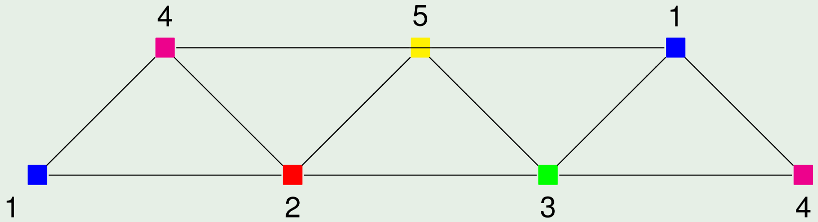
Standard colorings

Example (Standard coloring of the Möbius band MB)

Standard coloring of the Möbius band using 5 colors

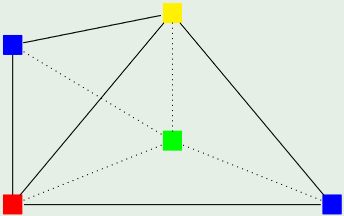


Example (Sudoku)

SUDOKU is an 8-dimensional simplicial complex with $9+9+9$ 8-dimensional maximal simplices. A sudoku problem consists in extending a given partial standard coloring to a full standard coloring of SUDOKU using 9 colors.

Standard Colorings

Example



Standard coloring of K
= Standard coloring of
1-skeleton of K

Theorem

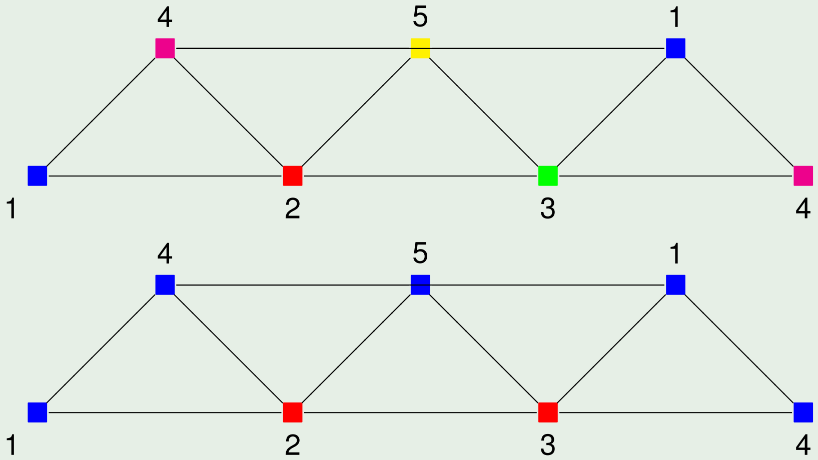
Standard coloring of K = Standard coloring of $\mathrm{sk}_1(K)$

Proof.

Two vertices are in the same simplex of K iff they are in the same 1-simplex, ie in the same simplex of of $\mathrm{sk}_1(K)$. □

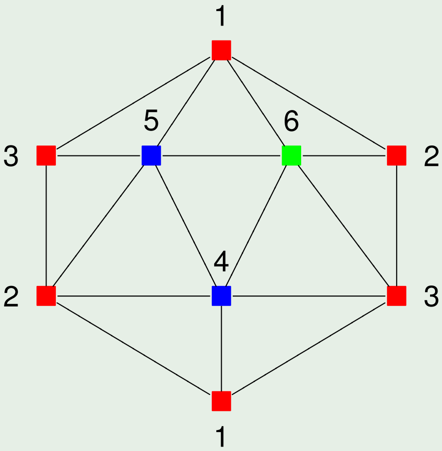
Standard and Relaxed Colorings

Example (Standard and Relaxed coloring of Möbius band MB)



Relaxed colorings

Example (Relaxed Coloring of projective plane P^2)



Relaxed coloring of P^2.
This is a $(3, 2)$-coloring.

A standard coloring of P^2 needs 6 colors.

Abstract Simplicial Complexes

Let V be a finite set (of vertices).

Definition (ASC)

An Abstract Simplicial Complex on set V is a set of subsets (simplices) σ of V that is closed under formation of subsets.

- $D[V]$, the ASC of all subsets of V, triangulates a simplex of dimension $|V| - 1$.
- $\partial D[V]$, the ASC of all proper subsets of V, triangulates a sphere of dimension $|V| - 2$.
- $K = \bigcup_{\sigma \in K} D[\sigma] \subset D[V]$, $1 + \dim K = \sup\{|\sigma| \mid \sigma \in K\}$.

Example

The maximal simplices of the 2-dimensional ASC P^2 are $\{1,3,5\}, \{1,5,6\}, \{1,2,6\}, \{2,3,5\}, \{2,4,5\}, \{4,5,6\}, \{3,4,6\}, \{2,3,6\}, \{1,2,4\}, \{1,3,4\}$.

Colorings of Simplicial Complexes

Let K be an ASC and P be a finite palette of r colors.

Definition

An (P, s)-coloring (or (r, s)-coloring) of K is a map $f\colon V \to P$ that is at most s-to-1 on all simplices of K.

Remark

An (r, s) coloring with

- $s = 1$ is a standard coloring using r colors
- $s > 1$ is a relaxed coloring using r colors

Theorem

(r, s)-colorings of K = (r, s)-colorings of $\mathrm{sk}_s(K)$

The Chromatic Numbers

Definition (The chromatic numbers)

The s-chromatic number, $\mathrm{chr}^s(K)$, is the least r such that K admits an (r, s) coloring.

- $|V| \geq \mathrm{chr}^1(K) \geq \mathrm{chr}^2(K) \geq \cdots \geq \mathrm{chr}^{1+\dim K}(K) = 1$
- $\mathrm{chr}^s(D[V]) = \left\lceil \frac{|V|}{s} \right\rceil$
- $K \supset K' \implies \mathrm{chr}^s(K) \geq \mathrm{chr}^s(K')$
- K admits a (r, s)-coloring $\implies$
  any maximal simplex of K admits a (r, s)-coloring $\implies$
  $1 + \dim K \leq rs$
- The computation of the chromatic numbers is NP-hard.

Example (The chromatic numbers of P^2)

$\mathrm{chr}^1(P^2) = 6$, $\mathrm{chr}^2(P^2) = 3$, and $\mathrm{chr}^3(P^2) = 1$

Davis–Januszkiewicz spaces

Definition (The Davis–Januszkiewicz space of K)

This definition is in three stages:

- Let $\mathrm{DJ}(D[V]) = \mathrm{map}(V, \mathbf{C}P^\infty) = \overbrace{\mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty}^{V}$
- For $\sigma \subset V$ consider $\mathrm{DJ}(D[\sigma]) = \mathrm{map}(V, V - \sigma; \mathbf{C}P^\infty, *)$ as the subspace of the σ-axises of $\mathrm{DJ}(D[V])$
- $\mathrm{DJ}(K) = \bigcup_{\sigma \in K} \mathrm{DJ}(D[\sigma])$

Example

$\sigma = \{1, 2\} \subset V = \{1, 2, 3\}$ and $\mathrm{DJ}(D[\sigma]) =$
$\mathbf{C}P^\infty \times \mathbf{C}P^\infty \times \{*\} \subset \mathrm{DJ}(D[V]) = \mathbf{C}P^\infty \times \mathbf{C}P^\infty \times \mathbf{C}P^\infty$.

- $K \subset K' \implies \mathrm{DJ}(K) \subset \mathrm{DJ}(K')$
- $\mathrm{DJ}(K) \subset \mathrm{DJ}(D[V]) = (\mathbf{C}P^\infty)^V$

Vector bundles over Davis–Januszkiewicz spaces

Definition (The canonical vector bundle λ_K)

The canonical vector bundle λ_K over $\mathrm{DJ}(K)$ is the pull-back

$$\begin{array}{ccc} \lambda_K & \longrightarrow & \lambda \times \cdots \times \lambda \\ \downarrow & & \downarrow \\ \mathrm{DJ}(K) & \hookrightarrow & \mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty \end{array} \qquad \dim \lambda_K = |V|$$

to $\mathrm{DJ}(K)$ of the product of the tautological line bundles.

Theorem (The canonical vector bundle ξ_K)

There exists a short exact sequence of vector bundles

$$0 \to \xi_K \to \lambda_K \to \mathbf{C}^{|V|-(1+\dim K)} \to 0$$

where dim$\xi_K = 1 + \dim K$.

Colorings and splittings of vector bundles

Theorem (Main theorem)

Assume that $1 + \dim K \leq rs$*. Then* K *admits an* (r, s)*-coloring if and only if there exists a lift in the diagram*

$$\begin{array}{ccc} & & BU(s)^r \\ & \overset{?}{\nearrow} & \big\downarrow \oplus \\ \mathrm{DJ}(K) & \xrightarrow[\xi_K]{} & BU(rs) \end{array}$$

The combinatorial problem of existence of an (r, s)-coloring of K has been translated to a topological problem of stably splitting the canonical vector bundle into r vectorbundles of dimension s.

Colorings of 2-Spherical complexes

Definition

A d-spherical complex is a triangulation of the d-sphere S^d.

Theorem (The 4-color theorem)

$$\sup\{\mathrm{chr}^1(K) \mid K \text{ is } 2\text{-spherical}\} \leq 4$$

Failed Proof.

$$\begin{array}{ccc} & & BU(1)^4 \\ & \overset{?}{\nearrow} & \big\downarrow \oplus \\ \mathrm{DJ}(K) & \xrightarrow[\xi_K]{} & BU(4) \end{array} \qquad (r, s) = (4, 1)$$

□

Colorings of other compact surfaces? $\left\lfloor \frac{7+\sqrt{49-24\chi}}{2} \right\rfloor$. The 5-color theorem.

Colorings of 3-Spherical complexes

3-Spherical complexes (cyclic polytopes) with large 1-skeletons

$\sup\{\mathrm{chr}^1(K) \mid K \text{ is 3-spherical }\} = \infty$

Question

Is $\sup\{\mathrm{chr}^2(K) \mid K \text{ is 3-spherical }\} < \infty$?

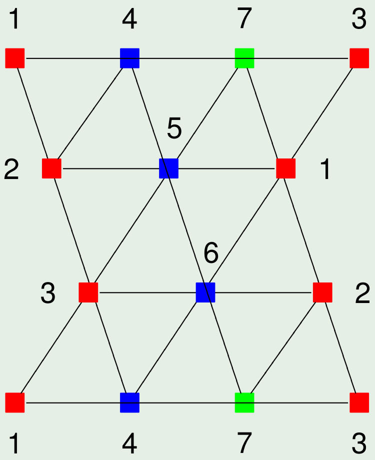
- Hundreds of 3-spherical complexes are $(4, 2)$-colorable.
- Hundreds of 3-manifold complexes from The Manifold Page are $(4, 2)$-colorable.
- I do not know of a single d-manifold complex that is not $(d + 1, 2)$-colorable.

The $(d + 1, 2)$-Conjecture

If K is a d-manifold complex then K admits a $(d + 1, 2)$-coloring.

Support for the $(d+1,2)$-Conjecture when $d=2$

Example



A $(3,2)$-coloring of the minimal triangulation of the torus.

A $(4,2)$-coloring of a 16-vertex triangulation of the Poincaré homology 3-sphere:

{1, 2, 4, 9}, {1, 2, 4, 15}, {1, 2, 6, 14}, {1, 2, 6, 15}, {1, 2, 9, 14}, {1, 3, 4, 12}, {1, 3, 4, 15}, {1, 3, 7, 10}, {1, 3, 7, 12}, {1, 3, 10, 15}, {1, 4, 9, 12}, {1, 5, 6, 13}, {1, 5, 6, 14}, {1, 5, 8, 11}, {1, 5, 8, 13}, {1, 5, 11, 14}, {1, 6, 13, 15}, {1, 7, 8, 10}, {1, 7, 8, 11}, {1, 7, 11, 12}, {1, 8, 10, 13}, {1, 9, 11, 12}, {1, 9, 11, 14}, {1, 10, 13, 15}, {2, 3, 5, 10}, {2, 3, 5, 11}, {2, 3, 7, 10}, {2, 3, 7, 13}, {2, 3, 11, 13}, {2, 4, 9, 13}, {2, 4, 11, 13}, {2, 4, 11, 15}, {2, 5, 8, 11}, {2, 5, 8, 12}, {2, 5, 10, 12}, {2, 6, 10, 12}, {2, 6, 10, 14}, {2, 6, 12, 15}, {2, 7, 9, 13}, {2, 7, 9, 14}, {2, 7, 10, 14}, {2, 8, 11, 15}, {2, 8, 12, 15}, {3, 4, 5, 14}, {3, 4, 5, 15}, {3, 4, 12, 14}, {3, 5, 10, 15}, {3, 5, 11, 14}, {3, 7, 12, 13}, {3, 11, 13, 14}, {3, 12, 13, 14}, {4, 5, 6, 7}, {4, 5, 6, 14}, {4, 5, 7, 15}, {4, 6, 7, 11}, {4, 6, 10, 11}, {4, 6, 10, 14}, {4, 7, 11, 15}, {4, 8, 9, 12}, {4, 8, 9, 13}, {4, 8, 10, 13}, {4, 8, 10, 14}, {4, 8, 12, 14}, {4, 10, 11, 13}, {5, 6, 7, 13}, {5, 7, 9, 13}, {5, 7, 9, 15}, {5, 8, 9, 12}, {5, 8, 9, 13}, {5, 9, 10, 12}, {5, 9, 10, 15}, {6, 7, 11, 12}, {6, 7, 12, 13}, {6, 10, 11, 12}, {6, 12, 13, 15}, {7, 8, 10, 14}, {7, 8, 11, 15}, {7, 8, 14, 15}, {7, 9, 14, 15}, {8, 12, 14, 15}, {9, 10, 11, 12}, {9, 10, 11, 16}, {9, 10, 15, 16}, {9, 11, 14, 16}, {9, 14, 15, 16}, {10, 11, 13, 16}, {10, 13, 15, 16}, {11, 13, 14, 16}, {12, 13, 14, 15}, {13, 14, 15, 16}

Run the magma program demo.prg from /home/moller/projects/simplicial/version04/presentation.

Support for the $(d+1,2)$-Conjecture when $d=4$

$[1,1,2,2,3,3,1,4,2,4,3]$,
$[1,1,1,2,2,2,2,3,3,3,3,1]$,
$[1,1,1,2,2,2,2,3,3,3,3,1]$,
$[1,1,1,2,2,2,2,3,3,3,3,1]$,
$[1,1,2,2,3,3,1,1,2,2,3,3]$,
$[1,1,2,2,3,3,1,1,2,2,3,3]$,
$[1,1,2,2,3,3,1,1,2,2,3,3]$,
$[1,1,2,2,1,3,2,3,4,4,5,5]$,
$[1,1,2,2,3,1,3,2,4,4,5,5]$,
$[1,1,2,2,3,3,1,1,2,2,3,3,4]$,
$[1,1,2,2,3,3,4,1,4,2,5,3,5]$,
$[1,1,2,2,3,3,4,1,4,2,4,3,5]$

$(5,2)$-colorings for some 4-manifolds from The Manifold Page

The Stanley–Reisner algebra

Definition (The Stanley–Reisner algebra of K)

$\mathrm{SR}(K;R) = H^*(\mathrm{DJ}(K);R)$

Theorem (Davis–Januszkiewicz)

$\mathrm{SR}(K;R) = R[V]/\langle D[V]-K\rangle$ is the quotient of the polynomial algebra on V (in degree 2) by the monomial ideal generated by the (minimal) non-simplices of K.

- $\mathrm{SR}(D[V];R) = R[V]$
- $\mathrm{SR}(\partial D[V];R) = R[V]/\langle V\rangle$
- $K \subset K' \implies \mathrm{SR}(K;R) \twoheadleftarrow \mathrm{SR}(K';R)$
- $R[V] \twoheadrightarrow \mathrm{SR}(K) = \lim(P(K)^{\mathrm{op}}; \mathrm{SR}(D[\sigma])) \subset \prod_{\sigma\in K} R[\sigma]$

Colorings and the Stanley–Reisner ring

The total Chern class of the canonical λ_K or ξ_K is

$$c(\lambda_K) = \prod_{v \in V} (1 + v) \in \mathrm{SR}(K; \mathbf{Z}) = R[V] / \langle D[V] - K \rangle$$

Theorem (Main theorem)

K admits an (r, s)-coloring iff there exists a partition $V = V_1 \cup \cdots \cup V_r$ of V into r subsets V_i such that

$$c(\lambda_K) = \prod_{1 \le i \le r} c_{\le s}(\lambda_{V_i}) \in \mathrm{SR}(K; \mathbf{Z})$$

The Stanley–Reisner ring of P^2

Example

Since $[1,2,3,4,5,6] \to [1,1,1,2,2,3]$ is a $(3,2)$-coloring, the identity

$$\prod_{1 \le i \le 6} (1 + v_i) = (1 + v_1 + v_2 + v_3 + v_2 v_3 + v_1 v_3 + v_1 v_2)(1 + v_4 + v_5 + v_4 v_5)(1 + v_6)$$

holds in the Stanley–Reisner ring

$$\mathrm{SR}(P^2; \mathbf{Z}) = \mathbf{Z}[v_1, \ldots, v_6]/(v_1 v_2 v_3, v_1 v_2 v_5, v_1 v_3 v_6, v_1 v_4 v_5, v_1 v_4 v_6, v_2 v_3 v_4, v_2 v_4 v_6, v_2 v_5 v_6, v_3 v_4 v_5, v_3 v_5 v_6)$$

of P^2.

What we learned today

- An (r, s)-coloring is a coloring of the vertices by r colors so that at most s vertices of any simplex has the same color
- (r, s)-colorings only depend on the s-skeleton
- (r, s)-coloring is equivalent to splitting the canonical vector bundle over the Davis–Januszkiewicz space
- (r, s)-coloring is equivalent to factorizing the total Chern class of the canonical vector bundle in the Stanley–Reisner ring

Extroduction

Questions

- Is the $(d+1,2)$-conjecture about colorings of manifolds true?
- With these tools, is it possible to find a topological proof of the 4-color theorem?
- or to solve the similar problem for other compact surfaces?
- Do the face numbers of K determine the chromatic numbers of K?

What I didn't talk about

- (L,s)-colorings of the vertices of K where L is some ASC
- (L,s)-colorings of the t-simplices of K
- Proofs