Chromatic polynomials of simplicial complexes

Two open problems

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1. log-concave sequences and falling factorials

2. Colorings of simplicial complexes
   - Chromatic numbers of simplicial complexes
   - The chromatic polynomial
   - Comparing chromatic polynomials of graphs and simplicial complexes

3. The $d$-chromatic lattice

4. Weighted colorings
Definition 1.1 (LC)

A finite sequence $a_1, a_2, \ldots, a_m$ of positive numbers is log-concave (LC) if $a_{j-1}a_{j+1} \leq a_ja_j$ for $1 < j < m$.

$(a_j)_{j=1}^m$ is log-concave $\iff \frac{a_1}{a_2} \leq \frac{a_2}{a_3} \leq \ldots \leq \frac{a_{m-1}}{a_m}$

$\iff \frac{\log a_{j-1} + \log a_{j+1}}{2} \leq \log a_j \Rightarrow (a_j)_{j=1}^m$ is unimodal

Example 1.2

Binomial sequence $j \mapsto \binom{m}{j}$ is LC

$1, 2, 5, 2, 1$ is unimodal but not LC
Falling factorials and Stirling numbers

Two bases for the polynomial ring $\mathbb{Z}[r]$

\[ [r]_j = r(r-1) \cdots (r-j+1), \quad r^j = r \cdot r \cdots r \]

falling factorial base (FFB) \hspace{1cm} monomial base (MOB)

\[ [r]_0, [r]_1, [r]_2, [r]_3, \ldots \]

base change \hspace{1cm} \[ r^0, r^1, r^2, r^3, \ldots \]

\[ [r]_m = \sum_{j=0}^{m} S_1(m, j)r^j \]

Stirling numbers 1st kind \hspace{1cm} \[ r^m = \sum_{j=0}^{m} S_2(m, j)[r]_j \]

Stirling numbers 2nd kind

$S_2(m, j)$ is the number of partitions of an $m$-set into $j$ blocks

\[ [r]_1 = r^1 \quad r^1 = [r]_1 \quad j \rightarrow |S_1(m, j)| \text{ is LC} \]

\[ [r]_2 = -r^1 + r^2 \quad r^2 = [r]_1 + [r]_2 \quad j \rightarrow S_2(m, j) \text{ is LC} \]

\[ [r]_3 = 2r^1 - 3r^2 + r^3 \quad r^3 = [r]_1 + 3[r]_2 + [r]_3 \]
Definition 2.1 (Colorings of simplicial complexes)

A (weak) \((r, d)\)-coloring of the simplicial complex \(K\) is a map

\[ \text{col}: F^0(K) \to \{1, 2, \ldots, r\} \]

such that

\[ |\text{col}(\sigma)| = 1 \implies \dim \sigma < d \]

for all simplices \(\sigma \in K\). \((K \neq \emptyset, d > 0.)\)
Definition 2.3 (The $d$-chromatic number of a simplicial complex $K$)

The $d$-chromatic number of $K$, $\text{chr}(K, d)$, is the minimal $r$ so that $K$ admits an $(r, d)$-coloring.

$$|F^0(K)| \geq \text{chr}(K, 1) \geq \text{chr}(K, 2) \geq \cdots \geq \text{chr}(K, \dim K) \geq 1$$

Example 2.4 (Do we know the chromatic numbers of any complexes?)

$$K = D[4]$$

$$(2, 2)$$-coloring

$\text{chr}(K, 1) = 4$

$\text{chr}(K, 2) = 2$

$\text{chr}(K, 3) = 2$

$\text{chr}(D[m], d) = \lceil \frac{m}{d} \rceil$
Definition 2.5 (The $d$-chromatic number of a compact manifold $M$)

$$\text{chr}(M, d) = \sup \{ \text{chr}(K, d) \mid K \text{ triangulates } M \}$$

$$\infty \geq \text{chr}(M, 1) \geq \text{chr}(M, 2) \geq \cdots \geq \text{chr}(M, \text{dim } M) \geq 1$$

Example 2.6 (Do we know the chromatic numbers of any manifolds?)

| $K$ | $|K| = S^2$ |
|-----|-------------|
| chr($S^2, 2$) $\geq$ | Is there a triangulation $K$ of $S^2$ with chr($K, 2$) $> 2$? |
| chr($K, 2$) $= 2$ | |

Theorem 2.7 (The 4-color theorem = chromatic numbers of $S^2$)

$$\text{chr}(S^2, 1) = 4 \text{ and } \text{chr}(S^2, 2) = 2$$
Problem 1: What are the chromatic numbers of $S^3$?

- $\text{chr}(S^3, 1) = \infty$ FOR SURE
- $\text{chr}(S^3, 2) = \infty$ PRESUMABLY
- $\text{chr}(S^3, 3) < \infty$ UNKNOWN

The standard triangulation $K = \partial D[5]$ of $S^3$ has $\text{chr}(K, 3) = 2$. There exists a triangulation $K, f(K) = (18, 143, 250, 125)$, of $S^3$ with 3-chromatic number $\text{chr}(K, 3) = 3$. Does there exist a triangulation $K$ of $S^3$ with 3-chromatic number $\text{chr}(K, 3) > 3$?

**Theorem 2.8 (Chromatic numbers of spheres)**

$\text{chr}(S^d, \lceil d/2 \rceil) = \infty$ when $d \geq 3$ PRESUMABLY
\( \chi(K, r, d) \) is the number of \((r, d)\)-colorings of \( K \)

\[
\begin{align*}
\chi(K, r, 1) & = 5 \\
\chi(K, 5, 1) & = 120 \\
\chi(K, r, 2) & = 2 \\
\chi(K, 2, 2) & = 10
\end{align*}
\]
$r \to \chi(K, r, d)$ is the $d$-chromatic polynomial of $K$.

$\chi(K, r, 1)$ and $\chi(K, r, 2)$ are the chromatic polynomials for different $d$ values.

- $\chi(K, 1, 1) = 5$
- $\chi(K, 5, 1) = 120$
- $\text{chr}(K, 1) = 5$
- $\text{chr}(K, 2) = 2$
Simplicial Stirling numbers

Compute the number $\chi(K, r, d)$ of $(r, d)$-colorings of $K$!

Definition 2.9 (Simplical Stirling numbers)

$S(K, j, d)$ is the number of partitions of $F^0(K)$ into $j$ blocks containing only $K$-simplices of dimension $< d$.

- $S(K, j, d) = S_2(m, j)$ when $K = \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$
- $K_1 \subseteq K_2 \implies S(K_1, j, d) \geq S(K_2, j, d)$ when $F^0(K_1) = F^0(K_2)$
- $S_2(m, j) \geq S(K, j, d) \geq S(D[m], j, d')$ with equality for $j = \lfloor F^0(K) \rfloor, \ldots, \lfloor F^0(K) \rfloor - d + 1$, $m = \lfloor F^0(K) \rfloor$
- $S(K, |F^0(K)| - d, d) = S_2(|F^0(K)|, |F^0(K)| - d) - f^d(K)$
- $S(K, j, d) = 0$ for $0 < j < \text{chr}(K, d)$
- $\text{chr}(K, d) = \min\{j \mid S(K, j, d) > 0\}$
**Theorem 2.10 (The $d$-chromatic polynomial of $K$)**

The number of $(r, d)$-colorings of $K$ is

\[
\chi(K, r, d) = \sum_{j=\text{chr}(K,d)} |F^0(K)| \cdot S(K, j, s)[r]_j
\]

where $S(K, j, d)[r]_j$ represents $(r, d)$-colorings of $K$ with $\left|\text{col}(F^0(K))\right| = j$, and $\text{S}(K, j, d)$ represents partitions of $F^0(K)$ into $j$ blocks without $d$-simplices. The theorem provides a formula for calculating the $d$-chromatic polynomial of a graph $K$. The notation $\text{chr}(K,d)$ refers to the number of colorings compatible with the chromatic number of $K$. The formula uses the concept of colorings and equivalence relations to express the number of colorings in terms of the number of blocks and simplices.
Colorings and equivalence relations

$r$ possible colors

$r - 1$ possible colors

$r - 2$ possible colors

3 blocks with no $d$-simplices can be colored in $[r]_3$ ways from a palette of $r$ colors

\[
\chi(K, r, d) = \frac{|F^0(K)|}{\sum_{j=\text{chr}(K,d)} S(K, j, d)[r]_j}
\]
\[ \chi(K, r, d) = \frac{|F^0(K)|}{\sum_{j=\text{chr}(K,d)} S(K, j, d)[r]_j} \]
\[
\chi(K, r, d) = \sum_{j=\text{chr}(K,d)} |F^0(K)| S(K, j, d)[r]_j
\]
Colorings and equivalence relations

\[ \chi(K, r, d) = \sum_{j=\text{chr}(K, d)} |F^0(K)| S(K, j, d)[r]_j \]
The two chromatic polynomials of a 2-complex

\[ \chi(\text{MB}, r, 1) = r^5 - 10r^4 + 35r^3 - 50r^2 + 24r^1 \]
\[ \text{chr}(\text{MB}, 1) = 5 \]

\[ \chi(\text{MB}, r, 2) = r^5 - 5r^3 + 5r^2 - r^1 \]
\[ 5[r]_2 + 20[r]_3 + 10[r]_4 + [r]_5 \]
\[ \text{chr}(\text{MB}, 2) = 2 \]
Example 2.11 (Specialization to graphs)

An \((r, 1)\)-coloring of \(K\) is an \(r\)-coloring of the simple graph \(K^1\), and the 1-chromatic number of \(K\) is the graph chromatic number of \(K^1\).

<table>
<thead>
<tr>
<th>Monomial basis (MOB)</th>
<th>Falling factorial basis (FFB)</th>
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<tbody>
<tr>
<td>(</td>
<td>F^0(K)</td>
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</table>

**Properties of 1-chromatic polynomials**

- **MOB is ±** The MOB coefficients \((b_j)\) alternate in sign
- **MOB is LC** The MOB coefficients \(|b_j|\) are LC
- **No roots < 0** \(\chi(K, r, 1)\) has no roots < 0
- **\(m, m - 1\) vals** \(\chi(K, m, 1) > e\chi(K, m - 1, 1)\), \(m = |F^0(K)|\)
- **FFB is LC** The FFB coefficients \((a_i)\) are LC **UNKNOWN**
Example of a 1-chromatic polynomial

\[ \chi(OG, r, 1) = -64r^1 + 154r^2 - 137r^3 + 58r^4 - 12r^5 + r^6 \]

1-chromatic polynomial in MOB and FFB

Properties of \( \chi(OG, r, 1) \)

- MOB is ±: Yes
- MOB is LC: Yes
- No roots < 0: Yes
- \( m, m-1 \) vals: Yes
- FFB is LC: Yes
Example of a 2-chromatic polynomial

\[ \chi(MT, r, 2) = 6r - 21r^2 + 7r^3 + 21r^4 - 14r^5 + r^7 \]

2-chromatic polynomial in MOB and FFB

Properties of \( \chi(MT, r, 2) \)

- MOB is ±: No
- MOB is LC: No
- No roots <0: No
- \( m, m-1 \) vals: No
- FFB is LC: Yes
Are the simplicial Stirling numbers LC?

Problem 2: Are the simplicial Stirling numbers

\[ j \rightarrow S(K, j, d), \quad \text{chr}(K, d) \leq j \leq |F^0(K)| \]

LC for fixed \( K \) and \( d \)? (Only property that might generalize!)

\[
\begin{align*}
\text{chr}(K, 1) &= 10 \\
\text{chr}(K, 2) &= 3 \\
\text{chr}(K, 3) &= 2
\end{align*}
\]
Theorem 2.12 (Equivalent conditions for colorability)

- $K$ admits an $(r, d)$-coloring
- There exists a lift such that

\[ \text{Davis–Januszkiewicz space} \]

\[ \text{DJ}(K) \subseteq \underbrace{BU(1) \times \cdots \times BU(1)}_{|F^0(K)|} \]

\[ \lambda_1 \times \cdots \times \lambda_1 \]

\[ \lambda_d \times \cdots \times \lambda_d \]

\[ BU(d) \times \cdots \times BU(d) \]

is homotopy commutative

- $\chi(K, r, d) > 0$
The $d$-chromatic lattice

$T \not\in L(K, 2)$ \hspace{2cm} $T \in L(K, 2), |\pi(T)| = 2$ \hspace{2cm} $T \in L(K, 2), |\pi(T)| = 1$

**Definition 3.1**

The $d$-chromatic lattice, $L(K, d)$, is the partially ordered set of monochrome subsets of $F^d(K)$ of the form

$$M^d(col) = \{ \sigma \in F^d(K) \mid |col(\sigma)| = 1 \} \subseteq F^d(K)$$

for some map $col: F^0(K) \rightarrow \{1, \ldots, |F^0(K)|\}$.

- $L(K, d)$ is a finite lattice with $\widehat{0} = \emptyset$ and $\widehat{1} = F^d(K)$
- $\mu$ is the Möbius function of $L(K, d)$
- $|\pi(T)|$ is the number of connected components of $T \in L(K, d)$
Theorem 3.2 (Relating simplicial and usual Stirling numbers)

\[
\chi(K, r, d) = \sum_{T \in L(K, d)} \mu(\hat{0}, T) r^{\pi(T)}
\]
\[
S(K, j, d) = \sum_{T \in L(K, d)} \mu(\hat{0}, T) S_2(|\pi(T)|, j)
\]

‘Dehn–Sommerville relations’ for simplicial Stirling numbers of manifold?
$L(K, d)$ is graded for $d = 1$ but not for $d > 1$.

Theorem 3.3

The reduced Euler characteristic of the open interval $(\hat{0}, \hat{1})$ in $L(K, d)$ is

\[
|F^0(K)| \sum_{j=\text{chr}(K, d)} (-1)^{j-1}(j - 1)! S(K, j, d)
\]
Integer sequences of Euler characteristics

The reduced Euler characteristics of $L(D[m], d)(\hat{0}, \hat{1})$ for $m - d = 2, 3, 4, \ldots$ are

$$
d = 1 : \quad 2, -6, 24, -120, 720, -5040, 40320, -362880, \ldots
$$

$$
d = 2 : \quad 3, -6, 0, 90, -630, 2520, 0, -113400, 1247400, \ldots
$$

$$
d = 3 : \quad 4, -10, 20, -70, 560, -4200, 25200, -138600, \ldots
$$

$$
d = 4 : \quad 5, -15, 35, -70, 0, 2100, -23100, 173250, -1051050, \ldots
$$

$$
d = 5 : \quad 6, -21, 56, -126, 252, -924, 11088, -126126, \ldots
$$

$$
d = 6 : \quad 7, -28, 84, -210, 462, -924, 0, 42042, -630630, \ldots
$$

$$
d = 7 : \quad 8, -36, 120, -330, 792, -1716, 3432, -12870, \ldots
$$

$$
d = 8 : \quad 9, -45, 165, -495, 1287, -3003, 6435, -12870, 0, \ldots
$$

The first sequence is the sequence $(-1)^{m-1}(m - 1)!$. The second sequence is A009014 from The On-Line Encyclopedia of Integer Sequences (OES). The remaining 6 sequences don’t match any sequences of the OES.
Weighted colorings

Let \( w : F^0(K) \to \mathbb{N} \) be a weight function on the vertices. The weight of a simplex \( \sigma \in K \) is the sum

\[
    w(\sigma) = \sum_{v \in \sigma} w(v)
\]

of the weights of its vertices. (Special case: \( w = 1 \).)

**Definition 4.1 (Weighted \((r, d)\)-coloring of \( K \))**

A \((r, w \leq d)\)-coloring of \( K \) is a function

\[
    \text{col} : F^0(K) \to \{1, 2, \ldots, r\}
\]

such that \(|\text{col}(\sigma)| = 1 \implies w(\sigma) \leq d\) for all simplices \( \sigma \in K \).

**Definition 4.2 (Weighted \(s\)-chromatic number of \( K \))**

The weighted \(d\)-chromatic number of \( K \), \( \text{chr}(K, w \leq d) \), is the minimal \( r \) so that \( K \) admits an \((r, w \leq d)\)-coloring.
**Weighted chromatic polynomials**

**Definition 4.3 (Weighted simplicial Stirling numbers)**

$S(K, j, w \leq d)$ is the number of partitions of $F^0(K)$ with $j$ classes containing only simplices $\sigma \in K$ of weight $w(\sigma) \leq d$.

**Theorem 4.4 (Weighted $d$-chromatic polynomial)**

The number of weighted $(r, d)$-colorings of $K$ is

$$\chi(K, r, w \leq d) = \left| F^0(K) \right| \sum_{j=\text{chr}(K, w \leq d)} S(K, j, w \leq d)[r]_j$$

**Problem 2:** Are the weighted simplicial Stirling numbers $j \rightarrow S(K, j, w \leq d)$, $\text{chr}(K, w \leq d) \leq j \leq |F^0(K)|$ LC for fixed $K$, $w$, and $d$?
Definition 4.5

An \((r, d)\)-coloring of \(K\) is a simplicial map

\[ \text{col}: K \rightarrow D[r] \]

such that \(\dim\{\sigma \in K \mid \text{col}(\sigma) = j\} < d\) for \(1 \leq j \leq r\)

Definition 4.6

An \((L, d)\)-coloring of \(K\) is a simplicial map

\[ \text{col}: K \rightarrow L \]

such that \(\dim \text{col}^{-1}(v) < d\) for all vertices \(v\) in \(L\).