Quillen complexes G finite grp p prime

$\mathcal{S}$ poset of subgrps $H \leq G$

$G/\mathcal{S}$ —"— cosets G/H for $H \in \mathcal{S}$

$\chi(\mathcal{S}) = \#\{\text{totally ordered odd subsets of } \mathcal{S}\} - \#\{\text{totally ordered even subsets of } \mathcal{S}\}$

$\boxed{\text{TOM}(G) \text{ determines Euler char.}}$

Statements about Eulers are statements about TOM

$\chi(G/\mathcal{S})$

~~How can we compute~~ The Euler chars $\chi(\mathcal{S})$ and $\chi(G/\mathcal{S})$ are determined by

$$\text{TOM}(G) = \left(|{}^K(G/H)| \right)_{H,K \leq G} = \mathcal{S}(\mathcal{O}_G) \qquad \text{TOM}_{[\mathcal{S}]}(G)$$

Prop $\chi(\mathcal{S}) = (1 \cdots 1) \cdot \text{TOM}_{[\mathcal{S}]}(G)^{-1} \cdot \begin{pmatrix} \vdots \\ |G:H| \\ \vdots \end{pmatrix}$

$\chi(G/\mathcal{S}) = \qquad \cdot \begin{pmatrix} \vdots \\ |G:H|^2 \\ \vdots \end{pmatrix}$

GAP
C = ConjugacyClassesSubgroups(G)
reps := List(C, Representative)
List(reps, order)

Exmp $\text{TOM}_{[1 \leq (2) < A_5]}(A_5) = \begin{pmatrix} 60 & 0 & 0 \\ 30 & 2 & 0 \\ 15 & 3 & 3 \end{pmatrix}$

2-subgps: 1, C_2 and $C_2 \times C_2$ = Sylow

$\chi(1 < (2) < A_5) = (1\ 1) \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 30 \\ 15 \end{pmatrix} = 5 \neq \cancel{0}$ nontrivial 2-subgrps

$\chi\left(A_5 / 1 \leq (2) < A_5\right) =$

$\chi\left(A_5 / 1 \leq () < A_5\right) = \cdots = 1561 \neq 1$ proper cosets in A_5

Why should we be interested in χ? <u>4 conjectures and 1 theorem</u>

<u>The Quillen conjecture</u> $\mathcal{S} = 1 < (p) < G$ $\quad O_pG = \bigcap S$

$\mathcal{S}$ is contractible $\quad \chi(\mathcal{S}) = 1$

$\mathcal{S}$ is equivariantly contractible ~~[illegible]~~$(\mathcal{S}) = 1$

~~[illegible]~~ $\chi(\mathcal{S}) = 1$

~~[illegible]~~ $\chi_r(\mathcal{S}, G) = 1$

$O_p(G) \neq 1$

This → Which matrices can be TOM(G)?

$\boxed{\text{Indeed true for } A_5}$

Remains to show: $\chi(O_pG < (p) < G) \neq 1$ for all groups G.

<u>The Weichner-Webb theorem</u> WW-thm $\quad \mathcal{S} = \{H \lneq G \mid |G:H| = p^i\}$

$O^p(G)$ = smallest normal of p-power index

$\mathcal{S}$ is contractible

$\mathcal{S}$ is G contractible

$\chi(\mathcal{S}) = 1$

$O^p(G) \neq G$

<u>Homotopy type Conjecture</u>

Need models for these homotopy types

<u>The ~~coset poset~~ non-contractibility conjecture</u> $\boxed{\text{True for } A_5}$

This is a statement about which groups can be TOM

$\chi\left(G/1 \leq (\ ) < G\right) \neq 1$ for all groups G

This will prove that the coset poset is never contractible (this is true)

It is known that $\chi\left(G/1 \leq (p) < G\right) \neq 1$ always.

Examples (very few known)

$1 < (2) < GL_3(\mathbb{F}_2) = \bigvee^{8} S^1$

OBS $\boxed{\text{Maybe we'll get better feeling for these conjectures!}}$

Homotopy type ~~Other~~ conjectures $\quad 1 < (p) < G, \; 1 < (\ ) < G, \; G/1 \leq (p) < G,$

$1 < (\ ) < GL_3(\mathbb{F}_2)$

Subgroup posets and coset posets are ~~homotopy equiv. to wedges of spheres~~? ~~What~~ $\boxed{\text{MODELS FOR the order complexes}}$? $\simeq \bigvee^{48}(S^1 \vee S^2)$

RECURSION VIEW on conjectures

Recursions

Subgroup poset

~~Let~~ $E(G) = -\tilde{\chi}(1 < (p) < G)$. Then $E(1) = 1$ and

$$E(G) + \sum_{H \in 1 < (p) < G} E(H) = 1$$

$$E(G) = \sum_{1 \leq (p^n) < G} (-1)^n p^{\binom{n}{2}}$$

is nonzero exactly when $O_p G \neq 1$

for $|G| > 1$.

Quillen conjecture: $E(G) = 0 \Leftrightarrow O_p G > 1$

Coset posets

~~Let~~ $E(G) = -\tilde{\chi}(G/1 \leq () < G)$ Euler char of coset poset

Then $E(1) = 1$ and

$$E(G) + \sum_{H \in 1 \leq () < G} E(H) = 1$$

reduces to this

$\boxed{\text{GEN LIDT}}$

Coset conjecture: $E(G) \neq 0$ for all G

$\mathbb{G}$ the set of all finite gps partially ordered set

Define ~~Euler~~: $\boxed{\varphi, \chi : \mathbb{G} \to \mathbb{Z}}$ by

What is $\varphi(\mathbb{G})$, $\chi(\mathbb{G})$?

$$\varphi(1) = 1 = \chi(1) \qquad \varphi(G) + \sum_{H \in (1 < (p) < G)} \varphi(H) = 1$$

Then ~~$\varphi(G)$~~

$\{G \mid \varphi(G) = 0\} = \{G \mid O_p G > 1\}$

$$\chi(G) + \sum_{H \in (1 \leq () < G)} \chi(H) |G : H| = 1$$

$0 \notin \chi(\mathbb{G})$

The nongenerating complex

$\sigma \subseteq G \quad \langle \sigma \rangle$

$NGSC(G) = \{ \sigma \subseteq G \mid \langle \sigma \rangle$ is a proper subgroup $\}$

$NGSC_p(G) = \{ \quad \mid$ —"— p-subgroup $\}$

Prop 1 The NGSCs are G-collapsible

2 The ~~quotient~~ orbit Δ-sets are contractible

3 $$f_d(NGSC_p(G)) = (1 \cdots 1)\, TOM_{1 \leq (\,) < G}(G)^{-1} \cdot \begin{pmatrix} \vdots \\ |G:H|\binom{|H|}{d} \\ \vdots \end{pmatrix}$$

number of nongenerating d-sets in G

Conj 4 These f-vectors are ~~to~~ always unimodal, even (conj.) log-concave except in some special cases

$\cancel{G = [G,G] \rtimes G/}$

$$1 \to \underbrace{[G,G]}_{\substack{\text{extension of} \\ \text{el. ab } p\text{-grps}}} \to G \to \underbrace{G_{ab}}_{\substack{\text{cyclic of} \\ \text{prime power order } q \\ p \nmid q}} \to 1$$

The Quillen simplex complex

$$QSC(G) = \{\sigma \in N \mid \sigma \not\ni e\}$$

$$QSC_p(G) = \{\sigma \in N_p \mid \sigma \not\ni e\}$$

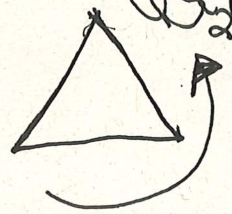
Prop ~~$\varphi_p G = 1$~~ (1) $QSC_p(G)$ is simple htpy equiv to the order complex of $1 < \langle p \rangle < G$.

(2) $QSC_p(G)/G$ is a contractible Δ-set

(3) $f_d + f_{d-1} = f_d(NGSC_p(G))$.

(4) $f_d(QSC_p(G))$ is unimodal even log-concave except ...

A_4 :



$A_4 \to C_3$

The simplicial version

~~$Q_p(G)$~~ $= QSC_p(G)$ is collapsible

$sd(QSC_p(G))$ G-collapsible

subdivision!! $\chi(\quad) = 1$

$\varphi_p G \neq 1$.

~~Conj~~ HTC

$QSC_p(G)$ is homotopy equivalent to a wedge of spheres.

Coset complexes

$[\sigma] =$

$COSC_p(G) = \{\sigma \subset G \mid [\sigma] \text{ is a proper } p\text{-}\cancel{\text{sub}}\text{coset in } G\}$

$COSC(G)$

Prp $COSC(G)$ is simple homotopy equiv. to the order complex of $G/$ the coset poset $G/1 \leq (\ ) < G$.

Conjecture $\chi(COSC(G)) \neq 1$ for all G. What can it be

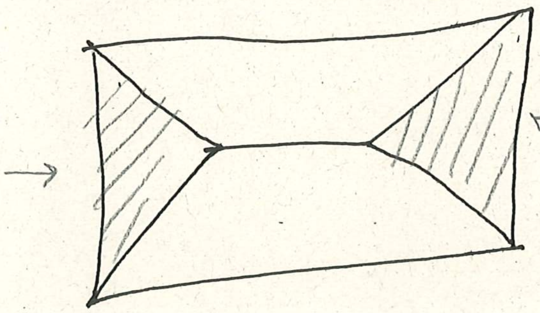
$COSC_p(G)$ is never contractible. f-vector $fd(CSSC_p(G)) = (1 \dots 1)\, TOM(G)^{-1} \begin{pmatrix} \vdots \\ |G:H| \binom{|H|}{d}^2 \end{pmatrix}$

HTC Htpy type Conject $COSC(G)$ is homotopy equiv. to a wedge of spheres

The non-contractibility conj

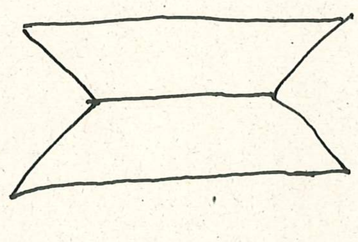
$CSS(G)$ is never collapsible.

Example $COSC(C_6)$



G-orbit of free faces

$\simeq$



$COSC(C_6) \simeq S^1 \vee S^1$

$C_6 \to GL_2(\mathbb{Z})$ what rep is it?