Non-equivariant case

(1)

P poset $\Delta_i(P)$ = i-simplices in P

$\chi(P) = \sum (-1)^i |\Delta_i(P)|$ $\quad \tilde{\chi}(P) = \chi(P) - 1$ $\quad$ same as $\tilde{\chi}(BP)$

Often we meet P poset with $\hat{0}$ and $\hat{1}$ $\quad \tilde{\chi}(P) = 0$!

$P^* = (\hat{0}, \hat{1})$ $\quad \tilde{\chi}(P^*) = \mu^P(\hat{0}, \hat{1})$

Examples

$B(n)$ subsets of an n-set $\quad \tilde{\chi}(B(n)) = (-1)^n$ (3.8.3)

$P(n)$ partitions of an n-set $\quad \tilde{\chi}(P(n)) = (-1)^{n-1}(n-1)!$

$L_n(q)$ subgroups of $\mathbb{F}_q^n$, q = prime power $\quad \tilde{\chi}(L_n(q)) = (-1)^n q^{\binom{n}{2}}$ (building)

$\mathcal{S}_G^p$ p-subgroups of group G, unknown in general

though $\tilde{\chi}\left(\mathcal{S}^{p+*}_{\Sigma(q)}\right) = (-1)^{r(\Sigma)} q^{|\Sigma^+|}$

Equivariant case

P G-poset with $\hat{0}$ and $\hat{1}$

$P^* = (\hat{0}, \hat{1})$ For every $r \geq 1$

$$\tilde{\chi}_r(P^*, G) = \frac{1}{|G|} \sum_{\substack{(X_1, \ldots, X_r) \\ X_i \in G,\ X_i X_j = X_j X_i}} \tilde{\chi}\left((P^*)^{\langle X_1, \ldots, X_r \rangle} \right) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, G)} \tilde{\chi}\left((P^*)^{X(\mathbb{Z}^r)} \right)$$ sequence of Eulers

p prime $$\tilde{\chi}_r^p(P^*, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^{r-1}, G)} \tilde{\chi}\left((P^*)^{X(\mathbb{Z} \times \mathbb{Z}_p^{r-1})} \right)$$ commuting elements of p-power order in G and one free element

p-primary Euler

TAKE $P^* = \phi$

I do not know if $\tilde{\chi}_r(P^*, G) = \mu_r^{(P, G)}(\hat{0}, \hat{1})$

<u>Manifesto</u>: Study P and $|P|$ equivariantly

$|P| \simeq \bigvee S^{n_i}$ is not G-equivariant, as a G-space $|P|$ has more structure

$|P(n)|$: Above

$\tilde{\chi}_r(P^*, G)$ is a primitive attempt to follow manifesto.

<u>Remarks</u> $P^* = \emptyset$ $\quad \tilde{\chi}_r(\emptyset, G) = -\dfrac{|C_r(G)|}{|G|}$ $\quad C_r(G)$ = set of commuting r-tuples in G

$\tilde{\chi}_r(P, G) = 0$

$\tilde{\chi}_1(P^*, G) = \tilde{\chi}(|P^*|/G)$

$\tilde{\chi}_r^p(P^*, G)$ = Euler of $|P^*|_{hG}$ computed in rth Morava K-theory at p
p-primary (Hopkins Kuhn-Ravenel)

Will now look at the four cases

<u>Theme</u>

$r = 1$ $\quad \tilde{\pi}_1(P^*G) \neq 0$, P^*/G contractible?

$r = 2$ $\quad$ interesting

$r > 2$ $\quad$ mysterious

$$\boxed{\mathcal{S}_G^{p+*} \circlearrowleft G}$$

$\underline{r=1}$ Peter Webb $\tilde{\chi}_1(\mathcal{S}_G^{p+*}, G) = \tilde{\chi}(|\mathcal{S}_G^{p+*}|/G) = 0$ (if $p \mid |G|$)

Symond's thm $|\mathcal{S}_G^{p+*}|/G \simeq *$ (if $p \mid |G|$)

$\underline{r=2}$ $z_p(G) = \#\{\chi \in Irr(G,\mathbb{C}) \mid |G|_p \mid \chi(1)\}$

Knörr-Robinson Conjecture: $\tilde{\chi}_2(\mathcal{S}_G^{p+*}, G) + z_p(G) = 0$

(Theveraz interpretation) $\forall G: KRC(G) \Leftrightarrow \forall G: AWC(G)$

Theveraz: OK for $GL_n(\mathbb{F}_q)$, A_n?, q a power of p, $z_p(GL_n(\mathbb{F}_q)) = q-1$

Computer: Any of the five Mathieu groups, J_1, J_2, J_3

Could possibly be verified for all sporadic groups

$KRC(G)$ is true if $O_p(G) \neq 1$, $p \nmid |G|$, G has a cyclic p-subgroup

$\underline{r>2}$ $\tilde{\chi}_3(\mathcal{S}_G^{p+*}, G)$ connected to some invariant of the group ?

→ table of $\tilde{\chi}_3$ for something

$p=2$	4	5	6	7	8	9	10
$\tilde{\chi}_3(\mathcal{S}_{A_n}^{p+*}, A_n)$	0	8	24	-2	32	20	92
Σ_n	0	2	12	-2	10	11	16

$$\boxed{L_n(q) \circlearrowright GL_n(\mathbb{F}_q) =} \qquad = \varphi^{st*} \circlearrowright GL_n(\mathbb{F}_q) \quad s = \mathrm{char}(\mathbb{F}_q) \quad GL_n(\mathbb{F}_q)$$

$$\tilde{\chi}_r(n,q) = \tilde{\chi}_r(L_n^*(\mathbb{F}_q), GL_n(\mathbb{F}_q))$$

$\cancel{n=1 \quad L_1^*(\mathbb{F}_q) = \emptyset \quad \tilde{\chi}_r(1,q) = \frac{1}{(q-1)}(q-1)^r - (q-1)^{r-1}}$ not needed

$r=1$ $\quad |L_n^*(\mathbb{F}_q)| / GL_n(\mathbb{F}_q) = \Delta^{n-2} \quad \tilde{\chi}_1(n,q) = \chi(\Delta^{n-2}) = \begin{cases} -1 & \text{for } n=1 \\ 0 & \text{for } n>1 \end{cases}$

General r:

$$F_r(x)(q) = 1 + \sum_{n \geq 1} \tilde{\chi}_r(n,q) x^n = 1 + (q-1)^{r-1} x + \cdots$$

Thm $\quad F_1(x)(q) = 1 - x$. What is F_r for $r > 1$?

Thm $\quad$ For $r \geq 1$: $F_{r+1}(x) = \dfrac{F_r(qx)}{F_r(x)} \in \mathbb{Z}[q][[x]]$ (q is a variable)

$$F_2(x) = \frac{1-qx}{1-x} = \frac{1-x+x-qx}{1-x} = 1 + (1-q)\frac{x}{1-x} = 1 + (1-q)\sum_{n \geq 1} x^n$$

$$-\tilde{\chi}_2(n,q) = 1-q = Z_S(GL_n(\mathbb{F}_q))$$

$$F_3(x) = \frac{(1-q^2x)(1-x)}{(1-qx)^2} \qquad \underbrace{\tilde{\chi}_2(n,q)} \qquad \underbrace{\tilde{\chi}_3(n,q)} \text{ are polynomials in } q$$

$$F_4(x) = 1 - (q-1)^3 \left(x + (3q^2+1)x^2 + (6q^4 - q^3 + 3q^2 + 1)x^3 + \cdots \right)$$

$$\tilde{\chi}_4(3,q) = -(q-1)^3 (6q^4 - q^3 + 3q^2 + 1) \text{ is polynomial in } q$$

Observation: $\tilde{\chi}_r(n,q) \Big/ \begin{matrix} (q-1)^{r-1} & r \text{ even} \\ (q-1)^{r-1} q^{n-1} & \text{if } r \text{ odd} \end{matrix}$ is irreducible for $n > 2$

q is a number a prime power

Thm $F_r(x)(q) = 1 + \sum_{n\geq 1} \tilde{\chi}_r(n,q)x^n = \exp\left(-\sum (q^n-1)^{r-1}\frac{x^n}{n}\right) = \prod (1-x^n)^{b_r(q)(n)}$

$$b_r(q)(n) = \sum_{d\mid n} \mu(n/d)(q^d-1)^{r-1}$$

$$\tilde{\chi}_r(n,q) = \begin{cases} 1 & n=0 \\ -\frac{1}{n}\sum_{1\leq j\leq n} (q^j-1)^{r-1}\tilde{\chi}_r(n-j,q) & \end{cases}$$

$r=2$: $b_2(q)(n)$ is the number of monic irreducible degree n polynomials in $\mathbb{F}_q[t]$

Thm

$F_r^p(x)(q) = 1 + \sum \tilde{\chi}_r^p(n,q)x^n = \exp\left(-\sum (q^n-1)_p^{r-1}\frac{x^n}{n}\right) = \prod (1-x^n)^{b_r^p(q)(n)}$

$r=2$ $b_2^p(q)(n)$ = number of monic irreducible degree n p-power order polynomials in $\mathbb{F}_q[t]$

Here is an explicit formula for $b_2^p(q)(n)$ $\boxed{F_r^p(x)(q) \text{ depends only on } \overline{\langle q\rangle} \leq \mathbb{Z}_p^\times}$

$r>2$?

Magic Lemma $A \leq GL_n(\mathbb{F}_q)$ abelian, A contains an element of order $s = \mathrm{char}(\mathbb{F}_q)$

$L_n^*(\mathbb{F}_q)^A \simeq *$ and $\tilde{\chi}(\ \ ) = 0$

Thus we only need to consider $(X_1, \dots, X_r)$ of order prime to q. (semi-simple elements)

+ Tits buildings?

$\boxed{B(n) \circlearrowleft \Sigma_n}$ Boolean case

Since $\tilde{\chi}(B^*(n)) = (-1)^n$ we get $(B^*(n))^A = B^*(n/A)$ so that

$$\tilde{\chi}(B^*(n)^A) = (-1)^{n/A}$$

and

$$\tilde{\chi}_r(n) = \tilde{\chi}_r(B^*(n), \Sigma_n) = \frac{1}{n!}\sum_{X \in Hom(\mathbb{Z}^r, \Sigma_n)} (-1)^{n/X(\mathbb{Z}^r)}$$

$\underline{r=1}$

$$\tilde{\chi}_1(B^*(n)\Sigma_n) = \frac{1}{n!}\sum_{g \in \Sigma_n} (-1)^{n/g} = \begin{cases} -1 & \text{for } n=1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

$$|B^*(n)|/\Sigma_n = \Delta^{n-2}$$

$\lambda_r(n)$ = index n subgroups of $\mathbb{Z}^r$

NOTE RELATION HERE — CFR $L_n^*(q)$?

Generating function $F_r(x) = 1 + \sum_{n \geq 1} \tilde{\chi}_r(n) x^n$, $F_1(x) = 1 - x$

$\underline{\text{Thm}}$ (arxive) $F_r(x) = \prod (1-x^n)^{\lambda_{r-1}(n)}$ $\quad F_r^p(x) = \prod_n (1-x^{p^n})^{\lambda_{r-1}(p^n)}$

Tamanoi in generalized version

$\lambda_r(n) = \#\{\text{index } n \text{ subgroups of } \mathbb{Z}^r\}$

$\underline{r=2}$ $\quad F_2(x)^{-1} = \prod_{n \geq 1} (1-x^n)^{-1} = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)$

$$= \sum_{n=1}^{\infty} p(n) x^n \text{ so } \tilde{\chi}_2(B^*(n), \Sigma_n) = \#\text{ partitions of } n$$

$r > 2$ The Euler transform of $\tilde{\chi}_r(n)$ is $\lambda_{r-1}(n)$

Consequence: $\displaystyle\sum_{i=0}^{n} \binom{n}{i} |Hom(\mathbb{Z}^r, \Sigma_{n-i})| \sum_{X \in Hom(\mathbb{Z}^r, \Sigma_i)} (-1)^{i/X(\mathbb{Z}^r)} = 0$ for $n > 0$