Equivariant Euler characteristic of G-poset

Conference in Louvain-la-Neuve

$\Pi \circlearrowleft G$ $\qquad C_r(G) = \{(g_1,\dots,g_r)\in G^r \mid g_i g_j = g_j g_i\}$

something with [illegible] on which G acts

$X \in C_r(G)$, $r \geq 1$ $\qquad = \mathrm{Hom}(\mathbb{Z}^r, G)$ $= \mathrm{GRP}(\mathbb{Z}^r, G)$

Stay about groups to this conference. Need a lot of questions from the audience

What is the character of $X \to \alpha_r($ $)$

$\Pi^X = X\text{-fixed poset} = \bigcap_{g\in X} \Pi^g = \Pi^{\langle X\rangle}$

$\tilde\chi(\Pi^X) \in \mathbb{Z}$

$$\tilde\chi_r(\Pi, G) = \frac{1}{|G|}\sum_{X\in C_r(G)} \tilde\chi(\Pi^X) \in \mathbb{Z}$$

Atiyah and Segal

$\#\{X \in C_r(A) \mid \langle X\rangle = A\}$

$$= \frac{1}{|G|}\sum_{\substack{A\leq G\\ A \text{ abelian}}} \tilde\chi(\Pi^A)\,\cancel{\varphi_r(A)}$$

~~$\varphi_r(A) = \{X \in C_r(A) \mid \langle X\rangle = A\}$~~

only have to worry about fixed points for abelian subgroups of G

$\tilde\chi_r(\Pi, G)$ is an integer sequence – what does it say about the G-poset Π?

Examples

0) Π has a $\hat{0}$ or a $\hat{1}$ $\tilde{\chi}(\Pi^{\times}) = 0$ and $\tilde{\chi}_r(\Pi, G) = 0$

1) $\Pi = \emptyset$ $\tilde{\chi}(\Pi^{\times}) = \tilde{\chi}(\emptyset) = -1$

$$\tilde{\chi}_r(\emptyset, G) = -\frac{1}{|G|}|C_r(G)|$$

What is Dirichlet series $\displaystyle\sum_{n=1}^{\infty} \frac{C_r(\Sigma_n)}{n^s} = \lambda_{r-1}$?

For symmetric group

exponential gen. function $\displaystyle\sum |C_r(\Sigma_n)| \frac{u^n}{n!} = \prod_{j=1}^{\infty} (1-u^j)^{-\lambda_{r-1}(j)}$

Example $\tilde{\chi}_2(\emptyset, \Sigma_n) = -\dfrac{|C_2(\Sigma_n)|}{n!} = -p(n)$ (number of partitions of n)

2) $G = 1$ (non-equivariant case) $\tilde{\chi}_r(\Pi, G) = \dfrac{1}{|G|}\tilde{\chi}(\Pi)$ constant sequence

$C_r(G) = 1$

In general, χ_r is a mix of contributions from Π and from G

General facts

$$\tilde{\chi}_1(\Pi, G) = \tilde{\chi}_1(B\Pi/G)$$

$$\tilde{\chi}_2(\Pi, G) = \tilde{K}_G\text{-Euler of } B\Pi \quad \text{(Atiyah–Segal)}$$

$$= \dim \tilde{K}_G^{\text{even}}(B\Pi) \otimes \mathbb{Q} - \dim \tilde{K}_G^{\text{odd}}(B\Pi) \otimes \mathbb{Q}$$

$\tilde{\chi}_3(\Pi, G) \overset{?}{=} \tilde{E}$-Euler char of $B\Pi$ for some exotic homology E?

Morava K-theory

Hopkins-Kuhn-Ravenel

Example: G finite group, p prime

$\mathcal{S}_G^{p+*}$ = non-trivial p-subgroups of G $\quad H \leq G,\ |H| = p^i,\ H \neq 1$

Quillen: $\mathcal{S}_G^{p+*}$ is not contractible $\Rightarrow O_p G = 1$

$$\tilde{\chi}_1(\mathcal{S}_G^{p+*}, G) = \tilde{\chi}_1\left(B\mathcal{S}_G^{p+*}/G\right) = 0 \quad \text{(Symonds, Thévenaz)}$$

$$\tilde{\chi}_2(\mathcal{S}_G^{p+*}, G) = -z_p(G) \quad \left(z_p(G) = \#\text{ of irreducible } \mathbb{C}\text{-reps of } p\text{-defect } 0\right) \quad \text{Alperin Weight Conjecture}$$

$$\tilde{\chi}_3(\mathcal{S}_G^{p+*}, G) = \ ?$$

The total sequence $\tilde{\chi}_r(\mathcal{S}_G^{p+*}, G)$ has not been $\boxed{\text{investigated}}$ project?

Depending on time

Digression

Dirichlet convolution ~~and~~ Dirichlet inverse, Dirichlet series

Estimate ~~time remain~~ remaining time

$f, g : \mathbb{N} \to \mathbb{C}$ $\quad f(1) = 1 = g(1)$

f is multiplicative if $GCD(n_1, n_2) = 1 \Rightarrow f(n_1 n_2) = f(n_1) f(n_2)$

Convolution: $$(f * g)(n) = \sum_{d_1 d_2 = n} f(d_1) g(d_2)$$

Mb inversion

$$\boxed{\mu * (1 * f) = f}$$

$\cancel{1*f}$

$f * g = g * f$

$f * \varepsilon = f$ $\quad \varepsilon = 1, 0, 0, \dots$ unit for

$$\boxed{1 * \mu = \varepsilon}$$

Dirichlet inverse: $f * f^{-1} = \varepsilon$

The most famous inverse: $1 = 1, 1, \dots$ $\quad 1^{-1} = \mu$ Möbius function

$$\sum_{d \mid n} \mu(d) = \sum_{d_1 d_2 = n} \mu(d_1) 1(d_2) = (\mu * 1)(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

f, g multiplicative $\Rightarrow f * g$ multiplicative

Dirichlet series: $$D(f)(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad D(\varepsilon)(s) = 1$$

$$\sum \frac{1}{n^s} = D(1)(s) = \zeta(s) \qquad D(\mu)(s) = \frac{1}{\zeta(s)}$$

$$D(f * g)(s) = D(f)(s) D(g)(s)$$

Partitions of a set (not of a number)

$\Pi(\{1,\dots,n\})$ = all partitions of $\{1,\dots,n\}$

= all equiv. relations on $\{1,\dots,n\}$ → rep. of $\{1,\dots,n\}$ with a finite set

$B(n)$ = number of partitions

~~Example~~ n=2

n=3

$$\begin{tikzcd}[column sep=small] & 12\text{-}3 \arrow[dr, dash] & \\ 1\text{-}2\text{-}3 \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & 13\text{-}2 \arrow[r, dash] & 123 \\ & 23\text{-}1 \arrow[ur, dash] & \end{tikzcd}$$

blocks

$\{1,2,3\}$

$\{\{1,2\},\{3\}\} = \pi \in \Pi(\{1,2,3\})$

$\pi_1 \leq \pi_2 \iff$ any π_1-block of π_1 ~~is contained~~ is a subset of a π_2-block

$\iff$ π_2-blocks are unions of π_1-blocks

$\hat{0} = 1\text{-}2\text{-}\cdots\text{-}n \qquad \hat{1} = 12\cdots n$

Usual Euler char

$$\boxed{\tilde{\chi}(\Pi_n) = (-1)^{n-1}(n-1)!}$$

$$\Pi(\{1,\dots,n\}) \circlearrowleft \Sigma_n$$

$$\Pi^*(\{1,\dots,n\}) = \Pi(\{1,\dots,n\}) - \{\hat{0},\hat{1}\}$$

$$\tilde{\chi}_r(\Pi^*_n, \Sigma_n) = ?$$

Computer:

$\tilde{\chi}_1(\Pi^*_n, \Sigma_n) = \tilde{\chi}(B\Pi^*_n/\Sigma_n) = 0$ for $n > 2$. (Kozlov)

r is fixed and n vary

$= \mu(n) - \mu(n/2)$ OEIS

$\tilde{\chi}_2(\Pi^*_n, \Sigma_n) = 1, -2, -1, 1, -1, 2, \dots$

not in OEIS !

$\tilde{\chi}_3(\Pi^*_n, \Sigma_n) = 1, -4, -4, 5, -6, 16$

<u>Thm</u> $\tilde{\lambda}_r(\Pi_n^*, \Sigma_n) = \frac{1}{n} c_r(n)$

What is c_r?

~~z = indic~~ $a(n) = (-1)^{n+1}$

$\lambda_r(n) = \#$ of index n subgroups of $\mathbb{Z}^r$

$\zeta_j(n) = n^j \quad j = 0, 1, \ldots$

z_2 = indicator function for powers of 2

$c_r^{-1} = z_2 * \zeta_1 * \cdots * \zeta_{r-1}$

$c_r = a * \lambda_r^{-1}$

$\lambda_r^{-1}(p^d) = (-1)^d p^{\binom{d}{2}} \binom{r}{d}_p$

Dirichlet series

$$\sum \frac{c_r(n)}{n^s} = \frac{2^s - 2}{2^s} \frac{1}{\underbrace{\zeta(s-1) \cdots \zeta(s-r+1)}_{r-1 \text{ factors}}}$$

~~Words about the~~ Words about the proof

$$\tilde{\chi}(\Pi^*(\{1,\dots,n\}),\Sigma_n) = \frac{1}{n!}\sum_{\substack{A \leq \Sigma_n \\ A \text{ abelian}}} \tilde{\chi}(\Pi^*(\{1,\dots,n\})^A)\,\varphi(A)$$

Need to find $\tilde{\chi}(\Pi^*(S)^G)$ for S a finite G-set and G an (abelian) group.

Step 1 S is a G-set, π a partition of S, $S \to \pi\backslash S$

π is a G-partition iff $S \to \pi\backslash S$ is a G-map, (the G-action on the set of blocks) so G-partitions on S descends to a G-action on the set of blocks $T \rightsquigarrow \mathcal{O}_G$ orbit category

$=$ G-maps from G-set S onto G-set T

$\rightsquigarrow \bar{\mathcal{O}}_G$ the additive extension of $\mathcal{O}_G$

Def S is an isotypical G-set if all G-orbits are isomorphic: $S = n\, H\backslash G$

π is an isotypical G-partition of S if the set $\pi\backslash S$ of blocks is ~~a~~ an isotypical G-set.

Arone's Lemma (cfr Quillen) $\Pi^*(S)^G$ is not contractible $\Rightarrow S$ is isotypical

Bouc's Thm (Quillen's Thm A) $\Pi^{*+\mathrm{iso}}(S)^G \hookrightarrow \Pi^*(S)^G$ is a homotopy equivalence.

Focus on $\Pi^{*+\mathrm{iso}}(S)^G$ i.e. G-maps $n\,H\backslash G \twoheadrightarrow k\,K\backslash G$

isotypical G-partitions of an isotypical G-set S

$\tilde{\chi}(\Pi^{*+\mathrm{iso}}(n\,H\backslash G)) = n H\backslash G \twoheadrightarrow kK\backslash G$

Step 2 $\quad \tilde{\chi}\left(\Pi^{*+\text{iso}}(nHG)^{S}\right) \rightsquigarrow nHG \twoheadrightarrow kKG$

Non equivariant case $\quad \tilde{\chi}\left(\Pi^{*}(\{1,\dots,n\})\right) = (-1)^{n-1}(n-1)!$

Stirling numbers of the 2nd kind

$$S(n,k) = \#\text{ partitions of } \{1,\dots,n\} \text{ into } k \text{ blocks}$$

$$B(n) = S(n,1) + S(n,2) + \dots + S(n,n)$$

Stirling matrix of 2nd kind

$$S = \big(S(n,k)\big) = \begin{pmatrix} S(1,1) & & \\ S(2,1) & S(2,2) & \\ S(3,1) & S(3,2) & S(3,3) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 3 & 1 & \\ 1 & 7 & 6 & 1 \end{pmatrix}$$

Stirling matrix of the first kind

$$s = S^{-1} = \begin{pmatrix} s(1,1) & & \\ s(2,1) & s(2,2) & \\ s(3,1) & s(3,2) & s(3,3) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1 & -1 & & \\ 2 & -3 & 1 & \\ -6 & 11 & -6 & 1 \end{pmatrix}$$

By accident

$$\tilde{\chi}\left(\Pi^{*}(\{1,\dots,n\})\right) = s(n,1)$$

Equivariant case

S, T G-orbit (transitive G-set)

$S_G(iS, jT)$ = # G-partitions of iS with block set isomorphic to jT

NB: $S_G(iS, jT)$ can be computed from $S(n,k)$ and O_G (table of marks)

$$S_G = \left(\left(S_G(iS, jT)\right)_{1 \le i, j \le n}\right)_{S, T \in [O_G]}$$

$$s_G = S_G^{-1}$$

Thm $\tilde{\chi}\left(\Pi^*(nS)^G\right) = s_G[nS, 1]$

Probably: $s_G[nS, kT] = \sum_{\pi \in S_G(nS,kT)} \tilde{\chi}\left(\hat{0}/\pi \, \Pi^*(nS)^G/\pi\right)$

Corollary G abelian $H \le G$

$$\tilde{\chi}\left(\Pi^*(nH\backslash G)^G\right) = \mu(H,G)\,|G:H|^{n-1} s(n,1)$$

$$\tilde{\chi}\left(\Pi^*(nH\backslash G)^{H\backslash G}\right) \qquad \mu(1, G/H)$$

G/H must be elementary abelian ... otherwise 0

Gen Thm $\tilde{\chi}\left(\Pi^*(nA)^A\right) = \mu(1,A)\,|A|^{n-1} s(n,1)$