Master Thesis December 6, 2001

Vector Fields on the Sphere.

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Thesis for the master degree in mathematics,
University of Copenhagen, Denmark.

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Typesetting: $\LaTeX\,2_\varepsilon$

Preface

This thesis represents my final work for the master degree in mathematics at Department of Mathematics, University of Copenhagen. It was written in the period from January 22nd to December 6th, 2001, under supervision of Jesper Michael Møller. The thesis is mostly self contained, and it should be possible to understand for anyone with only limited preliminary knowledge of homology and topology, assuming of course extensive knowledge of mathematics.

I am grateful for the all help and support I have received during the work on this thesis. In particular I would like to thank Jesper Michael Møller for his tireless efforts to make me understand, Ana Babus for her caring and keeping up my spirit, Mads Keinicke, Rasmus Ejlers Møgelberg and Morten Misfeldt for the mathematical discussions and suggestions, Rebecca Svane for her linguistic assistance and the people of S15b for the mathematical (and non-mathematical) discussions along the way. Finally I wish to thank the Troglodytes at Studentergaarden for their comfort and tolerance.

Erik Nørgaard
Copenhagen, 2001.

Contents

Introduction

The existence and non-existence of vector fields on the sphere is a classical problem arising in the theory of fibre bundles and homotopy theory with implications in many areas of mathematics.

A tangent vector field on S^{n-1} is a continuous function which assigns a tangent vector to S^{n-1} to each point $x \in S^{n-1}$. Given k such vector fields we say that they are linearly independent, if the k vectors are linearly independent for all points $x \in S^{n-1}$.

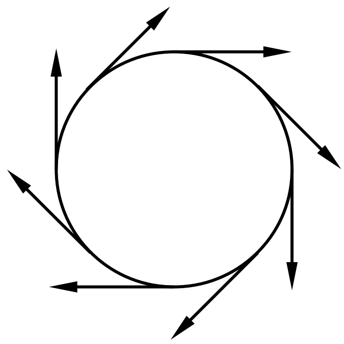


Figure 1: Vector fields on sphere of dimension 1

To illustrate the problem, consider first the one dimensional sphere, the circle. As illustrated in figure 1 there is one obvious tangent vector field everywhere non-zero, and clearly, any other tangent vector field is linearly dependent on this.

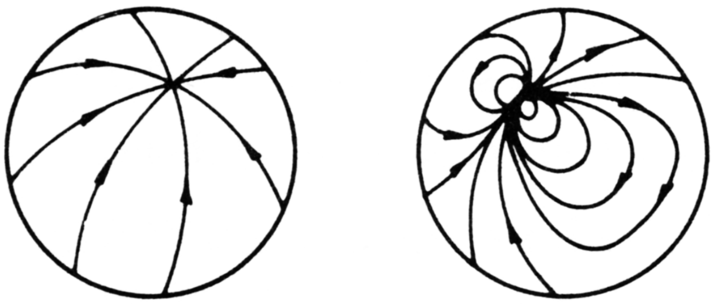


Figure 2: Attempts of constructions of vector fields on the 2-sphere

For the two dimensional sphere, there are no everywhere non-trivial vector fields at all, as illustrated in the figure 2. It is a classical result of algebraic topology (the hairy ball theorem) that for the 2-sphere, any vector field has a singularity in at least two points. In fact, for any even dimensional sphere (i.e. n odd) there exist no everywhere non-trivial vector field.

The problem of vector fields on the sphere is to determine the maximum number of such linearly independent vector fields on S^{n-1}. Since Gram-Schmidt orthonormalisation is a continuous operation, we may restrict our studies to orthonormal vector fields. Then the existence of vector fields on the sphere S^{n-1} is equivalent to the existence of a cross section of the appropriate Stiefel fibrering.

It is known from a result by Hurwitz, Radon and Eckmann from 1942, that for $n = (2a+1)2^{c+4d}$, with a, c, d integers and $0 \leq c \leq 3$ the sphere S^{n-1} admits $\rho(n)-1$ independent vector fields, where $\rho(n) = 2^c + 8d$.

In the following we state and prove the converse: The sphere S^{n-1} does not admit $\rho(n)$ linearly independent vector fields. This was first proved in 1962 by use of K-theory[2]. This is the approach we will follow.

The line of proof may seem indirect, it is in fact a pincer movement: Using the theory of Thom spaces we reduce the problem to a problem of coreducibility of certain stunted projective spaces. Subsequently with the aid of K-theory, we prove that such a coreduction cannot exist.

Chapters 1 and 2 serve to provide an introduction to vector bundles and determine the homotopy properties of vector bundles. These chapters provide the natural framework of our further studies.

In chapter 3, we construct the algebraic and cohomological structures making up K-theory, using the structures of vector bundles introduced in the first chapters. Chapter 4 is a fairly standard account of the Bott periodicity theorem, and in chapter 5 we introduce additional algebraic structures of K-theory, first of all the Adams operations, and state some results connecting K-theory with ordinary cohomology. We conclude the chapter with some results on spectral sequences due to Atiyah and Hirzebruch which are an important tool for determining the K-rings in chapter 7.

The remaining two chapters provide the actual solution of the non-existence problem. Chapter 6 gives an account of Thom spaces and reducibility. In this chapter, we provide the proof of a well established reduction of the problem of non-existence of vector fields on the sphere to a problem of coreducibility of certain stunted projective spaces. The results of chapter 6 motivate further studies on stunted projective spaces in chapter 7. In this chapter, we apply K-theory to arrive at our main result.

Throughout the thesis we keep notation simple and consistent, and to the extent possible in accordance with standard notation. However to ease notation we make use of "dynamic" notation, the meaning of the notation is context dependent. Thus, if we denote a vector bundle by ξ, we may also refer to the isomorphism class determined by ξ as ξ in the context of vector bundle isomorphism classes, or to the element of $K(X)$ or $\tilde{K}(X)$ determined by ξ by that same symbol. This abuse of notation should not lead to any confusion and will prove convenient.

1

Vector bundles

Vector bundles arise naturally in differential geometry, but may also be viewed as general bundles with more structure. In this chapter we introduce the basic concepts and results from the theory of vector bundles, but only to the extend needed for our further studies. A more thorough exposition of the subject can be found in [15, chp. 2–8].

In section 1, we define vector bundles and give some simple yet important examples of vector bundles. In section 2, we define morphisms of vector bundles and form categories of vector bundles. Section 3 is devoted to the study of pullback of vector bundles, which is important in order to establish the cohomology structure of K-theory. We end the chapter studying constructions of vector bundles from which we will define the algebraic structures of K-theory which will be introduced in the following chapters.

1.1 Definitions and examples

1.1.1 Definition. An $\mathbb{F}$-*vector bundle* is a triple (E, p, X) where E and X are spaces and $p : E \to X$ is a map such that the following condition of local triviality is satisfied: There is an open covering $\{U_\alpha\}$ of X such that for each α there is an integer n and a homeomorphism $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^n$ and for each $x \in U_\alpha$ the restriction $\psi_\alpha : p^{-1}(x) \to \mathbb{F}^n$ is a vector space isomorphism.

The space E is called the *total space*, X the *base space* and p the *projection* of the bundle. The vector spaces $p^{-1}(x)$ are called the *fibres*, and the map ψ_α a *local trivialisation*. Some times the inverse ψ_α^{-1} is used, and borrowing from differential geometry we then speak of a *local chart*. For $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ we speak of real, complex or quaternionic vector bundles respectively.

It is common to denote bundles by Greek letters, $\xi, \eta, \zeta \ldots$ Also it is common notation to denote the base space of the vector bundle ξ by $B(\xi)$ and the total space by $E(\xi)$, and we will denote the projection map by p_ξ.

1.1.2 Remark. With the definition above, the dimension of the vector spaces may differ on each connected component. If the dimension of $p^{-1}(x)$ is n for all $x \in X$, e.g. if the base space is connected, we speak of an *n-dimensional* vector bundle.

1.1.3 Example. Clearly, for a space X the triple $(X \times \mathbb{F}^n, \pi, X)$ with π the projection onto the first factor is a vector bundle, the *product vector bundle*. This is also known as the

n-dimensional *trivial* vector bundle, and we will denote it ε^n, in particular, if $n = 1$ we speak of a *line bundle.* A space X may be regarded as the zero dimensional vector bundle (X, id, X) and a vector space V may be regarded as a vector bundle over a point with a constant map as projection.

1.1.4 Definition. Let $\xi = (E, p, X)$ be a vector bundle, and let $A \subset X$. The *restriction* of ξ to A is the vector bundle $\xi|_A = (E', p', A)$ where $E' = p^{-1}(A)$ and $p' = p|_{E'}$.

1.1.5 Definition. A *vector subbundle* of a vector bundle $\xi = (E, p, X)$ is a vector bundle (E', p', X) such that $E' \subset E$ and $p' = p|_{E'}$.

1.1.6 Definition. A *cross section* of a vector bundle $\xi = (E, p, X)$ is a map $s : X \to E$ such that $p \circ s = 1_X$, that is for each $x \in X$, $s(x) \in p^{-1}(x)$. In particular we define the *zero section* as the cross section mapping X into the zero vectors of the fibres of E.

1.1.7 Example. The *tangent vector bundle* $\tau(S^n) = (T, p, S^n)$ over S^n is the subbundle of the product bundle $(S^n \times \mathbb{R}^{n+1}, p, S^n)$ where the total space T is the space $T = \{(x, v) \in S^n \times \mathbb{R}^{n+1} | \langle x|v \rangle = 0\}$. The bundle $\tau(S^n)$ has a natural vector space structure on the fibres since $p^{-1}(x)$ is an n-dimensional vector subspace of $\mathbb{R}^{n+1}$ for every $x \in S^n$. For an element $(x, v) \in T$, v is called a *tangent vector* to S^n at x and a cross section is called a *tangent vector field* of S^n.

Similarly we define the *normal vector bundle* $\nu(S^n) = (N, q, S^n)$ as the subbundle of $(S^n \times \mathbb{R}^{n+1}, q, S^n)$ with total space $N = \{(x, v) \in S^n \times \mathbb{R}^{n+1} | v = sx, s \in \mathbb{R}\}$. For elements $(x, v) \in N$, v is called a *normal vector* to S^n at x, and the fibre $q^{-1}(x)$ is of dimension 1. A cross section of $\nu(S^n)$ is called a *normal vector field* of S^n.

1.1.8 Example. A bundle of k-frames $\tau_k(S^n)$ over S^n for $k \leq n$ is a subbundle of the product bundle $(S^n \times (S^n)^k, p, S^n)$ where the total space $E = E(\tau_k(S^n))$ is the subspace $E(\tau_k(S^n)) = \{(x, v_1, \ldots, v_k) \in S^n \times (S^n)^k | \langle x|v_i \rangle = 0 \text{ and } \langle v_i|v_j \rangle = \delta_{ij}, 1 \leq i, j \leq k\}$. Thus an element of E is a system of k orthonormal tangent vectors to S^n at x. Clearly, this is not a vector bundle.

1.1.9 Definition. The *Stiefel variety* of k-frames in $\mathbb{R}^n$ is the space $V_k(\mathbb{R}^n) = \{(v_1, \ldots, v_k) \in (S^{n-1})^k | \langle v_i|v_k \rangle = \delta_{ij}\}$. $V_k(\mathbb{R}^n)$ is compact since it is a closed subspace of the compact space $(S^{n-1})^k$.

With each k-frame $(v_1, \ldots, v_k) \in V_k(\mathbb{R}^n)$ we associate the k-dimensional subspace of $\mathbb{R}^n$ spanned by the vectors $v_1, \ldots, v_k$, and denote this space $\langle v_1, \ldots, v_k \rangle$.

1.1.10 Definition. The *Grassmann variety* of k-dimensional subspaces of $\mathbb{R}^n$ denoted $G_k(\mathbb{R}^n)$ is the set of k-dimensional subspaces of $\mathbb{R}^n$ with the quotient topology obtained by the map $(v_1, \ldots, v_k) \mapsto \langle v_1, \ldots, v_k \rangle$ of $V_k(\mathbb{R}^n)$ to $G_k(\mathbb{R}^n)$. Thus $G_k(\mathbb{R}^n)$ is a compact subspace since $V_k(\mathbb{R}^n)$ is compact.

1.1.11 Remark. We define the Stiefel and Grassmann variety over $\mathbb{C}^n$ and $\mathbb{H}^n$ analogously.

1.1.12 Example. Consider the product bundle $\xi = (G_k(\mathbb{F}^n) \times \mathbb{F}^n, p, G_k(\mathbb{F}^n))$. We define the *canonical k-dimensional vector bundle* γ_k^n on $G_k(\mathbb{F}^n)$ as the subbundle of the product bundle ξ with total space $E(\gamma_k^n) = \{(V, v) \in G_k(\mathbb{F}^n) \times \mathbb{F}^n | v \in V\}$. This bundle has a natural vector space structure on the fibres, since the fibre over $V \in G_k(\mathbb{F}^n)$ is the subspace V of $\mathbb{F}^n$.

To define local trivialisations on γ_k^n we define maps $\pi : G_k(\mathbb{F}^n) \times \mathbb{F}^n \to \mathbb{F}^n$ where $\pi(V, v)$ is the orthogonal projection of v onto V, and define for $H \subset \{1, \dots, n\}$ a subset of k elements the map $u_H : \mathbb{F}^n \to \mathbb{F}^n$ by placing zero's on the coordinates not in H. Let U_H be an open subset of $G_k(\mathbb{F}^n)$ with elements $V \in G_k(\mathbb{F}^n)$ such that the map $\pi(V, \cdot) : u_H(\mathbb{F}^n) \to V$ is a bijection. Then $h_H : U_H \times \mathbb{F}^n \to p^{-1}(U_H)$ defined by the formula $\psi_H(V, v) = (V, \pi(V, v))$ is a isomorphism linear on each fibre. For more details on this construction cf. [13, lem. 1.7, p. 13].

Similarly we define the orthogonal complement ${}^*\gamma_k^n$ as the subbundle of ξ with total space $E({}^*\gamma_k^n) = \{(V, v) \in G_k(\mathbb{F}^n) \times \mathbb{F}^n \,|\, \langle V|x\rangle = 0\}$.

By natural inclusion we have $G_k(\mathbb{F}^n) \subset G_k(\mathbb{F}^{n+1})$ and define $G_k(\mathbb{F}^\infty) = \cup_{k \leq n} G_k(\mathbb{F}^n)$ with the inductive topology. Then γ_k^n us defined for $k \leq n \leq \infty$. If $n = \infty$ we simply write γ_k.

1.1.13 Example. As special cases, let $k = 1$. Note $V_1(\mathbb{R}^n) = S^{n-1}$ and $G_1(\mathbb{R}^n) = \mathbb{R}\mathrm{P}^{n-1}$. The canonical bundle on $G_1(\mathbb{R}^n)$ is called the *canonical line bundle.*

The canonical vector bundles are important for the classification of vector bundles (which explains the term canonical), we return to this in section 2.2.

1.2 Morphisms of vector bundles

1.2.1 Definition. Let $\xi = (E, p, X)$ and $\xi' = (E', p', X')$ be vector bundles. A *vector bundle morphism* is a pair of maps, $u : E \to E'$ and $f : X \to X'$ satisfying $p' \circ u = f \circ p$ and such that the restriction $u : p^{-1}(x) \to p'^{-1}(f(x))$ is linear for each $x \in X$.

Thus a vector bundle morphism is a pair of maps (u, f) such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{f} & X' \end{array}$$

commutes, and such that when restricting to the fibres, u is linear. Given a map $u : E \to E'$, such that u is linear when restricting to the fibres, $p^{-1}(x)$, then u defines a map $f : X \to X'$ by commutativity of the above diagram, and the pair (u, f) defines a vector bundle morphism. Thus we may speak of the vector bundle morphism $u : E \to E'$ letting f be given implicitly, and we will write $u : \xi \to \xi'$.

1.2.2 Definition. Let $\xi = (E, p, X)$ and $\xi' = (E', p', X)$ be vector bundles. An *X-vector bundle morphism* is a map, $u : \xi \to \xi'$ such that u induces the identity map on the base space.

1.2.3 Example. Let ξ be the product bundle $(X \times \mathbb{F}^n, p, X)$ and let ξ' be the product bundle $(X \times \mathbb{F}^m, p, X)$. Then the X-morphisms, $u : \xi \to \xi'$ are on the form $u(x, v) = (x, f(x, v))$, where $f : X \times \mathbb{F}^n \to \mathbb{F}^m$ is a map linear in v.

Let $\mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ denote the vector space of linear transformations, $\mathbb{F}^n \to \mathbb{F}^m$. Then $f : X \times \mathbb{F}^n \to \mathbb{F}^m$ is continuous if and only if $x \mapsto f(x, \cdot) \in \mathbf{L}(\mathbb{F}^n, \mathbb{F}^m)$ is continuous.

Clearly, the pair of identity maps $(1_E, 1_X)$ is a vector bundle morphism of the vector bundle (E, p, X). Also composition of vector bundle morphisms form a vector bundle morphism. Hence, the set of vector bundles forms a category denoted **VB**, with vector bundle morphisms as morphisms. We form the natural subcategories of n-dimensional vector bundles denoted $\mathbf{VB}^n$, the subcategory of vector bundles over X denoted $\mathbf{VB}_X$ with X-vector bundle morphisms as morphisms, and the subcategory of n-dimensional vector bundles over X, $\mathbf{VB}_X^n = \mathbf{VB}_X \cap \mathbf{VB}^n$.

1.2.4 Definition. An *isomorphism* of vector bundles $\xi = (E, p, X)$ and $\xi' = (E', p', X')$ is a vector bundle morphism $u : \xi \to \xi'$ such that there exists a vector bundle morphism $u' : \xi' \to \xi$ with $u'u = 1_E$ and $uu' = 1_{E'}$.

An *X-vector bundle isomorphism* is an X-vector bundle morphism $u : \xi \to \xi'$ of vector bundles over X that is an isomorphism of vector bundles.

1.2.5 Definition. A vector bundle $\xi = (E, p, X)$ is called *trivial* if for some $n \in \mathbb{N}$ it is isomorphic to the trivial bundle $\varepsilon^n = (X \times \mathbb{F}^n, \pi, X)$, where $\pi : X \times \mathbb{F}^n \to X$ is the projection onto the first factor. An X-vector bundle isomorphism, $u : \xi \to \varepsilon^n$ is also called a *trivialisation*.

1.2.6 Proposition. *Let $u : \xi \to \xi'$ be an X-vector bundle morphism of the vector bundles ξ and ξ' over X. Then u is an isomorphism if and only if the restriction $u : p^{-1}(x) \to p'^{-1}(x)$ is a vector space isomorphism for every $x \in X$.*

Proof: The direct implication is clear. For the converse, let $u' : \xi' \to \xi$ be a function defined by the condition that the restriction $u' : p'^{-1}(x) \to p^{-1}(x)$ is the inverse of the restriction $u : p^{-1}(x) \to p'^{-1}(x)$.

We need to prove that u' is continuous. Let $\psi : p^{-1}(U) \to U \times \mathbb{F}^n$ and $\psi' : p'^{-1}(U) \to U \times \mathbb{F}^n$ be local trivialisations of ξ and ξ' respectively. Then $\psi' u \psi^{-1} : U \times \mathbb{F}^n \to U \times \mathbb{F}^n$ is a map of the form $(x, v) \mapsto (x, f_x(v))$ where $x \mapsto f_x$ is a map $U \to GL(n, \mathbb{F})$. Similarly, $\psi u \psi'^{-1} : U \times \mathbb{F}^n \to U \times \mathbb{F}^n$ is a map of the form $(x, v) \mapsto (x, f_x^{-1}(v))$ where $x \mapsto f_x^{-1}$ is a map $U \to GL(n, \mathbb{F})$.

Thus, the restriction $u' : p'^{-1}(U) \to p^{-1}(U)$ is continuous. It now follows that u' is continuous, since by definition there exists a family of local trivialisations (ψ_α, U_α) and (ψ'_α, U_α) of ξ and ξ' respectively, such that $\{U_\alpha\}$ is an open covering of X. □

1.3 Pullback of vector bundles

Let $\xi = (E, p, X)$ be a vector bundle over X and let $f : X' \to X$ be a map. Define $f^*(E) = \{(x', v) \in X' \times E | f(x') = p(v)\}$. With this subspace of $X' \times E$ we obtain the commutative diagram

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

where $p' : f^*(E) \to X'$ is the projection onto the first factor, and $\tilde{f} : f^*(E) \to E$ is defined by $\tilde{f}(x', v) = v$. Clearly $f^*(\xi) = (f^*(E), p', X')$ is a vector bundle as we may view $f^*(\xi)$ as the restriction of the product bundle $(X' \times E, 1_{X'} \times p, X' \times X)$ to the graph of f, $\{(x', f(x')) \in X' \times X\}$ which we identify with X' via the projection $(x', f(x')) \mapsto x'$. Since $\tilde{f}(x', v) = v$, $\tilde{f}$ is a vector space isomorphism when restricted to the fibres of $f^*(\xi)$.

1.3.1 Definition. Let $\xi = (E, p, X)$ be a vector bundle over X and let $f : X' \to X$ be a map. The vector bundle $f^*(\xi) = (f^*(E), p', X')$ is called the *pullback* vector bundle.

1.3.2 Example. A trivial example is the inclusion $\imath$ of the subspace $X' \subset X$. Then $\imath^*(E)$ is isomorphic to $p^{-1}(X')$ via the map $(x', v) \mapsto v$ since the condition $\imath(x') = p(v)$ simply states that $v \in p^{-1}(x')$. Thus restriction of vector bundles may be viewed as a special case of pullback.

1.3.3 Remark. If $u : \xi' \to \xi$ is a vector bundle morphism, then u factors as a composition, $u = \tilde{f}u'$,

$$\begin{array}{ccccc} & & f^*(\xi) & & \\ & {}^{u'}\nearrow & & \searrow^{\tilde{f}} & \\ \xi' & & \xrightarrow{\quad u \quad} & & \xi \end{array}$$

where $u'(y) = (p_{\xi'}(y), u(y))$. Further, u' is a vector bundle morphism over X'.

1.3.4 Theorem. *Let ξ and ξ' be vector bundles. For a map $f : X' \to X$ the bundles $f^*(\xi)$ and ξ' are X'-isomorphic if and only if there exists a morphism $u : \xi' \to \xi$ such that u is an isomorphism of vector spaces when restricted to the fibres, $u : p'^{-1}(x') \to p^{-1}(f(x'))$ for every $x' \in X'$.*

Proof: Assume $f^*(\xi)$ and ξ' are X'-isomorphic and let $u' : \xi' \to f^*(\xi)$ be an X'-isomorphism. Then by the above factorisation we have a vector bundle morphism $u : \xi' \to \xi$ given by $u = \tilde{f}u'$. Since u' is an X'-isomorphism, the restriction, $u' : p'^{-1}(x') \to p'^{-1}(x')$ is a vector space isomorphism. Since $\tilde{f}$ is a vector bundle isomorphism it now follows that the restriction $u : p'^{-1}(x') \to p^{-1}(f(x'))$ is a vector space isomorphism.

Conversely, assume $u : \xi' \to \xi$ is a vector bundle morphism, such that the restriction to the fibres is a vector space isomorphism. Then u factors through $f^*(\xi)$ by the composition $\tilde{f}u'$. Since the restriction of u to the fibres is a vector space isomorphism, so is u' and it follows that u' is an X'-isomorphism. □

If $u : \xi \to \xi'$ is an X-morphism and $f : X' \to X$ a map, then it follows from the formula $f^*(u)(x', v) = (x, u(v))$ that $f^*(u) : f^*(\xi) \to f^*(\xi')$ is an X'-vector bundle morphism. Thus linearity of u over $f(x')$ implies linearity of $f^*(u)$ over x'. With these definitions it follows that $f^* : \mathbf{VB}_X \to \mathbf{VB}_{X'}$ is a functor.

For vector bundles ξ, ξ_1, ξ_2 over X, X_1 and X_2 respectively, $1^*(\xi)$ and ξ are X-isomorphic and for maps, $f : X_1 \to X$ and $g : X_2 \to X_1$, $g^*(f^*(\xi))$ and $(fg)^*(\xi)$ are X_2 isomorphic. It is easy to show that if $\xi_1 \approx \xi_2$, with ξ_1 and ξ_2 vector bundles over X, then $f^*(\xi_1) \approx f^*(\xi_2)$.

1.4 Operations on vector bundles

1.4.1 Definition. Let $\xi_1 = (E_1, p_1, X)$ and $\xi_2 = (E_2, p_2, X)$ be vector bundles over X. We then define the *fibre product*, $\xi_1 \oplus \xi_2 = (E_1 \oplus E_2, q, X)$, where $E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 | p_1(v_1) = p_2(v_2)\}$ and $q(v_1, v_2) = p_1(v_2) = p_2(v_2)$.

The fibre $q^{-1}(x)$ of $\xi_1 \oplus \xi_2$ for $x \in X$ is the product of the fibres, $q^{-1}(x) = p_1^{-1}(x) \times p_2^{-1}(x)$ which explains the term fibre product.

For the fibre product $\xi_1 \oplus \xi_2$ of two vector bundles we define a vector space structure on the fibres $q^{-1}(x)$ by the direct sum of the two vector spaces. Thus if $\psi_1 : p_1^{-1}(U) \to U \times \mathbb{F}^m$ and $\psi_2 : p_2^{-1}(U) \to U \times \mathbb{F}^m$ are local trivialisations of ξ_1 and ξ_2 respectively, then $\psi_1 \oplus \psi_2 : U \times \mathbb{F}^{m+n} \to q^{-1}(U)$ is a local trivialisation of $\xi_1 \oplus \xi_2$.

1.4.2 Definition. The *Whitney sum* of two vector bundles ξ_1 and ξ_2, denoted $\xi_1 \oplus \xi_2$, is the fibre product of the bundles ξ_1 and ξ_2 with the above vector space structure on the fibres.

1.4.3 Example. For the canonical vector bundle γ_k^n and its dual ${}^*\gamma_k^n$ we have $\gamma_k^n \oplus {}^*\gamma_k^n \approx \varepsilon^1$.

1.4.4 Example. Let $\tau(\mathbb{R}\mathrm{P}^{k-1})$ be the tangent bundle over the real projective space $\mathbb{R}\mathrm{P}^{k-1}$ and let ξ_{k-1} denote the canonical line bundle over $\mathbb{R}\mathrm{P}^{k-1}$. Then there is an isomorphism $\phi : k\xi_{k-1} \to \tau(\mathbb{R}\mathrm{P}^{k-1}) \oplus \varepsilon^1$.

First note that we may view the tangent bundle $\tau(\mathbb{R}\mathrm{P}^{k-1})$ as the quotient of the sphere bundle $\tau(S^{k-1})$, with total space $E(\tau(\mathbb{R}\mathrm{P}^{k-1})) = \{\pm(x, v) | x \in S^{k-1}, v \in \mathbb{R}\}$.

Let $\langle\cdot|\cdot\rangle$ be the standard inner product of $\mathbb{R}^{k-1}$. We then have linear maps, $\pi_x : \mathbb{R}^{k-1} \to \mathbb{R}$ and $\nu_x : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ defined by

$$\begin{aligned} \pi_x(v) &= \langle v|x\rangle / \langle x|x\rangle \\ \nu_x(v) &= v - \pi_x(v)x \end{aligned}$$

For $\pm x \in \mathbb{R}\mathrm{P}^{k-1}$ we then define the bundle morphism $u : k\xi_{k-1} \to \tau(\mathbb{R}\mathrm{P}^{k-1}) \oplus \varepsilon^1$ by

$$u(\pm x, a_1 x, \ldots, a_k x) = (\pm(x, \nu_x(a_1, \ldots, a_k)), (\pm x, \pi_x(a_1, \ldots, a_k)))$$

u is well defined since $-\nu_x(v) = \nu_{-x}(-v)$ and $-\pi_x(v) = \pi_x(v)$, and u is invertible since $v = \nu_x(v) + \pi_x(v)x$. Hence we have an isomorphism of vector bundles.

The Whitney sum may be extended to define a functor on the category of vector bundles over X. Let ξ_1, ξ_2, η_1 and η_2 be vector bundles over X, and let $u_1 : \xi_1 \to \eta_1$ and $u_2 : \xi_2 \to \eta_2$ be X-vector bundle morphisms. Then we define the X-vector bundle morphism $u_1 \oplus u_2 : \xi_1 \oplus \xi_2 \to \eta_1 \oplus \eta_2$ by the relation $u_1 \oplus u_2(v_1, v_2) = (u_1(v_2), u_2(v_2))$. Since $p_{\eta_1} u_1(v_1) = p_{\xi_1}(v_1) = p_{\xi_2}(v_1) = p_{\eta_2} u_2(v_2)$, $u_1 \oplus u_2$ is well defined. Clearly, $1_{E_1} \oplus 1_{E_2} = 1_{E_1 \oplus E_2}$. If ζ_1 and ζ_2 are vector bundles over X, and $u_1' : \eta_1 \to \zeta_1$ and $u_1' : \eta_1 \to \zeta_1$ are vector bundle morphisms, then $(u_1' \oplus u_2')(u_1 \oplus u_2') = (u_1' u_1) \oplus (u_2' u_2)$.

We define other operations on vector bundles similarly by adapting the operations from vector spaces to operations on the fibres of vector bundles. The *tensor product* is of particular importance for the development of K-theory in the following chapters.

Let $\xi_i = (E_i, p_i, X)$ be n_i-dimensional real or complex vector bundles over X, for $i = 1, 2$. We wish to define the bundle, $(E_1 \otimes E_2, p, X)$, with fibres, $p_1^{-1}(x) \otimes p_2^{-1}(x)$ for $x \in X$.

As a set, the total space of the bundle is

$$\coprod_{x\in X} p_1^{-1}(x)\otimes p_2^{-1}(x)$$

and p maps $p_1^{-1}(x)\otimes p_2^{-1}(x)$ to x, $\forall x\in X$. To define a topology on the set, choose an open cover $\{U_\alpha\}$ of X, such that for $i=1,2$ there exists a local trivialisation, $\psi_\alpha^i : p_i^{-1}(U_\alpha)\to U_\alpha\times\mathbb{R}^{n_i}$. For each $x\in X$ we then have a linear isomorphism of vector spaces,

$$\psi_{\alpha,x}^1\otimes\psi_{\alpha,x}^2 : p_1^{-1}(x)\otimes p_2^{-1}(x)\to\mathbb{R}^{k_1}\otimes\mathbb{R}^{k_2}=\mathbb{R}^{n_1 n_2}$$

Hence, we obtain the commutative diagram,

$$\begin{array}{ccc} \coprod_{x\in U_\alpha} p_1^{-1}(x)\otimes p_2^{-1}(x) & \xrightarrow{\psi_\alpha} & U_\alpha\times\mathbb{R}^{n_1 n_2} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ U_\alpha & \xrightarrow[\text{id}]{} & U_\alpha \end{array}$$

where ψ_α is a bijection given fibrewise by $\psi_{\alpha,x}^1\otimes\psi_{\alpha,x}^2$. We then adopt the topology of $U_\alpha\times\mathbb{R}^{n_1n_2}$ to $\coprod_{x\in U_\alpha} p_1^{-1}(x)\otimes p_2^{-1}(x)$ by the map ψ_α.

Before proceeding the reader should note the following properties of Whitney sum and tensor products which can easily be derived from the corresponding properties of the operations on vector spaces:

i) $\xi_1\oplus\xi_2\approx\xi_2\oplus\xi_1$

ii) $\xi_1\oplus(\xi_2\oplus\xi_3)\approx(\xi_1\oplus\xi_2)\oplus\xi_3$

iii) $\xi_1\otimes\xi_2\approx\xi_2\otimes\xi_1$

iv) $\xi_1\otimes(\xi_2\otimes\xi_3)\approx(\xi_1\otimes\xi_2)\otimes\xi_3$

v) $\xi_1\otimes(\xi_2\oplus\xi_3)\approx(\xi_1\otimes\xi_2)\oplus(\xi_1\otimes\xi_3)$

vi) $f^*(\xi_1\oplus\xi_2)\approx f^*(\xi_1)\oplus f^*(\xi_2)$

vii) $f^*(\xi_1\otimes\xi_2)\approx f^*(\xi_1)\otimes f^*(\xi_2)$

The properties involving tensor product only holds for real and complex vector bundles. Clearly, $\varepsilon^1\oplus\cdots\oplus\varepsilon^1\approx\varepsilon^n$ for the n-fold sum of trivial line bundles, hence $\varepsilon^m\oplus\varepsilon^n\approx\varepsilon^{m+n}$. Also, $\varepsilon^m\otimes\varepsilon^n\approx\varepsilon^{mn}$. Hence we may write n for ε^n. For the n-fold sum of the bundle ξ with itself we will write $n\xi$ and for the n-fold tensor product we write ξ^n. Note that $\xi\oplus\varepsilon^0\approx\xi$ and $\xi\otimes\varepsilon^1\approx\xi$.

In case that ξ and η are vector bundles over X and Y respectively we may define the *exterior tensor product* as follows. Let $\pi_X : X\times Y\to X$ and $\pi_Y : X\times Y\to Y$ be the standard projections, we then define the exterior tensor product by $\xi\hat{\otimes}\eta=\pi_X^*(\xi)\otimes\pi_Y^*(\eta)$.

Using the results above, we construct the *exterior product* of vector bundles, $\lambda^k(\xi)$ (this should not be confused with the exterior tensor product above). First recall for a vector space

V, the k'th exterior product $\lambda^k(V)$ is given by the k-fold tensor product of V, $V \otimes \cdots \otimes V$ modulo the subspace generated by vectors on the form $v_1 \otimes \cdots \otimes v_k - \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$ where σ is a permutation and $\operatorname{sgn}(\sigma)$ is the sign of σ. If V has dimension n then $\lambda^k(V)$ has dimension $\binom{n}{k}$.

For a vector bundle ξ we first form the disjoint union of the exterior product of the fibres, $\coprod \lambda^k(p^{-1}(x))$. To define a topology for this set, we note that a linear map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ induces a linear map $\lambda^k(\varphi) : \lambda^k(\mathbb{R}^n) \to \lambda^k(\mathbb{R}^n)$. This map is continuously dependent on φ since it is a quotient map of the k-fold tensor product of φ with itself. We can then use arguments analogous to those for the tensor product.

The exterior product of vector bundles has the following properties,

i) $\lambda^k(\xi_1 \oplus \xi_2) = \bigoplus_i (\lambda^i(\xi_1) \otimes \lambda^{k-i}(\xi_2))$.

ii) $\lambda^0(\xi) = \varepsilon^1$

iii) $\lambda^1(\xi) = \xi$

iv) $\lambda^k(\xi) = \varepsilon^0$ for $k > \max \dim(p^{-1}(x))$.

v) $f^*(\lambda^k(\xi)) = \lambda^k(f^*(\xi))$.

The first property follows from the distributive properties of tensor products and Whitney sum stated above. Properties ii) and iii) are clear and property iv) follows from the corresponding fact for vector spaces, and v) follows from properties vi) and vii) of Whitney sum and tensor product together with property i) above.

2

Homotopy properties of vector bundles

In this chapter we develop the homotopy properties of vector bundles. Using these we classify vector bundles up to homotopy, and construct new vector bundles. The study of the homotopy properties of vector bundles will provide us with the extra tools needed to study the cohomological properties of K-theory in the following chapters.

2.1 Homotopy properties

2.1.1 Lemma. *Let ξ be a vector bundle with base space $X \times [0,1]$. Then ξ is trivial if for some $t \in [0,1]$ the restrictions $\xi_0 = \xi|_{X\times[0,t]}$ $\xi_1 = \xi|_{X\times[t,1]}$ are trivial.*

Proof: Let $\psi_0 : E_0 \to X_0 \times \mathbb{F}^n$ and $\psi_1 : E_1 \to X_1 \times \mathbb{F}^n$ be trivialisations of ξ_0 and ξ_1 respectively. These trivialisations may not agree on $p^{-1}(X \times \{t\})$. Define $\varphi_0 = \psi_0|_{X\times\{t\}}$ and $\varphi_1 = \psi_1|_{X\times\{t\}}$. Then $\varphi_1\varphi_0^{-1} : X\times\{t\}\times\mathbb{F}^n \to X\times\{t\}\times\mathbb{F}^n$ is an isomorphism of trivial vector bundles. Thus we may write, $\varphi_1\varphi_0^{-1}(x,t,y) = (x,t,\tau(x)y)$, where $(x,t,y) \in X \times \{t\} \times \mathbb{F}^n$ and $\tau : X \to GL(n,\mathbb{F})$.

We prolong $\varphi_1\varphi_0^{-1}$ to a $X \times [t,1]$-isomorphism of trivial bundles, $\omega : X \times [t,1] \times \mathbb{F}^n \to X \times [t,1] \times \mathbb{F}^n$, defined by the formula $\omega(x,s,y) = (x,s,\tau(x)y)$ for $(x,s,y) \in X \times [t,1] \times \mathbb{F}^n$. Then $\omega\psi_1 : E_1 \to X \times [t,1] \times \mathbb{F}^n$ is a vector bundle isomorphism, and ψ_0 and $\omega\psi_1$ agree on $p^{-1}(X \times \{t\})$. Thus there exists an isomorphism $\psi : E \to X \times [0,1] \times \mathbb{F}^n$ with $\psi|_{E_0} = \psi_0$ and $\psi|_{E_1} = \omega\psi_1$. □

2.1.2 Lemma. *Let ξ be a vector bundle over $X \times I$. Then there exists an open covering $\{U_\alpha\}$ of X such that the restrictions, $\xi|_{U_\alpha\times I}$ are trivial.*

Proof: By definition there exists an open cover $\{V_\beta \times I_\beta\}$ such that the restriction $\xi|_{V_\beta\times I_\beta}$ is trivial. Thus for any $x \in X$ we can choose a family of neighbourhoods, $\{V_{\beta,x} \times I_{\beta,x}\}$ such that $\xi|_{V_{\beta,x}\times I_{\beta,x}}$ is trivial, and $\cup I_{\beta,x} = I$. Since I is compact we may choose a finite family of neighbourhoods with these properties.

Let $U_\alpha = \cap V_\beta$. Since I is locally compact Hausdorff, we can choose a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $U_\alpha \times [t_{i-1}, t_i] \subset V_{\beta,x} \times I_{\beta,x}$ for some β. Then $\xi|_{U_\alpha\times[t_{i-1},t_i]}$ is trivial for all $1 \leq i \leq k$. Thus by lemma 2.1.1, $\xi|_{U_\alpha\times I}$ is trivial. □

2.1.3 Proposition. *Let ξ be a vector bundle over $X \times I$ where X is paracompact. Then the restrictions of ξ to the base spaces $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.*

Proof: By lemma 2.1.2 we can choose an open cover $\{U_\alpha\}$ of X, such that $\xi|_{U_\alpha \times I}$ is trivial. Since X is paracompact there exists a countable cover $\{V_k\}_{k\geq 1}$ of X and a partition of unity $\{\varphi_k\}$ such that V_k is a disjoint union of open sets with each connected component contained in U_α for some α, and for each $k \geq 1$ φ_k has support in V_k. Thus $\xi|_{V_k \times I}$ is trivial for all k.

Define for $k \geq 0$, $\psi_k = \sum_{i=1}^k \varphi_i$ and let $\psi_0 = 0$. Let $X_k = \{(x, \psi_k(x)) \in X \times I\}$ be the graph of ψ_k and let $\xi_k = \xi|_{X_k}$. A trivialisation of ξ over $V_k \times I$ then lifts to an isomorphism, $h_k : \xi_k \to \xi_{k-1}$ which is the identity outside $p^{-1}(V_k \times I)$.

For any $x \in X$, there is a neighbourhood U of x such that only finitely many φ_i's are nonzero, and thus for large k, $h_k|_U = 1_U$. It follows that the infinite composition, $h = h_1 h_2 \cdots$ is a well defined isomorphism of vector bundles, $h : \xi|_{X\times\{0\}} \to \xi|_{X\times\{1\}}$. □

2.1.4 Theorem. *Let ξ be a vector bundle over Y and let $f, g : X \to Y$ be homotopic maps. If X is paracompact, then the pullback bundles $f^*(\xi)$ and $g^*(\xi)$ are X-isomorphic.*

Proof: Let $h : X \times I \to Y$ be a homotopy of f and g with $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for $x \in X$, and define the vector bundle $\xi \times I = (E \times I, p' \times 1_I, Y \times I)$.

Then the pullback $h^*(\xi)$ is a bundle over $X \times I$. By restriction, we have $f^*(\xi) \approx h^*(\xi)|_{X\times\{0\}}$ and $g^*(\xi) \approx h^*(\xi)|_{X\times\{1\}}$, and thus by proposition 2.1.3, $f^*(\xi) \approx g^*(\xi)$. □

2.1.5 Corollary. *Every vector bundle over a contractible paracompact space X is trivial.*

Proof: Let 1_X be the identity map and let $f : X \to X$ be the constant map. Then for each k-dimensional vector bundle ξ over X, $1_X^*(\xi)$ is X-isomorphic to ξ and $f^*(\xi)$ is X-isomorphic to the product bundle $(X \times \mathbb{F}^k, p, X)$. Since 1_X and f are homotopic it follows from theorem 2.1.4 that ξ is isomorphic to the product bundle $(X \times \mathbb{F}^k, p, X)$. □

2.2 The universal bundle

2.2.1 Definition. Let ξ be a k-dimensional $\mathbb{F}$-vector bundle. A *Gauss map* of ξ is a map $G : E \to \mathbb{F}^n$ with $k \leq n \leq +\infty$ such that G is a linear monomorphism on the fibres of ξ.

Recall that for the canonical vector bundle γ_k^n, the total space E is the subspace $\{(V, v) \in G_k(\mathbb{F}^n) \times \mathbb{F}^n | v \in V\}$. Thus for the canonical vector bundle, we obtain a Gauss map by projection on the second factor $G(V, v) = v$.

2.2.2 Proposition. *Let $u : \xi \to \gamma_k^n$ be a vector bundle morphism that is an isomorphism on the fibres. Then Gu is a Gauss map of ξ. Conversely if there is a Gauss map $g : \xi \to \mathbb{F}^n$ then there exists a vector bundle morphism $u : \xi \to \gamma_k^n$ such that $Gu = g$.*

Proof: The first statement is clear. For the converse statement, we construct a vector bundle morphism as follows. Let $f(x) = g(p^{-1}(x)) \in G_k(\mathbb{F}^n)$ for $x \in X$ and define $u(v) = (f(p(v)), g(v)) \in E(\gamma_k^n)$ for $v \in E(\xi)$. Then from the local trivialisations of ξ it follows that f is continuous, and thus that u is continuous. □

2.2.3 Corollary. *For a vector bundle ξ over X there exists a Gauss map if and only if ξ is X-isomorphic to $f^*(\gamma_k^n)$ for some map $f : X \to G_k(\mathbb{F}^n)$.*

2.2.4 Proposition. *let ξ be a vector bundle over a paracompact space X, and let $\{U_i\}$ be an open covering of X such that $\xi|_{U_i}$ is trivial for all i. Then there exists a countable covering $\{V_j\}$ such that $\xi|_{V_j}$ is trivial for all j.*

Proof: This follows immediately by paracompactness of X. □

2.2.5 Theorem. *Let ξ be a k-dimensional vector bundle over a paracompact space X. Then there exists a Gauss map $g : E \to \mathbb{F}^\infty$.*

Proof: Let $\{U_i\}$ be a countable open covering of X such that $\xi|_{U_i}$ is trivial for all i, with trivialisations $\psi_i : \xi|_{U_i} \to U_i \times \mathbb{F}^k$. Let $\{\varphi_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$, and let $\imath_i : \mathbb{F}^k \to \mathbb{F}^\infty$ be the inclusion of $\mathbb{F}^k$ into the i'th k-dimensional subspace of $\mathbb{F}^\infty$. We then define a map $g : E \to \mathbb{F}^\infty$ by the sum $g = \sum_i g_i$ where $g_i = \varphi_i \imath_i \pi \psi_i$ and $\pi : X \times \mathbb{F}^k \to \mathbb{F}^k$ is the projection onto the second factor.

Clearly g_i is a monomorphism on the fibres over $x \in X$ where $\varphi_i(x) > 0$, and since the g_i's maps to complementary subspaces g is a monomorphism. □

2.2.6 Corollary. *Every k-dimensional vector bundle ξ over a paracompact space X is X-isomorphic to $f^*(\gamma_k)$ for some map $f : X \to G_k(\mathbb{F}^\infty)$.*

2.2.7 Proposition. *Let $f_0, f_1 : X \to G_k(\mathbb{F}^\infty)$ be maps such that $f_0^*(\gamma_k) \approx f_1^*(\gamma_k)$. Then f_0 and f_1 are homotopic.*

Proof: Let ξ be a vector bundle over X such that $\xi \approx f_0^*(\gamma_k)$ and $\xi \approx f_1^*(\gamma_k)$. Then f_0 and f_1 induce Gauss maps $g_0, g_1 : E(\xi) \to \mathbb{R}^\infty$ that are linear injections on the fibres.

Define $L_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $L_t(x_0, x_1, \dots) = (1-t)(x_0, x_2, dots) + t((x_0, 0, x_1, 0, \dots)$. Composing with g_0 we obtain a homotopy of the image of g_0 into the odd numbered coordinates of $\mathbb{R}^\infty$. Similarly we have a homotopy mapping the image of g_1 into the even numbered coordinates of $\mathbb{R}^\infty$. Hence, we may assume that the Gauss maps g_0 and g_1 maps to the odd numbered and even numbered coordinates respectively.

We then have a homotopy from g_0 to g_1 defined by $g_t = (1-t)g_0 + tg_1$. From this we obtain a homotopy f_t from f_0 to f_1 defined by $f_t(x) = g_t(p^{-1}(x))$ where $p : E(\xi) \to X$ is the projection map of ξ. □

2.2.8 Definition. A vector bundle ξ over X is of *finite type* if there exists a finite open covering $\{U_i\}$ of X such that $\xi|_{U_i}$ is trivial for all i.

2.2.9 Proposition. *Let ξ be a k-dimensional vector bundle over X. Then the following statements are equivalent*

i) ξ is of finite type.

ii) There exists a map $f : X \to G_k(\mathbb{F}^n)$ for some n such that ξ and $f^*(\gamma_k^n)$ are X-isomorphic.

iii) There exists a vector bundle η over X such that $\xi \oplus \eta$ is trivial.

Proof: From theorem 2.2.5 it follows that i) implies ii). Since $\gamma_k^n \oplus {}^*\gamma_k^n$ is trivial over $G_k(\mathbb{F}^n)$ it follows that $f^*(\gamma_k^n \oplus {}^*\gamma_k^n)$ is X-isomorphic to ε^n. Let $\eta = f^*({}^*\gamma_k^n)$. From property iv) of Whitney sums we have that $f^*(\gamma_k^n \oplus {}^*\gamma_k^n)$ and $(f^*(\gamma_k^n) \oplus f^*({}^*\gamma_k^n)$ are X-isomorphic. Hence ii) implies iii). Let η be a vector bundle such that $\xi \oplus \eta$ is trivial. Then the composition $\xi \hookrightarrow \xi \oplus \eta \to X \times \mathbb{F}^n \to \mathbb{F}^n$ is a Gauss map. □

Let $\mathrm{Vect}_{\mathbb{F}}(X)$ denote the set of X-isomorphism classes of $\mathbb{F}$-vector bundles over X. We denote the set of X-isomorphism classes of k-dimensional $\mathbb{F}$-vector bundles over X by $\mathrm{Vect}_{\mathbb{F}}^k(X)$ and the isomorphism class of a vector bundle ξ over X by $\{\xi\}$.

Let $f : Y \to X$ be a map. The homotopy class determined by f defines a morphism $\mathrm{Vect}_{\mathbb{F}}([f]) : \mathrm{Vect}_{\mathbb{F}}(X) \to \mathrm{Vect}_{\mathbb{F}}(Y)$ by the formula, $\mathrm{Vect}_{\mathbb{F}}([f])(\{\xi\}) = \{f^*(\xi)\}$. This is well defined by theorem 2.1.4. Thus, by the properties of pullback $\mathrm{Vect}_{\mathbb{F}}$ is a cofunctor from the category of spaces and maps to the category of isomorphism classes of vector bundles and homotopy classes of vector bundle morphisms.

Let $[X, Y]$ denote the set of homotopy classes of maps $f : X \to Y$. For a space X we define the function $\phi_X : [X, G_k(\mathbb{F}^\infty)] \to \mathrm{Vect}_{\mathbb{F}}^k(X)$ by the formula, $\phi_X([f]) = \{f^*(\gamma_k)\}$, where γ_k is the canonical k-dimensional vector bundle. From theorem 2.1.4 if $f, g : X \to G_k(\mathbb{F}^\infty)$ are homotopic maps, then $f^*(\gamma_k)$ and $g^*(\gamma_k)$ are X-isomorphic. Hence, hence ϕ_X is well defined.

2.2.10 Theorem. *For a paracompact space X the map $\phi_X : [X, G_k(\mathbb{F}^\infty)] \to \mathrm{Vect}_{\mathbb{F}}^k(X)$ is a bijection.*

Proof: By proposition 2.2.7, if $f_0, f_1 : X \to G_k(\mathbb{F}^\infty)$ are maps such that $f_0^*(\gamma_k)$ and $f_1^*(\gamma_k)$ are X-isomorphic, then f_0 and f_1 are homotopic. Hence, ϕ_X is injective.

For a vector bundle ξ over X there exists a map $f : X \to G_k(\mathbb{F}^\infty)$ such that ξ is X-isomorphic to $f^*(\gamma_k)$ by corollary 2.2.6. Hence, ϕ_X is surjective. □

With this theorem the problem of classifying vector bundles has been reduced to calculation of homotopy classes of maps $[X, G_k(\mathbb{F}^\infty)]$. However the theorem does not provide any tools for calculating $[X, G_k(\mathbb{F}^\infty)]$. Its significance is more of theoretical implications, since general statements of vector bundles may be deduced from statements of universal bundles, i.e. vector bundles over $G_k(\mathbb{F}^\infty)$.

2.3 Collapsing and clutching constructions

We have in the previous chapter studied how to combine vector bundles to form new ones using the vector space structure on the fibres. In this section, we will study other ways to construct vector bundles by altering the base space, namely collapsing and clutching of vector bundles.

Collapsing of vector bundles is a process of identifying the fibres over a closed subset to a single fibre over a point. This operation is needed to establish the cohomological properties of K-theory to be presented in the next chapter.

2.3.1 Definition. Let ξ be a vector bundle over X and A a subset of X such that there exists a trivialisation $t : \xi|A \to A \times \mathbb{F}^n$. A *collapsing* of ξ with respect to t is a triple

$(\xi/t, u, r)$ where ξ/t is a vector bundle over X/A, $u : \xi \to \xi/t$ is a vector bundle morphism and $r : \xi/t|_* \to \mathbb{F}^n$ is a vector space isomorphism, such that the restriction $u : \xi_x \to \xi/t|_{\pi(x)}$ for $x \in X$ is a linear isomorphism where $\pi : X \to X/A$ is the standard projection, and such that $t = ru$ when restricted to $\xi|_A$.

The next proposition states under which conditions such a collapsing exists,

2.3.2 Proposition. *Let ξ be a vector bundle over X and let A be a closed subset of X such that there exists a trivialisation of ξ over A. Then there exists a collapsing $(\xi/t, u, r)$ of ξ with respect to t if and only if there exists an open neighbourhood U with $A \subset U$ and a trivialisation t' of U such that t' prolongs t to U. The collapsing is unique up to isomorphism.*

Proof: Assume that there exists a collapsing, $(\xi/t, u, r)$, then there exists a local trivialisation $\varphi : \xi/t|_V \to V \times \mathbb{F}^n$ with $* \in V$, such that $\varphi|_* = r$. Let $t' = \varphi u$, then t' prolongs t.

For the converse we construct the total space E/t of ξ/t as the total space E of ξ with the following identification: For $x, x' \in p^{-1}(A)$ let $t(x) = (p^{-1}(x), v)$ and $t(x') = (p^{-1}(x'), v')$. Then we set $x \sim x'$ if $v = v'$. With this identification, ξ/t is a vector bundle over X/A with the induced projection. Let $u : \xi \to \xi/t$ be the quotient map, then u is a fibrewise linear isomorphism. The prolongation t' of t then defines a local trivialisation of ξ/t in a neighbourhood of $*$, and for neighbourhoods U with $* \notin U$ we have local trivialisations from ξ.

For uniqueness, assume $((\xi/t)', u', r')$ is another collapsing of ξ with respect to t. Then we have an isomorphism $\psi : \xi/t \to (\xi/t)'$ defined by $\psi = u'u^{-1}$ for $x \neq *$ and using local trivialisations that restricts to r and r' respectively, we have $\psi = r'^{-1}r$ in a neighbourhood of $*$. □

2.3.3 Remark. Note that the proposition applies to CW-pairs (X, A), i.e. X is a CW-complex and A is a subcomplex of X.

2.3.4 Proposition. *Let ξ and ξ' be vector bundles over X and X' with trivialisations t and t' over A and A' respectively. Let $v : \xi \to \xi'$ be a vector bundle morphism such that the image of A is a subset of A' and $t = t'v$. Then there exists an unique vector bundle morphism $w : \xi/t \to \xi'/t'$ such that the diagram*

$$\begin{array}{ccc} \xi & \xrightarrow{v} & \xi' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ \xi/t & \xrightarrow[w]{} & \xi'/t' \end{array}$$

commutes. Further if v is an isomorphism carrying A into A' then w is an isomorphism.

Proof: The diagram defines w uniquely. If v is an isomorphism, then the inverse of v defines an inverse of w. □

2.3.5 Proposition. *Let ξ and ξ' be vector bundles over X with trivialisations t and t' over A. Then $\xi/t \oplus \xi'/t' \approx (\xi \oplus \xi')/(t \oplus t')$ and $\xi/t \otimes \xi'/t' \approx (\xi \otimes \xi')/(t \otimes t')$.*

We end the section studying clutching of vector bundles. For a CW-complex X, we define a *CW-triad* as a triple (X, X_0, X_1), where X_0 and X_1 are subcomplexes of X such that $X = X_0 \cup X_1$. Given vector bundles ξ_0 and ξ_1 over X_0 and X_1 respectively we now wish to construct a vector bundle over X by gluing these together. This process is important for the proof of the Bott periodicity theorem in chapter 4. The following proposition states to what extent this is possible.

2.3.6 Proposition. *Let (X, X_0, X_1) be a CW-triad with $A = X_0 \cap X_1$ and let ξ_i be a vector bundle over X_i, $i = 0, 1$. If $\alpha : \xi_0|_A \to \xi_1|_A$ is a vector bundle isomorphism, then there exists a triple (ξ, u_0, u_1) such that ξ is a vector bundle over X, $u_i : \xi_i \to \xi|_{X_i}$ is a vector bundle isomorphism for $i = 0, 1$ and $u_0 = u_1\alpha$ over A. Further, the vector bundle ξ is unique up to isomorphism.*

Proof: Let E be the space obtained by the disjoint union of E_0 and E_1 identifying $x \in p_0^{-1}(A)$ with $\alpha(x) \in p_1^{-1}(A)$. There is a natural projection $E \to X$ induced by the projections of ξ_0 and ξ_1, a natural vector space structure and natural vector bundle isomorphisms $u_i : \xi_i \to \xi|_{X_i}$, $i = 0, 1$.

Thus, we only need to prove local triviality of ξ. For $x \notin A$ the existence of local trivialisations is clear. Let $\xi \in A$ and choose an open neighbourhood U of x in X such that there are local trivialisations $\varphi_i : \xi_i|_{U\cap X_i} \to U \cap X_i \times \mathbb{F}^n$ and such that there is a retraction $r : U \cap X_0 \to U \cap A$.

$$
\begin{array}{ccccccc}
U \cap A \times \mathbb{F}^n & \xleftarrow{\varphi_0} & E_0|_{U\cap A} & \xrightarrow{\alpha} & E_1|_{U\cap A} & \xrightarrow{\varphi_1} & U \cap A \times \mathbb{F}^n \\
\downarrow{\scriptstyle \pi} & & \searrow{\scriptstyle p_0} & & \swarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\
U \cap A & \longleftarrow & & U \cap A & & \longrightarrow & U \cap A
\end{array}
$$

Over the set $U \cap A$ we then have $\varphi_1 \alpha \varphi_o^{-1} = 1_{U\cap A} \times M$ where $1_{U\cap A}$ is the identity on $U \cap A$ and $M \in GL(n, \mathbb{F})$ is a matrix continuously dependent on $x \in U \cap A$. Composing φ_0 with the retraction we can assume that $\varphi_0 = \varphi_1\alpha$ over $U \cap A$. We then have a local trivialisation $\varphi : \xi|_U \to U \times \mathbb{F}^n$ and $\varphi_i u_i = \varphi|_{U\cap X_i}$.

For uniqueness assume (η, v_0, v_1) is some other triple satisfying the same conditions. Then we obtain an isomorphism $w : \eta \to \xi$ by the compositions, $u_i v_i^{-1} : \eta|_{X_i} \to \xi|_{X_i}$. The isomorphism is well defined since $E(\eta)$ is the union of the closed subsets $E(\eta|_{X_i})$. □

2.3.7 Corollary. *Let ξ be a bundle over a CW-complex X and let X_0 and X_1 be subcomplexes of X such that $X = X_0 \cup X_1$. Then $\xi \approx \xi|_{X_0} \bigcup_1 \xi|_{X_1}$.*

2.3.8 Definition. The vector bundle $\xi_0 \bigcup_\alpha \xi_1$ is called the *clutching* of ξ_0 and ξ_1 along A, the map α is called the *clutching map* and the triple (ξ_0, α, ξ_1) is called the *clutching data*

Directly from the definition we have,

2.3.9 Proposition. *Let (ξ_0, α, ξ_1) be clutching data over the CW-triad (X, X_0, X_1), let (η_0, β, η_1) be clutching data over the CW-triad (Y, Y_0, Y_1), and let $u_i : \xi_i \to \eta_i$ be vector bundle morphisms for $i = 0, 1$ such that $u_0\alpha = \beta u_1$ over A. Then there exists a vector* *bundle morphism* $u : \xi_0 \bigcup_\alpha \xi_1 \to \eta_0 \bigcup_\beta \eta_1$ *such that the diagram*

$$\begin{array}{ccc} \xi_i & \xrightarrow{u_i} & \eta_i \\ \downarrow & & \downarrow \\ \xi_0 \bigcup_\alpha \xi_1 & \xrightarrow{u} & \eta_0 \bigcup_\beta \eta_1 \end{array}$$

commutes for $i = 0, 1$ *where the vertical maps are inclusions. Further, the morphism* u *is unique with respect to this property, and if* u_0 *and* u_1 *are vector bundle isomorphisms, then so is* u*.*

2.3.10 Proposition. *Let* (ξ_0, α, ξ_1) *and* (η_0, β, η_1) *be two sets of clutching data over the CW-triad* (X, X_0, X_1)*. Then* $(\xi_0 \oplus \eta_0) \bigcup_{\alpha\oplus\beta} (\xi_1 \oplus \eta_1)$ *and* $(\xi_0 \bigcup_\alpha \xi_1) \oplus (\eta_0 \bigcup_\beta \eta_1)$ *are isomorphic. Similarly,* $(\xi_0 \otimes \eta_0) \bigcup_{\alpha\otimes\beta} (\xi_1 \otimes \eta_1)$ *and* $(\xi_0 \bigcup_\alpha \xi_1) \otimes (\eta_0 \bigcup_\beta \eta_1)$ *are isomorphic.*

2.3.11 Proposition. *Let* (X, X_0, X_1) *be a CW-triad with* ξ_i *vector bundles over* X_i *for* $i = 0, 1$*. Let* $\alpha_t : \xi_0|_A \to \xi_1|_A$ *be a homotopy of clutching functions. Then* $\xi_0 \bigcup_{\alpha_0} \xi_1$ *and* $\xi_0 \bigcup_{\alpha_1} \xi_1$ *are isomorphic.*

Proof: We may view α as a clutching function of the bundles $\xi_0 \times I$ and $\xi_1 \times I$. Then $\xi_0 \bigcup_{\alpha_0} \xi_1$ is isomorphic to the restriction $(\xi_0 \times I \bigcup_\alpha \xi_1 \times I)|_{X\times\{0\}}$ and $\xi_0 \bigcup_{\alpha_1} \xi_1$ is isomorphic to the restriction $(\xi_0 \times I \bigcup_\alpha \xi_1 \times I)|_{X\times\{1\}}$. Since $\xi_0 \times I \bigcup_\alpha \xi_1 \times I$ is isomorphic to $\eta \times I$ for some η the proposition follows from proposition 2.1.3. □

3

K-theory

The history of K-theory traces back to Grothendieck, who developed tools to solve the Riemann-Roch theorem. His method was first applied to topology by Borel and Hirzebruch[8] in 1958.

The idea of K-theory is to make the Whitney sum and tensor product operations on vector bundles over a fixed base space into addition and multiplication operations of a ring. For quaternionic vector bundles we can only form a group this way with addition defined by Whitney sum of vector bundles, but since we will only need real and complex vector bundles, we will discard this case, and so $\mathbb{F}$ refers to either $\mathbb{R}$ or $\mathbb{C}$.

The spaces we are concerned with are compact CW-complexes, and thus have all the nice properties we may wish for, namely Hausdorff and paracompactness properties. For the remaining part of the thesis all spaces are assumed compact CW-complexes.

3.1 The K-cofunctor

Transferring the operations of vector bundles, $\mathrm{Vect}_{\mathbb{F}}(X)$ forms a semiring, that is $\mathrm{Vect}_{\mathbb{F}}(X)$ satisfies the axioms of a ring except for the existence of an additive inverse, with sum defined by the Whitney sum of bundles, $\{\xi\} + \{\eta\} = \{\xi \oplus \eta\}$, and multiplication defined by tensor product, $\{\xi\} \times \{\eta\} = \{\xi \otimes \eta\}$, with neutral elements of addition and multiplication, $\{\varepsilon^0\}$ and $\{\varepsilon^1\}$ respectively.

The notation introduced will soon prove inconvenient. Since we will only be working with isomorphism classes of vector bundles and homotopy classes of maps, we will simply write ξ for the isomorphism class determined by the vector bundle ξ and similarly, write f for the homotopy class of maps, $[f]$, determined by f. This should not lead to any confusion.

There is a standard method to extend a semiring to a ring. We will apply this to $\mathrm{Vect}_{\mathbb{F}}(X)$, but first we need a definition.

3.1.1 Definition. Let ξ and η be vector bundles over X. We say that ξ and η are *stably equivalent* and write $\xi \approx_s \eta$, if for some integer n, $\xi \oplus \varepsilon^n \approx \eta \oplus \varepsilon^n$. If for some integers m, n, $\xi \oplus \varepsilon^m \approx \eta \oplus \varepsilon^n$, we say that ξ and η are *s-equivalent* and write $\xi \sim \eta$. A bundle ξ s-equivalent to ε^0 is called *stably trivial* or *s-trivial*.

Consider the set of $\approx_s$-equivalence classes, $\mathrm{Vect}_{\mathbb{F}}(X)/\approx_s$. For the Whitney sum operation, only the ε^0-class has an inverse, namely ε^0, since $\xi \oplus \xi' \approx_s \varepsilon^0$ implies $\xi \oplus \xi' \oplus \varepsilon^n \approx_s \varepsilon^n$, hence ξ is 0-dimensional. However, since for any ξ there exists a bundle ξ' such that $\xi \oplus \xi' \approx \varepsilon^n$ by proposition 2.2.9, we do have the following cancellation property: If $\xi_1 \oplus \xi_2 \approx_s \xi_1 \oplus \xi_3$ adding ξ_1' to both sides we then obtain $\xi_1 \oplus \xi_1' \oplus \xi_2 \approx_s \xi_1 \oplus \xi_1' \oplus \xi_3$, which reduces to $\xi_2 \oplus \varepsilon^n \approx_s \xi_3 \oplus \varepsilon^n$. It then follows that $\xi_2 \approx_s \xi_3$.

Now consider the set of pairs of classes of vector bundles $\mathrm{Vect}_{\mathbb{F}}(X) \times \mathrm{Vect}_{\mathbb{F}}(X)$, we will write the elements as formal differences $\xi - \xi'$, and define the following equivalence relation, $\xi_1 - \xi_1' = \xi_2 - \xi_2'$ if and only if $\xi_1 \oplus \xi_2' \approx_s \xi_2 \oplus \xi_1'$. It is easy to show that this is an equivalence relation using the cancellation property above.

For elements $\xi - \xi', \eta - \eta' \in \mathrm{Vect}_{\mathbb{F}}(X) \times \mathrm{Vect}_{\mathbb{F}}(X)$ define addition and multiplication by $(\xi - \xi') + (\eta - \eta') = \xi \oplus \eta - \xi' \oplus \eta'$ and $(\xi - \xi')(\eta - \eta') = \xi \otimes \eta - \xi \otimes \eta' - \xi' \otimes \eta + \xi' \otimes \eta' = (\xi \otimes \eta) \oplus (\xi' \otimes \eta') - (\xi \otimes \eta') \oplus (\xi' \otimes \eta)$. With these operations $\mathrm{Vect}_{\mathbb{F}}(X) \times \mathrm{Vect}_{\mathbb{F}}(X)$ forms a ring with identity ε^1 and zero element ε^0.

Defining the morphism $\Delta : \mathrm{Vect}_{\mathbb{F}}(X) \to \mathrm{Vect}_{\mathbb{F}}(X) \times \mathrm{Vect}_{\mathbb{F}}(X)$ by the formula $\Delta(\xi) = \xi - \varepsilon^0$, the semiring $\mathrm{Vect}_{\mathbb{F}}(X)$ is included in $\mathrm{Vect}_{\mathbb{F}}(X) \times \mathrm{Vect}_{\mathbb{F}}(X)$. Thus we have obtained the *ring completion* of $\mathrm{Vect}_{\mathbb{F}}(X)/\approx_s$. It is possible to show that ring completion is unique up to isomorphism [15, p. 115].

3.1.2 Definition. For a space X we define $K_{\mathbb{F}}(X)$ as the ring completion of $\mathrm{Vect}_{\mathbb{F}}(X)/\approx_s$.

3.1.3 Remark. Any element of $K_{\mathbb{F}}(X)$ can be represented as $\xi - \varepsilon^n$ for some n. For $\zeta - \eta \in K_{\mathbb{F}}(X)$ we have an element η' such that $\eta \oplus \eta' \approx \varepsilon^n$. Adding η' to both terms we have $\zeta \oplus \eta' - \varepsilon^n$. Since $\zeta \oplus \eta' - \varepsilon^n \approx_s \zeta - \eta$ setting $\xi = \zeta \oplus \eta'$ gives the result. Since $\xi \oplus \varepsilon^0 \approx \xi$ we will write ξ for an element $\xi - \varepsilon^0$.

For a map $f : Y \to X$ we defined the semiring morphism $\mathrm{Vect}_{\mathbb{F}}(f) : \mathrm{Vect}_{\mathbb{F}}(X) \to \mathrm{Vect}_{\mathbb{F}}(Y)$. Similarly, we now define the morphism $K_{\mathbb{F}}(f) : K_{\mathbb{F}}(X) \to K_{\mathbb{F}}(Y)$ by commutativity of the following diagram,

$$\begin{array}{ccc} \mathrm{Vect}_{\mathbb{F}}(X) & \xrightarrow{\Delta} & K_{\mathbb{F}}(X) \\ {\scriptstyle \mathrm{Vect}_{\mathbb{F}}(f)}\downarrow & & \downarrow{\scriptstyle K_{\mathbb{F}}(f)} \\ \mathrm{Vect}_{\mathbb{F}}(Y) & \xrightarrow{\Delta} & K_{\mathbb{F}}(Y) \end{array}$$

Thus for vector bundles ξ, η over X, we have $K_{\mathbb{F}}(f)(\xi - \eta) = f^*(\xi) - f^*(\eta)$. It is easy to show that this is a ring homomorphism using the properties of pullback. If $g : Z \to Y$ is a second map we obtain $K_{\mathbb{F}}(fg) = K_{\mathbb{F}}(g)K_{\mathbb{F}}(f)$ and clearly $K_{\mathbb{F}}(1_X)$ is the identity on $K_{\mathbb{F}}(X)$. Thus $K_{\mathbb{F}}$ is a cofunctor.

Choose a basepoint $x_0 \in X$. We then define the morphism, called the *rank*, $rk : \mathrm{Vect}(X) \to \mathbb{Z}$ by $rk(\xi) = \dim(\xi|_{x_0})$, that is rk maps the vector bundle ξ over X to the dimension of the vector space over x_0. This induces a morphism $rk : K_{\mathbb{F}}(X) \to \mathbb{Z}$ defined by $rk(\xi - \eta) = rk(\xi) - rk(\eta)$.

Conversely, since any element of $K_{\mathbb{F}}(X)$ may be represented $\xi - \varepsilon^n$ for some n there is a natural morphism $K_{\mathbb{F}}(X) \to \mathrm{Vect}_{\mathbb{F}}(X)/\sim$ defined by $\xi - \varepsilon^n \mapsto \xi$. Clearly this morphism is surjective, and the kernel consists of elements on the form $\varepsilon^m - \varepsilon^n$. These elements form a subring of $K_{\mathbb{F}}(X)$ isomorphic to $\mathbb{Z}$.

Hence we have a splitting, $K_{\mathbb{F}}(X) = \tilde{K}_{\mathbb{F}}(X) \oplus \mathbb{Z}$ depending on the choice of x_0. If X is path connected, $\tilde{K}_{\mathbb{F}}(X)$ is independent of x_0.

3.1.4 Definition. For a pointed space X we define $\tilde{K}_{\mathbb{F}}(X)$ by the splitting $K_{\mathbb{F}}(X) = \tilde{K}_{\mathbb{F}}(X) \oplus \mathbb{Z}$. $\tilde{K}_{\mathbb{F}}(X)$ is called the *reduced $K_{\mathbb{F}}$-ring.*

Let X and Y be pointed spaces. For a basepoint preserving map $f : Y \to X$ we define $\tilde{K}_{\mathbb{F}}(f)$ as the restriction of $K_{\mathbb{F}}(f)$ to $\tilde{K}_{\mathbb{F}}(X)$. Thus, with the cofunctorial properties of $K_{\mathbb{F}}$ it follows that $\tilde{K}_{\mathbb{F}}$ is a cofunctor.

In the beginning of the chapter we mentioned that we will consider both real and complex vector bundles. Unless specified explicitly, the results in this chapter apply to both real and complex vector bundles, and thus we can drop the $\mathbb{F}$ from the notation for the remaining part of the chapter.

Also, we will simplify notation for the remaining part of the thesis further by writing $f^\#$ for $K(f)$ or $\tilde{K}(f)$ leaving it to the reader to determine which is meant from the context.

3.2 Cohomological properties

K-theory was originally known as extraordinary cohomology. In this section we explore some of the cohomological properties of K-theory, namely exact sequences.

For a space X let A be a closed subspace of X. With the natural inclusion and projection maps we then obtain the sequence of spaces,

$$A \xrightarrow{\imath} X \xrightarrow{\pi} X/A$$

If X is a CW-complex and A a subcomplex of X also refer to this sequence as the sequence of the pair (X, A).

3.2.1 Theorem. *For a closed subspace $A \subset X$ the sequence $A \xrightarrow{\imath} X \xrightarrow{\pi} X/A$ induces an exact sequence $\tilde{K}(X/A) \xrightarrow{\pi^\#} \tilde{K}(X) \xrightarrow{\imath^\#} \tilde{K}(A)$.*

Proof: The inclusion $\mathrm{im}\pi^\# \subset \ker \imath^\#$ follows immediately from the composition $A \to A/A \hookrightarrow X/A$ and the fact that $\tilde{K}(A/A) = 0$.

For the opposite inclusion, $\ker \imath^\# \subset \mathrm{im}\pi^\#$, let ξ be a vector bundle over X. Since A is closed there exists a collapsing ξ/t. We then have the commutative diagram,

$$\begin{array}{ccc} E & \longrightarrow & E/t \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow{\pi} & X/A \end{array}$$

and it remains only to verify that $\xi \approx \pi^*(\xi/t)$. This follows since the map $E \to E/t$ is a linear isomorphism on the fibres. □

3.2.2 Corollary. *If $A \subset X$ is contractible, the quotient map $\pi : X \to X/A$ induces an isomorphism $\pi^\# : \tilde{K}(X/A) \to \tilde{K}(X)$.*

Proof: Since A is contractible to a point there exists a retract $F : A \times I \to A$ such that $F(a, 0) = a$ and $F(a, 1) = *$ for $a \in A$. Using this we obtain a homotopy $X/* \to X/A$. Thus by proposition 2.1.3 the vector bundles over $X/*$ and X/A are isomorphic. □

With the following definition the above results have analogs in unreduced K-theory:

3.2.3 Definition. For a finite CW-pair, (X, A) we define $K(X, A) = \tilde{K}(X/A)$.

3.2.4 Corollary. *Let (X, A) be a finite CW-pair. We then have the exact sequence,*

$$K(X,A) \xrightarrow{\pi^\#} K(X) \xrightarrow{\imath^\#} K(A)$$

Proof: This follows since $K(X, A) = \tilde{K}(X/A)$ and the inclusion $A \hookrightarrow X$ induces morphisms $K(X) \to K(A)$ and $\tilde{K}(X) \to \tilde{K}(A)$ with the same kernel. □

Before we proceed we will recall some common constructions of spaces from homotopy theory. For a space X we define the *cylinder* of X, $ZX = X \times I$, and we obtain the *cone* of X by identifying the top of the cylinder to a point, $CX = X \times I/X \times \{1\} = X \wedge I$. We continue the process and collapse the base of the cone to obtain the *unreduced suspension*, $\Sigma X = X \times I/X \times \partial I$.

For a pointed space X we form the *reduced suspension* $S \wedge X = S \times X/S \vee X$. This can be obtained from the unreduced suspension by collapsing the arc through the basepoint of X, $\{x_0\} \times I$ to a point.

For CW-complexes this is a homotopy equivalence [10, chp. VII, thm. 1.9, p. 436] and since we are only working with CW-complexes we will not distinguish, and for short write SX. Since $S \wedge S = S^2$ we will write $S^n X$ for the n-fold suspension of X.

If $f : X \to Y$ is a map, we define the *mapping cylinder* $Z_f = Y \cup_f X \times I$ as the disjoint union of Y and $X \times I$ identifying the bottom of the cylinder with the image of f, that is $(x, 0) \sim f(x)$ and $Z_f = Y \amalg X \times I/\sim$. Similarly we define the *mapping cone* $C_f = Y \cup_f CX$. The map f is called the *attaching map*.

For a map $f : X \to Y$ we define the *cone map* of f, $Cf : CX \to CY$ by $Cf = f \times 1/f(X) \times \{1\}$, and similarly we define the *reduced suspension map*, $Sf : SX \to SY$.

Let (X, A) be a finite CW-pair with A a closed subcomplex of X. Consider the sequence

$$A \xrightarrow{i} X \xrightarrow{j} C_i \xrightarrow{k} C_j \xrightarrow{l} C_k$$

where i, j, k and l are the natural inclusion maps.

Since CA is contractible to a point there exists a homotopy equivalence [10, thm. 1.6, p. 433] $h : C_i \to X/A$. By similar arguments we have homotopy equivalences $C_j \to C_j/CX = C_i/X = SA$, and $C_k \to C_k/C_j = C_j/C_i = SX$. Thus we have the homotopy equivalent sequence,

$$A \longrightarrow X \longrightarrow X/A \longrightarrow SA \longrightarrow SX$$

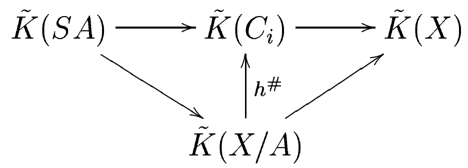
This sequence is known as the *Puppe sequence* of spaces. Now apply the $\tilde{K}$-cofunctor.

3.2.5 Theorem. *Let X be a finite CW-complex and A a closed subcomplex of X with the natural inclusion map $\imath : A \hookrightarrow X$. Then there exists a morphism $\partial : \tilde{K}(SA) \to \tilde{K}(X/A)$ such that the sequence,*

$$\tilde{K}(SX) \longrightarrow \tilde{K}(SA) \xrightarrow{\partial} \tilde{K}(X/A) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(A)$$

is exact.

Proof: From the discussion above we have a homotopy equivalence $h : C_i \to X/A$, thus we obtain an isomorphism $h^{\#} : \tilde{K}(X/A) \to \tilde{K}(C_i)$. Since we have $C_j/CX = C_i/X = SA$ the diagram,



commutes and by theorem 3.2.1 the top row is exact. Since $h^{\#}$ is an isomorphism the result follows. □

Since $SC_f = C_{Sf} \simeq SX/SA \simeq S(X/A)$, the Puppe sequence can be extended to form a long sequence of spaces

$$A \longrightarrow X \longrightarrow X/A \longrightarrow SA \longrightarrow SX \longrightarrow S(X/A) \longrightarrow \cdots$$

and thus we can extend the previous theorem,

$$\cdots \longrightarrow \tilde{K}(S(X/A)) \longrightarrow \tilde{K}(SX) \longrightarrow \tilde{K}(SA) \longrightarrow \tilde{K}(X/A) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(A)$$

Now let $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$ and $\tilde{K}^{-n}(X/A) = \tilde{K}(S^n(X/A))$, then the sequence can be rewritten

$$\cdots \longrightarrow \tilde{K}^{-1}(X/A) \longrightarrow \tilde{K}^{-1}(X) \longrightarrow \tilde{K}^{-1}(A) \longrightarrow \tilde{K}(X/A) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(A)$$

where we have chosen negative indices so that the coboundary maps ∂ introduced in theorem 3.2.5 increase dimension as in ordinary cohomology.

As in corollary 3.2.4 we have an unreduced version of the sequence.

$$\cdots \longrightarrow K^{-1}(X,A) \longrightarrow K^{-1}(X) \longrightarrow K^{-1}(A) \longrightarrow K(X,A) \longrightarrow K(X) \longrightarrow K(A)$$

3.3 Products in K-theory

For spaces X, Y let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ denote the natural projections of the product space. We then define a ring morphism $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ by $\mu(\xi \otimes \eta) = p_X^{\#}(\xi) p_Y^{\#}(\eta)$. This is simply the exterior tensor product and we will write $\xi \hat{\otimes} \eta$. Clearly μ is a ring homomorphism since for vector bundles ξ, ξ' over X and η, η' over Y we have $(\xi \hat{\otimes} \eta) \otimes (\xi' \hat{\otimes} \eta') = (\xi \otimes \xi') \hat{\otimes} (\eta \otimes \eta')$ and $(\xi \hat{\otimes} \eta) \oplus (\xi' \hat{\otimes} \eta') = (\xi \oplus \xi') \hat{\otimes} (\eta \oplus \eta')$

3.3.1 Definition. For spaces X, Y we define the *external K-cup product* by the morphism $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ defined as above.

For pointed spaces X, Y, we wish to define an external $\tilde{K}$-cup product in a similar way. This is done essentially by restriction of the external K-cup product.

3.3.2 Proposition. *Let X, Y be pointed spaces and let $\imath_X : X \to X \vee Y$ and $\imath_Y : Y \to X \vee Y$ be the natural inclusion maps. Then the group morphism $(\imath_X^{\#}, \imath_Y^{\#}) : \tilde{K}(X \vee Y) \to \tilde{K}(X) \oplus \tilde{K}(Y)$ is an isomorphism.*

Proof: With the natural inclusion and projection maps we have the sequence of spaces,

$$X \xrightarrow{\imath_X} X \vee Y \xrightarrow{\pi_X} X \vee Y/X \simeq Y$$

and thus from theorem 3.2.1 we obtain the sequence

$$\tilde{K}(Y) \xrightarrow{\pi_X^\#} \tilde{K}(X \vee Y) \xrightarrow{\imath_X^\#} \tilde{K}(X)$$

Similarly we obtain the sequence

$$\tilde{K}(X) \xrightarrow{\pi_Y^\#} \tilde{K}(X \vee Y) \xrightarrow{\imath_Y^\#} \tilde{K}(Y)$$

Thus by exactness $\pi_Y^\#$ is inverse of $\imath_X^\#$ and $\pi_X^\#$ is inverse of $\imath_Y^\#$. Hence, we have a splitting $\tilde{K}(X \vee Y) \approx \tilde{K}(X) \oplus \tilde{K}(Y)$. □

3.3.3 Proposition. *With the natural inclusion $X \vee Y \hookrightarrow X \times Y$ the exact sequence*

$$0 \longrightarrow \tilde{K}(X \wedge Y) \longrightarrow \tilde{K}(X \times Y) \longrightarrow \tilde{K}(X \vee Y) \longrightarrow 0$$

splits.

Proof: With the natural projections defined above, the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K}(X \wedge Y) & \longrightarrow & \tilde{K}(X \times Y) & \longrightarrow & \tilde{K}(X \vee Y) & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle p_X^\# \oplus p_Y^\#} & \swarrow{\scriptstyle (\imath_X^\#, \imath_Y^\#)} & & & \\ & & & & \tilde{K}(X) \oplus \tilde{K}(Y) & & & & \end{array}$$

Thus the composition $(p_X^\# \oplus p_Y^\#)(\imath_X^\#, \imath_Y^\#)$ is a splitting map and so $\tilde{K}(X \times Y) \to \tilde{K}(X \vee Y)$ is an epimorphism. □

It follows that $\tilde{K}(X \times Y) \approx \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y)$. Viewing $\tilde{K}(X)$ as a subring of $K(X)$ and $\tilde{K}(Y)$ as a subring of $K(Y)$ we can apply the external K-cup product to the elements $\xi \in \tilde{K}(X)$ and $\eta \in \tilde{K}(Y)$. Since $rk(\xi \hat{\otimes} \eta) = rk(\xi) rk(\eta)$, $\xi \hat{\otimes} \eta \in \tilde{K}(X \times Y)$.

3.3.4 Proposition. *With the restricted K-cup product as the first morphism the composition*

$$\tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \times Y) \longrightarrow \tilde{K}(X \vee Y)$$

vanish.

Proof: Since $p_X^\#$ vanish on $\tilde{K}(Y)$ and $p_Y^\#$ vanish on $\tilde{K}(X)$, the product vanish on both $\tilde{K}(X)$ and $\tilde{K}(Y)$, hence on $\tilde{K}(X \vee Y)$. □

In particular, since $\tilde{K}(X \vee Y) \approx \tilde{K}(X) \oplus \tilde{K}(Y)$, there is a unique morphism $\tilde{\mu} : \tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$ which composed with the morphism $\tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y)$ is identical to the restricted K-cup product of proposition 3.3.4.

3.3.5 Definition. We define the *external $\tilde{K}$-cup product* (or the *reduced K-cup product*) by the morphism $\tilde{\mu} : \tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$ defined above. We will write $\xi \hat{\otimes} \eta$ also for the external $\tilde{K}$-cup product, we leave it to the reader to determine which is meant.

We now obtain the commutative diagram,

$$\begin{array}{ccc} K(X) \otimes K(Y) & \xrightarrow{\approx} & (\tilde{K}(X) \otimes \tilde{K}(Y)) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z} \\ \mu \downarrow & & \downarrow \\ K(X \times Y) & \xrightarrow{\approx} & (\tilde{K}(X) \wedge \tilde{K}(Y)) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z} \end{array}$$

where the second vertical arrow is the identity on the last three summands and the external $\tilde{K}$-cup product on the first.

By use of the reduced external K-cup product we can now define the external K-cup product for finite CW-pairs, (X, A) and (Y, B), using the natural identification $X \times Y/(X \times B \cup A \times Y) = X/A \wedge Y/B$,

3.3.6 Definition. For pairs (X, A) and (Y, B) of finite CW-complexes, we define the *relative K-cup product* as the morphism $K(X, A) \otimes K(Y, B) \to K(X \times Y, X \times B \cup A \times Y)$ defined by the reduced external K-cup product, $\tilde{K}(X/A) \otimes \tilde{K}(Y/B) \to \tilde{K}(X/A \wedge Y/B)$.

3.4 Relations between real and complex K-theory

In this section we introduce some morphisms relating real and complex K-theory. To do this we apply well known operations on vector spaces to the fibres of vector bundles and so obtain the morphisms relating real and complex K-theory.

Recall that for a real vector space V of dimension k there is a natural inclusion $V \hookrightarrow V'$ into the complex vector space V' of dimension k mapping a basis $v_1, \ldots, v_k$ of V into a basis $v'_1, \ldots, v'_k$ of V'. That is, essentially we replace real coefficients by complex coefficients. Applying this operation to the fibres of vector bundles over X we obtain a sequence of maps $C_k : \mathrm{Vect}^k_{\mathbb{R}}(X) \to \mathrm{Vect}^k_{\mathbb{C}}(X)$.

Clearly, for vector bundles ξ and η of dimension k and l respectively we have,

$$C_{k+l}(\xi \oplus \eta) = C_k(\xi) \oplus C_l(\eta)$$
$$C_{kl}(\xi \otimes \eta) = C_k(\xi) \otimes C_l(\eta)$$

and so the sequence defines a homomorphism of semirings, $C : \mathrm{Vect}_{\mathbb{R}}(X) \to \mathrm{Vect}_{\mathbb{C}}(X)$. This homomorphism induces a ring homomorphism, *complexification*, $C : K_{\mathbb{R}}(X) \to K_{\mathbb{C}}(X)$ defined by the formula $C(\xi - \eta) = C(\xi) - C(\eta)$.

For a complex vector space V of dimension k, let $v_1, \ldots, v_k$ be a basis. Then there is a natural inclusion of V into the real vector space V' of dimension $2k$ defined by $v_j \mapsto v'_{2j-1}$ and $iv_j \mapsto v'_{2j}$, where $v'_1, \ldots, v'_{2k}$ is a basis of V'. Applying this operation to the fibres of vector bundles over X we obtain a sequence of maps $R_k : \mathrm{Vect}^k_{\mathbb{C}}(X) \to \mathrm{Vect}^{2k}_{\mathbb{R}}(X)$.

Clearly, for vector bundles ξ and η of dimension k and l respectively we have

$$R_{k+l}(\xi \oplus \eta) = R_k(\xi) \oplus R_l(\eta)$$

but the morphism does not have multiplicative properties like C since $R_{kl}(\xi \otimes \eta)$ is of dimension $2kl$ whereas $R_k(\xi) \otimes R_l(\eta)$ is of dimension $4kl$.

Thus the sequence defines a homomorphism of semigroups $R : \mathrm{Vect}_{\mathbb{C}}(X) \to \mathrm{Vect}_{\mathbb{R}}(X)$ which induces a group homomorphism, *realification*, $R : K_{\mathbb{C}}(X) \to K_{\mathbb{R}}(X)$ defined by $R(\xi - \eta) = R(\xi) - R(\eta)$.

Finally, let $v_1, \dots, v_k$ be a basis for the complex vector space V. Then we define a map $V \to V$ which maps each vector v to its complex conjugate, that is if $v = a_1 v_1 + \cdots + a_k v_k$ then $\bar{v} = \bar{a}_1 v_1 + \cdots + \bar{a}_k v_k$. Transferring this operation to the fibres of complex vector bundles we obtain a sequence of maps, $T_k : \mathrm{Vect}^k_{\mathbb{C}}(X) \to \mathrm{Vect}^k_{\mathbb{C}}(X)$ which maps the fibres of a vector bundle to the complex conjugate. Clearly, this sequence induces a ring homomorphism, *conjugation* $T : K_{\mathbb{C}}(X) \to K_{\mathbb{C}}(X)$.

From the defining operations on vector spaces we obtain the following relations for the homomorphisms defined above

$$\begin{aligned} RC &= 2 \\ CR &= 1 + T \end{aligned}$$

4

The Bott periodicity theorem

The Bott periodicity theorem is a key result in K-theory, the first proof of Bott appeared in 1959 used purely geometric arguments, later J. C. Moore proved the theorem using homology theory. In 1964 Bott and Atiyah gave a proof for the complex case of the theorem using only very basic theory of vector bundles and complex analysis, and Atiyah gave later (in 1966) a similar proof for the real case.

The theorem is stated in versions for both real and complex K-theory, but we will only prove the complex case, the reader may refer to [16, sec. 3.5] for a proof of the real version. The line of proof remains unchanged from the presentation of Atiyah and Bott [5], the presentation given in this chapter follows [15, chp. 11] closely.

4.1 The Bott periodicity theorem

4.1.1 Theorem (Bott periodicity theorem). *Let X be a compact space. Then the external cup product $K_{\mathbb{F}}(X) \otimes K_{\mathbb{F}}(S^k) \to K_{\mathbb{F}}(X \times S^k)$ is an isomorphism for $k = 2$ if $\mathbb{F} = \mathbb{C}$ and for $k = 8$ if $\mathbb{F} = \mathbb{R}$.*

Consider the sequence

$$X \vee S^k \longrightarrow X \times S^k \longrightarrow X \wedge S^k$$

From this we obtain the commutative diagram

$$\begin{array}{ccc} K(X) \otimes K(Y) & \xrightarrow{\approx} & (\tilde{K}(X) \otimes \tilde{K}(Y)) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z} \\ \downarrow \mu & & \downarrow \\ K(X \times Y) & \xrightarrow{\approx} & (\tilde{K}(X) \wedge \tilde{K}(Y)) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z} \end{array}$$

where the vertical arrow on the right is the identity on the last three summands and the external $\tilde{K}$-cup product on the first. From this we obtain,

4.1.2 Proposition. *The external K-cup product $K(X) \otimes K(S^k) \to K(X \times S^k)$ is an isomorphism if and only if the external $\tilde{K}$-cup product $\tilde{K}(X) \otimes \tilde{K}(S^k) \to \tilde{K}(X \wedge S^k)$ is an isomorphism.*

Thus the periodicity theorem transfers to a periodicity theorem in reduced K-theory. Whereas it is far easier to specify the isomorphism explicitly for reduced K-theory, it is easier to prove the theorem in unreduced K-theory. We will only prove the theorem for complex K-theory, and for the remaining part of the chapter all vector bundles are complex vector bundles.

The strategy of the proof is to use the clutching construction introduced in section 2.3. We first show that any vector bundle over $X \times S^2$ can be decomposed as the clutching of two (isomorphic) vector bundles over $X \times D$ for some clutching map α. We then gradually reduce the clutching map to simpler clutching maps. With this decomposition we can specify an inverse to the periodicity morphism defined by the external K-cup product.

4.2 Complex bundles over $X \times S^2$

In this section, we establish the general setup for the proof of the periodicity theorem. We view S^2 as the Riemann sphere, and D_0 denote the disk $|z| \leq 1$ and D_∞ denote the disk $|z| \geq 1$. Then $D_0 \cap D_\infty = S^1$ and $D_0 \cup D_\infty = S^2$.

For a compact space X we define the natural projections on X, $\pi_0 : X \times D_0 \to X$, $\pi_\infty : X \times D_\infty \to X$, and $\pi_1 : X \times S^1 \to X$. Defining $s : X \to X \times S^2$ by the formula $s(x) = (x, 1)$ we have $\pi_1 s = 1_X$.

The following proposition asserts that any vector bundle over $X \times S^2$ can be described as the clutching of two vector bundles over $X \times D$ for some clutching function.

4.2.1 Proposition. *Let ξ be a vector bundle over $X \times S^2$ and let $\eta = s^*(\xi)$. Then there is an automorphism $\alpha : \pi_1^*(\eta) \to \pi_1^*(\eta)$ such that ξ is isomorphic to $\pi_0^*(\eta) \cup_\alpha \pi_\infty^*(\eta)$ and $\alpha : \pi_1^*(\eta)|_{X\times\{1\}} \to \pi_1^*(\eta)|_{X\times\{1\}}$ is homotopic to the identity. Further α is unique with respect to these properties.*

Proof: Consider $s : X \to X \times D_0$. Then the composition $s\pi_0 : X \times D_0 \to X \times D_0$ is a homotopy equivalence. Thus, since the disk is contractible, the natural isomorphism $\xi|_{X\times\{1\}} \to \pi_1^*(\eta)|_{X\times\{1\}}$ prolongs to an isomorphism, $\alpha_0 : \xi|_{X\times D_0} \to \pi_0^*(\eta)$. Similarly for $s : X \to X \times D_\infty$.

The extensions over $X \times D_0$ (and $X \times D_\infty$) differ by an automorphism which is the homotopic to the identity on $X \times \{1\}$, and since D_0 is contractible, the automorphism is homotopic to the identity on $X \times D_0$. Choosing $\alpha = \alpha_\infty^{-1}\alpha_0$ as our clutching function it follows from proposition 2.3.9 that there is an isomorphism $\xi \to \pi_0^*(\eta) \cup_\alpha \pi_\infty^*(\eta)$. □

We will denote the clutching construction defined by proposition 4.2.1 by $[\eta, \alpha]$. The following proposition asserts that any vector bundle over $X \times S^2$ can be obtained as the clutching with a certain clutching function defined as follows,

4.2.2 Definition. A *Laurent polynomial clutching map* α for the vector bundle η is a map of the form

$$\alpha(x, z) = \sum_{|k|\leq n} a_k(x) z^k$$

where $a_k : \eta \to \eta$ are vector bundle endomorphisms of η and $|z| = 1$. In particular, a *linear* clutching map is a map of the form $\alpha(x, z) = a(x) + b(x)z$.

4.2.3 Proposition. *Let ξ be a vector bundle over $X \times S^2$, and let $\eta = s^*(\xi)$. Then $\xi \approx \pi_0^*(\eta) \cup_\alpha \pi_\infty^*(\eta)$ where α is a Laurent polynomial clutching map.*

Proof: Assume ξ is isomorphic to $\pi_0^*(\eta) \cup_{\alpha'} \pi_\infty^*(\eta)$, for some clutching map α'. We define the morphism $a_k : \eta \to \eta$ by the integral

$$a_k(x)\frac{1}{2\pi i}\int_{S^1} z^{-k}\alpha'(x,z)\frac{dx}{z}$$

and define

$$s_k(x,z) = \sum_{|j|\leq k} a_j(x)z^j$$

and

$$\alpha'_n(x,z) = \sum_{0\leq k\leq n} s_k(x,z)$$

Then α'_n is the n'th partial *Cesaro sum* of a Fourier sequence, and by Fejer's theorem [13, lem. 2.10, p. 40] α'_n converges uniformly in x and z to α'.

Since the homotopy classes of clutching maps are open sets with the uniform topology α'_n is a clutching map homotopic to α' for n sufficiently large. Hence choose $\alpha = \alpha'_n$ for some sufficiently large integer n. Then there exists an isomorphism $\xi \to \pi_0^*(\eta) \cup_\alpha \pi_\infty^*(\eta)$. □

4.2.4 Example. Let γ denote the canonical line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$, we then wish to decompose γ as a clutching of vector bundles. We obtain $\mathbb{C}\mathrm{P}^1$ as the quotient space $\mathbb{C}^2 - \{0\}/\sim$ where $(z_0,z_1) \sim (z'_0,z'_1)$ if $(z_0,z_1) = (\lambda z'_0, \lambda z'_1)$ for some $\lambda \in \mathbb{C} - \{0\}$. Let $\langle z_0, z_1\rangle$ denote the class determined by (z_0,z_1). We may also represent points of $\mathbb{C}\mathrm{P}^1$ as ratios, $z = z_1/z_0 \in \mathbb{C}\cup\{\infty\} = S^2$. Then points in D_0 can be represented uniquely as $\langle 1, z_1/z_0\rangle = \langle 1, z\rangle$ with $|z| \leq 1$ and points in D_∞ can be represented uniquely as $\langle z_0/z_1, 1\rangle = \langle z^{-1}, 1\rangle$ with $|z^{-1}| \leq 1$.

Over D_0 and D_∞ we have sections $\langle 1, z_1/z_0\rangle \mapsto (1, z_1/z_0)$ and $\langle z_0/z_1, 1\rangle \mapsto (z_0/z_1, 1)$ respectively, which determine trivialisations of the canonical bundle over the disks. We pass from $S^1 \subset D_\infty$ to $S^1 \subset D_0$ by multiplication with $z = z_1/z_0$. Thus we obtain the canonical line bundle over $\mathbb{C}\mathrm{P}^1$ as the clutching $D_0 \cup_\alpha D_\infty$ where $\alpha(z) = z$.

Similarly, the dual line bundle ${}^*\gamma$ can be obtained with the clutching function $\alpha(z) = z^{-1}$. Then $\gamma = [\varepsilon^1, z]$ and ${}^*\gamma = [\varepsilon^1, z^{-1}]$, where ε^1 is the trivial line bundle over a point. We have $\gamma \otimes {}^*\gamma \approx \varepsilon^1$ where ε^1 is the trivial line bundle over $\mathbb{C}\mathrm{P}^1 = S^2$.

4.2.5 Proposition. *Let ξ be isomorphic to $[\eta, \alpha]$ over $X \times S^2$ for some vector bundle η over X and Laurent polynomial clutching map α, then $[\eta, z^n\alpha]$ is isomorphic to $\xi\hat{\otimes}\gamma^n$ and $[\eta, z^{-n}\alpha]$ is isomorphic to $\xi\hat{\otimes}{}^*\gamma^n$ where γ^n denotes the n-fold tensor product of γ and $\hat{\otimes}$ is the external tensor product.*

Proof: By the properties of the tensor product for clutching maps, we have that $\pi_0^*(\eta) \cup_{z^n\alpha} \pi_\infty^*(\eta)$ is isomorphic to $(\pi_0^*(\eta) \otimes \varepsilon^1) \cup_{\alpha\otimes z^n} (\pi_\infty^*(\eta) \otimes \varepsilon^1)$. The proposition now follows since $(\pi_0^*(\eta) \otimes \varepsilon^1) \cup_{\alpha\otimes z^n} (\pi_\infty^*(\eta) \otimes \varepsilon^1)$ and $\xi\hat{\otimes}\gamma^n$ are isomorphic. Likewise for ${}^*\gamma$. □

It follows that for a monomial clutching map az^n we have $[\eta, az^n] \approx \eta\hat{\otimes}\gamma^n$.

4.3 Linear clutching maps

Let η be a vector bundle over X. Any Laurent polynomial clutching function α for η can be written $\alpha = z^{-n}p$ for some polynomial clutching function $p = \sum_{0\leq k\leq n} p_k z^k$ where $p_k : \eta \to \eta$ is a vector bundle endomorphism for $|k| \leq n$. Then we have $[\eta, z^{-n}p] \approx [\eta, p]\hat{\otimes}^* \gamma^n$. The next step is to linearise the polynomial clutching map p.

For a vector bundle η with polynomial clutching function $p = \sum_{0\leq k\leq n} p_k z^k$ let $L^n(p)$ denote the linear polynomial clutching function for the vector bundle $L^n(\eta) = (n+1)\eta$ given by the matrix,

$$\begin{bmatrix} p_0 & p_1 & \dots & p_{n-1} & p_n \\ -z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -z & 1 \end{bmatrix}$$

Then $L^n(p)$ can be obtained as the product of three matrices,

$$\begin{bmatrix} 1 & p_1^* & \dots & p_n^* \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} p & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ -z & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

where $p_l^*(z) = \sum_{l\leq k\leq n} p_k z^{k-l}$ and $p_l^* - z p_{l+1}^* = p_l$. We then have $L^n(p) = (1+N_1)(p \oplus 1)(1+N_2)$ where N_1 and N_2 are nil-potent. Hence, we obtain a homotopy of clutching maps $L_t^n(p) = (1+tN_1)(p\oplus 1)(1+tN_2)$ of $L^n(\eta)$, from $p \oplus 1$ to $L^n(p)$. This gives the following proposition,

4.3.1 Proposition. *For a vector bundle η over X and polynomial clutching function $p = \sum_{0\leq k\leq n} p_k z^k$, the vector bundles $[L^n(\eta), L^n(p)]$ and $[L^n(\eta), p\oplus 1]$ are isomorphic.*

For the following propositions, we may view $p = \sum_{0\leq k\leq n} p_k z^k$ as a polynomial clutching map of degree $n+1$ with $p_{n+1} = 0$.

4.3.2 Proposition. *Let η be a vector bundle over X and let $p = \sum_{0\leq k\leq n} p_k z^k$ be a polynomial clutching map of η. Then $[L^{n+1}(\eta), L^{n+1}(p)]$ is isomorphic to $[L^n(\eta), L^n(p)] \oplus [\eta, 1]$.*

Proof: The proposition follows using the homotopy of clutching maps,

$$\begin{bmatrix} p_0 & p_1 & \dots & p_n & 0 \\ -z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(1-tz) & 1 \end{bmatrix}$$

Then for $t = 0$ we have $[L^{n+1}(\eta), L^{n+1}(p)]$ and for $t = 1$ we have $[L^n(\eta), L^n(p)] \oplus [\eta, 1]$. □

4.3.3 Proposition. *Let η be a vector bundle over X and let $p = \sum_{0\leq k\leq n} p_k z^k$ be a polynomial clutching map of η. Then $[L^{n+1}(\eta), L^{n+1}(zp)]$ is isomorphic to $[L^n(\eta), L^n(p)] \oplus [\eta, z]$.*

Proof: As above, we obtain the result using the homotopy of clutching maps,

$$\begin{bmatrix} 0 & p_0 & p_1 & \dots & p_n \\ -z & 1-t & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -z & 1 \end{bmatrix}$$

Then for $t = 0$ we have $[L^{n+1}(\eta), L^{n+1}(zp)]$ and for $t = 1$ we have $[L^n(\eta), L^n(p)] \oplus [\eta, z]$. □

As a corollary we obtain,

4.3.4 Corollary. *For the canonical line bundle γ over $S^2 = \mathbb{C}P^1$ and the dual bundle ${}^*\gamma$ we have $\gamma^2 \oplus \varepsilon^1 \approx \gamma \oplus \gamma$ or equivalently ${}^*\gamma^{-2} \oplus \varepsilon^1 \approx {}^*\gamma^{-1} \oplus {}^*\gamma^{-1}$*

Proof: By proposition 4.3.3 there is an isomorphism from $[\varepsilon^2, z^2] = [L^2(\varepsilon^1), z^2]$ to $[\varepsilon^1, z] \oplus [\varepsilon^1, z] = \gamma \oplus \gamma$, and by proposition 4.3.1 there is an isomorphism from $[\varepsilon^2, z^2]$ to $[\varepsilon^1, z^2] \oplus [\varepsilon^1, 1] = \gamma^2 \oplus \varepsilon^1$. □

Let η be a vector bundle over X and let $p(x, z) = a(x)z + b(x)$ be a linear clutching map of η. We wish to prove that η decomposes as a Whitney sum $\eta = \eta^0_+ \oplus \eta^0_-$ such that the restrictions $p|_{\eta^0_+} : \eta^0_+ \to \eta^0_+$ are nonsingular for all $x \in X$ and $|z| \geq 1$, and $p|_{\eta^0_-} : \eta^0_- \to \eta^0_-$ is nonsingular for all $x \in X$ and $|z| \leq 1$. Then the vector bundle $[\eta, a(x)z + b(x)]$ is isomorphic to $[\eta^0_+, z] \oplus [\eta^0_-, 1]$.

The linear clutching map $p(x, z) = a(x)z + b(x)$ is nonsingular for all x and $|z| = 1$. By compactness of X, there exists a $\delta > 0$ such that p is singular for all x and $1-\delta \leq |z| \leq 1+\delta$.

We define,

$$p_0(x) = \frac{1}{2\pi i} \int_{|z|=1} [a(x)z + b(x)]^{-1} a(x) dz$$
$$p_\infty(x) = \frac{1}{2\pi i} \int_{|z|=1} a(x)[a(x)z + b(x)]^{-1} dz$$

where $p_0 : \eta \to \eta$ and $p_\infty : \eta \to \eta$ are bundle morphisms independent of z.

4.3.5 Proposition. *For a vector bundle η over X and a linear clutching map $p(x, z)$, let p_0 and p_∞ be defined as above. Then $p(x, z)p_0(x) = p_\infty(x)p(x, z)$ for all $x \in X$ and $|z| = 1$. Further p_0 and p_∞ are projections, i.e. $p_0(x)p_0(x) = p_0(x)$ and $p_\infty(x)p_\infty(x) = p_\infty(x)$.*

Proof: First, let z, w denote complex variables. Then for $z \neq w$ we have the following relation,

$$\begin{aligned}[aw + b]^{-1} a [az + b]^{-1} &= [az + b]^{-1} a [aw + b]^{-1} \\ &= \frac{[az + b]^{-1}}{w - z} + \frac{[aw + b]^{-1}}{z - w}\end{aligned}$$

This follows using the symmetry of z and w since,

$$\begin{aligned}\frac{[az+b]^{-1}}{w-z}+\frac{[aw+b]^{-1}}{z-w}&=[aw+b]^{-1}\frac{aw+b}{z-w}[az+b]^{-1}+[aw+b]^{-1}\frac{az+b}{w-z}[az+b]^{-1}\\&=[aw+b]^{-1}\left[\frac{aw+b}{w-z}-\frac{az+b}{w-z}\right][az+b]^{-1}\\&=[aw+b]^{-1}a[az+b]^{-1}\end{aligned}$$

Note: This relation also holds if $w=z$.

To establish the relation $p(x,z)p_0(x)=p_\infty(x)p(x,z)$ we multiply by $[az+b]$ on the left,

$$\begin{aligned}[az+b]p_0&=\frac{1}{2\pi i}\int_{|z|=1}[az+b][aw+b]^{-1}adw\\&=\frac{1}{2\pi i}\int_{|z|=1}[az+b][aw+b]^{-1}a[az+b]^{-1}[az+b]dw\\&=\frac{1}{2\pi i}\int_{|z|=1}[az+b][az+b]^{-1}a[aw+b]^{-1}[az+b]dw\\&=\frac{1}{2\pi i}\int_{|z|=1}a[aw-b]^{-1}[az+b]dw\\&=p_\infty[az+b]\end{aligned}$$

To show that p_0 is a projection, chose $1-\delta<r_1<r_2<1+\delta$. Then for $|z|=r_2>r_1$ we have,

$$\int_{w=|r_1|}\frac{dw}{w-z}=0$$

From this we obtain,

$$\begin{aligned}p_0p_0&=\frac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}[az+b]^{-1}a[aw+b]^{-1}adzdw\\&=\frac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}\frac{[az+b]^{-1}}{w-z}a+\frac{[aw+b]^{-1}}{z-w}adzdw\\&=\frac{1}{2\pi i}\int_{|z|=r_1}[aw+b]^{-1}adw\\&=p_0\end{aligned}$$

The proof for p_∞ follows the same pattern. □

For $[\eta,p]$ let p_0 and p_∞ be defined as above. We denote the vector bundle $\mathrm{im}p_0$ by η_+^0 and $\ker p_0$ by η_-^0, hence, $\eta=\eta_+^0\oplus\eta_-^0$. Similarly, $\eta_+^\infty=\mathrm{im}p_\infty$, $\eta_-^\infty=\ker p_\infty$, and $\eta=\eta_+^\infty\oplus\eta_-^\infty$. From the relation $p(x,z)p_0(x)=p_\infty(x)p(x,z)$ we then have the following restrictions of $p(x,z)$,

$$\begin{aligned}p_+(\cdot,z)&:\eta_+^0\to\eta_+^\infty\\p_-(\cdot,z)&:\eta_-^0\to\eta_-^\infty\end{aligned}$$

4.3.6 Proposition. *The restriction $p_+(\cdot, z) : \eta_+^0 \to \eta_+^\infty$ is an isomorphism for $|z| \geq 1$ and $p_-(\cdot, z) : \eta_-^0 \to \eta_-^\infty$ is an isomorphism for $|z| \leq 1$.*

Proof: Let v be in the fibre of η over x such that $[a(x)w + b(x)]v = 0$ for $|w| \neq 1$. Then $[a(x)w+b(x)]v = (z-w)a(x)v$ and $[a(x)z+b(x)]^{-1}a(x)v = (z-w)^{-1}v$ for $|z| = 1$. Integrating over S^1 we get the relations,

$$p_0(x)v = \begin{cases} v & \text{for } |w| < 1 \\ 0 & \text{for } |w| > 1 \end{cases}$$

If $v \in \ker[a(x)w + b(x)]$ and $|w| < 1$ then $v \in \eta_+^0$ and p_- is an monomorphism for $|z| \leq 1$. If $v \in \ker[a(x)w + b(x)]$ and $|w| > 1$ then $v \in \eta_-^0$. Hence, p_+ is an monomorphism for $|z| \geq 1$. For dimensional reasons, p_+ and p_- are epimorphisms for $|z| \geq 1$ and $|z| \leq 1$ respectively. Hence, p_+ and p_- are isomorphisms on D_∞ and D_0 respectively. □

4.3.7 Proposition. *Let $p_+ = a_+z + b_+$ and $p_- = a_-z + b_-$ where p_+ and p_- are defined as above, and let $p^t = p_+^t + p_-^t$ where $p_+^t = a_+z + tb_+$ and $p_- = ta_-z + b_-$ for $0 \leq t \leq 1$. Then p^t is a homotopy of linear clutching maps from $a_+z + b_-$ to p. Further, the vector bundles $[\eta, p]$ and $[\eta_+^0, z] \oplus [\eta_-^0, 1]$ are isomorphic.*

Proof: By proposition 4.3.6, p_+^t and p_-^t are isomorphisms onto their respective images for all $0 \leq t \leq 1$. Then $[\eta, p]$ is isomorphic to $(\eta_+^0 \cup_{a_+z} \eta_+^\infty) \oplus (\eta_-^0 \cup_{b_-} \eta_-^\infty)$ over $X \times S^2$. Since $a_+ : \eta_+^0 \to \eta_+^\infty$ and $b_- : \eta_-^0 \to \eta_-^\infty$ are isomorphisms, there are isomorphisms $[\eta_+^0, z] \to \eta_+^0 \cup_{a_+z} \eta_+^\infty$ and $[\eta_-^0, 1] \to \eta_-^0 \cup_{b_-} \eta_-^\infty$. □

To sum up the results of the section, let η be a vector bundle over X with polynomial clutching map p with $\deg p \leq n$. Then the vector bundle $L^n(\eta) = (n+1)\eta$ decomposes with respect to the linear clutching map $L^n(p)$. We denote this vector bundle by,

$$L^n(\eta) = L^n(\eta, p)_+ \oplus L^n(\eta, p)_-$$

By proposition 4.3.7 we have $[L^n(\eta), L^n(p)] \approx [L^n(\eta, p)_+, z] \oplus [L^n(\eta, p)_-, 1]$. Thus,

4.3.8 Proposition. *Let η be a vector bundle over X and let $p(x, z)$ be a polynomial clutching map for η with $\deg p \leq n$. Then for $L^{n+1}(p)$ we have,*

$$\begin{aligned} L^{n+1}(\eta, p)_+ &\approx L^n(\eta, p)_+ \\ L^{n+1}(\eta, p)_- &\approx L^n(\eta, p)_- \oplus \eta \end{aligned}$$

and for $L^{n+1}(zp)$ we have,

$$\begin{aligned} L^{n+1}(\eta, zp)_+ &\approx L^n(\eta, p)_+ \oplus \eta \\ L^{n+1}(\eta, zp)_- &\approx L^n(\eta, p)_- \end{aligned}$$

4.4 The periodicity isomorphism

The periodicity homomorphism $\mu : K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2) \to K_{\mathbb{C}}(X \times S^2)$ is the external K-cup product as defined in section 3.3. With the results of the previous sections we can now define a morphism $\nu : K_{\mathbb{C}}(X \times S^2) \to K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2)$ and show that this is an inverse of μ.

Let η be a vector bundle over X and let α be a clutching map. By proposition 4.2.3 there exists a Laurent polynomial clutching map $z^{-n}p_n(x,z)$ homotopic to α, where $p_n(x,z)$ is a polynomial with $\deg p_n \leq 2n$. We then define

$$\nu_n(\eta, \alpha) = L^{2n}(\eta, p_n)_+ \otimes ({}^*\gamma^{n-1} - {}^*\gamma^n) + \eta \otimes {}^*\gamma^n$$

Since $(1 - {}^*\gamma){}^*\gamma = (1 - {}^*\gamma)$ by corollary 4.3.4 this reduces to

$$\nu_n(\eta, \alpha) = L^{2n}(\eta, p_n)_+ \otimes (1 - {}^*\gamma) + \eta \otimes {}^*\gamma^n$$

This defines a morphism $\nu_n : K(X \times S^2) \to K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2)$

4.4.1 Proposition. *The morphism $\nu_n : K(X \times S^2) \to K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2)$ is independent of n, i.e. $\nu_n(\eta, \alpha) = \nu_{n+1}(\eta, \alpha)$.*

Proof: From proposition 4.3.8 we have

$$L^{2n+2}(\eta, p_{n+1})_+ = L^{2n+2}(\eta, zp_{n+1})_+ = L^{2n+1}(\eta, zp_n)_+ = L^{2n}(\eta, p_n)_+ \oplus \eta$$

Hence, we obtain

$$\begin{aligned}
\nu_{n+1}(\eta, \alpha) &= L^{2n+2}(\eta, p_{n+1})_+ \otimes ({}^*\gamma^n - {}^*\gamma^{n+1}) + \eta \otimes {}^*\gamma^{n+1} \\
&= L^{2n}(\eta, p_n)_+ \otimes (1 - {}^*\gamma) + \eta \otimes ({}^*\gamma^n - {}^*\gamma^{n+1}) + \eta \otimes {}^*\gamma^{n+1} \\
&= L^{2n}(\eta, p_n)_+ \otimes (1 - {}^*\gamma) + \eta \otimes {}^*\gamma^n \\
&= \nu_n(\eta, \alpha)
\end{aligned}$$

□

Since ν_n is independent of n we can drop the index from notation and write ν for this morphism. Let α and α' be clutching functions of the vector bundle η such that $[\eta, \alpha] \approx [\eta, \alpha'] \approx \xi$ and let $z^{-n}p$ and $z^{-n}p'$ be Laurent polynomial clutching maps for α and α' respectively. Then the line segment joining p with p' defines a homotopy of clutching maps. Hence, $\nu([\eta, \alpha]) = \nu([\eta, \alpha'])$ and we simply write $\nu(\xi)$.

Note that for vector bundles ξ, ξ' over $X \times S^2$ we have $\nu(\xi \oplus \xi') = \nu(\xi) + \nu(\xi)$. Hence, $\nu : K_{\mathbb{C}}(X \times S^2) \to K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2)$ is a well defined homomorphism.

4.4.2 Theorem. *The external K-cup product $\mu : K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2) \to K_{\mathbb{C}}(X \times S^2)$ is an isomorphism with $\nu : K_{\mathbb{C}}(X \times S^2) \to K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2)$ as inverse.*

Proof: We first prove that $\nu\mu = 1$. It is sufficient to show that $\nu\mu(\eta \otimes {}^*\gamma^n) = \eta \hat{\otimes} {}^*\gamma^n$ where η is a vector bundle over X. Since $\mu(\eta \otimes {}^*\gamma^n) = [\eta, z^{-n}]$ by proposition 4.2.5 we have,

$$\nu([\eta, z^{-n}]) = L^{2n}(\eta, 1)_+ \otimes ({}^*\gamma^{n-1} - {}^*\gamma^n) + \eta \otimes {}^*\gamma^n = \eta \otimes {}^*\gamma^n$$

Since $L^{2n}(\eta,1)_+ = L^0(\eta,1)_+ = 0$. Hence, $\nu\mu = 1$.

For $\mu\nu = 1$. we have the following relations by proposition 4.3.7

$$[L^{2n}(\eta,p_n)_+, z] = [L^{2n}(\eta), L^{2n}(p_n)] - [L^{2n}(\eta,p_n)_-, 1]$$
$$[L^{2n}(\eta,p_n)_-, z^{1-n}] = [L^{2n}(\eta), L^{2n}(p_n)] \otimes {}^*\gamma^n - [L^{2n}(\eta,p_n)_-, 1] \otimes {}^*\gamma^n$$

where the second relation follows from proposition 4.3.2. Now,

$$[L^{2n}(\eta), L^{2n}(p_n)] = [\eta, p_n(x,z)] + 2n\eta$$

Let ξ be a vector bundle over $X \times S^2$ with $\xi \approx [\eta, \alpha]$ for some vector bundle η over X and clutching map α. We then have,

$$\nu(\xi) = \nu([\eta,\alpha]) = L^{2n}(\eta,p_n)_+ \otimes ({}^*\gamma^{n-1} - {}^*\gamma^n) + \eta \otimes {}^*\gamma^n$$

Hence,

$$\begin{aligned}
\mu\nu(\xi) &= [L^{2n}(\eta,p_n)_+, z^{1-n}] - [L^{2n}(\eta,p_n), z^{-n}] + [\eta, z^{-n}] \\
&= [L^{2n}(\eta), L^{2n}(p_n)] \otimes {}^*\gamma^n - [L^{2n}(\eta,p_n)_-, 1] \otimes {}^*\gamma^n - [L^{2n}(\eta,p)_+, 1] \otimes {}^*\gamma^n + [\eta, z^{-n}] \\
&= 2n(\eta \otimes {}^*\gamma^n) + [\eta, z^{-n}p_n] - [L^{2n}(\eta), 1] \otimes {}^*\gamma^n + \eta \otimes {}^*\gamma^n \\
&= [\eta, z^{-n}p_n(x,z)] \\
&= \xi
\end{aligned}$$

This proves the theorem. □

4.4.3 Example. From the periodicity theorem we have that $K_{\mathbb{C}}(S^2) = \mathbb{Z} \oplus \mathbb{Z}$. This is generated by $\langle 1, \gamma \rangle$ where γ is the canonical vector bundle over $S^2 = \mathbb{C}\mathrm{P}^1$, with the multiplication determined by corollary 4.3.4, $\gamma^2 = 2\gamma - 1$. We may rewrite this as $(\gamma - 1)^2 = 0$. Hence, $K_{\mathbb{C}}(S^2)$ is the truncated polynomial ring $\mathbb{Z}[\gamma]/(\gamma-1)^2$.

Regarding $\tilde{K}_{\mathbb{C}}(S^2)$ as the kernel of $K_{\mathbb{C}}(S^2) \to K_{\mathbb{C}}(x_0)$, then $\tilde{K}_{\mathbb{C}}(S^2)$ is generated by $(\gamma - 1)$. Multiplication is trivial since we have the relation $(\gamma-1)^2 = 0$

Using the Bott periodicity we may extend the long exact sequence of chapter 3 and define rings $K^n(X,Y)$ for positive n,

$$\cdots \longrightarrow K(X,A) \longrightarrow K(X) \longrightarrow K(A) \longrightarrow K^1(X,A) \longrightarrow K^1(X) \longrightarrow K^1(A) \longrightarrow \cdots$$

and it is common to define $K^*_{\mathbb{C}}(X) = K^0_{\mathbb{C}}(X) \cup K^1_{\mathbb{C}}(X)$.

For the sphere S^n, $\tilde{K}_{\mathbb{F}}(S^n)$ is determined by the homotopy group $\pi_{n-1}(U_{\mathbb{F}})$ where $U_{\mathbb{F}} = \cup_n U(n)$ for $\mathbb{F} = \mathbb{C}$ and $U_{\mathbb{F}} = \cup_n O(n)$ for $\mathbb{F} = \mathbb{R}$ [15, thm. 5.1, p. 120]. Since $S^n = S^n(*)$, this yields,

$n \equiv$	0	−1	−2	−3	−4	−5	−6	−7	−8
$\tilde{K}^n_{\mathbb{C}}(*) =$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\tilde{K}^n_{\mathbb{R}}(*) =$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

5

Algebraic structures of K(X)

In this chapter, we introduce further algebraic operations on the rings $K_{\mathbb{F}}(X)$, namely a set of morphisms $\Psi^k_{\mathbb{F}} : K_{\mathbb{F}}(X) \to K_{\mathbb{F}}(X)$. These operations are known as the Adams operations and were introduced by J. F. Adams in 1962, [2]. The Adams operations are studied in the first section. In the second section we study the Chern classes and Chern character which provide a connection with ordinary cohomology. Finally we briefly consider spectral sequences and state some results due to Atiyah and Hirzebruch [6].

5.1 The Adams operations

For the construction of the Adams operations, Adams used representation theory to obtain the results for real and complex vector bundles [2, sec. 4]. Restricting to complex vector bundles the Adams operations can be obtained using the splitting principle, which in turn relies on the Leray-Hirsch theorem [15, thm. 1.1, p. 245].

It is beyond the scope of this thesis to give a full account on the construction of the Adams operations, we will only give an account for the complex case. However, unless explicitly stated otherwise, the properties of the Adams operations also hold for real vector bundles. We state the splitting principle without proof, for a proof we refer to [16, chp. IV, thm. 2.15].

5.1.1 Theorem (The splitting principle). *Let ξ be a complex vector bundle over X. Then there is a space Y and a map $f : Y \to X$ such that*

i) $f^*(\xi)$ decomposes as a Whitney sum of line bundles.

ii) $f^\# : K_{\mathbb{C}}(X) \to K_{\mathbb{C}}(Y)$ is injective.

5.1.2 Corollary. *For any finite set of vector bundles $\xi_1, \ldots, \xi_n$ there is a space Y and a map $f : Y \to X$ such that*

i) For all i, $f^*(\xi_i)$ decomposes as a Whitney sum of line bundles.

ii) $f^\# : K_{\mathbb{C}}(X) \to K_{\mathbb{C}}(Y)$ is injective.

Proof: This follows by repeated use of the splitting principle and the naturality of f^* since line bundles are pulled back to line bundles. □

5.1.3 Theorem. *For a compact Hausdorff space X there exists ring homomorphisms $\Psi^k_{\mathbb{F}}$: $K_{\mathbb{F}}(X) \to K_{\mathbb{F}}(X)$, defined for $k \geq 0$ satisfying the following properties,*

i) For a map $f : X \to Y$, $\Psi^k_{\mathbb{F}} f^\# = f^\# \Psi^k_{\mathbb{F}}$

ii) For a line bundle γ, $\Psi^k_{\mathbb{F}}(\gamma) = \gamma^k$

Proof: To prove the theorem we first construct operations for the class of sums of line bundles satisfying the above properties for the isomorphism classes of vector bundles. Then, using the splitting principle, we show that these operations satisfies the properties above when applied to arbitrary classes of vector bundles. Finally we show that these operations induce operations on $K_{\mathbb{F}}(X)$ satisfying the properties stated in the theorem.

First note that for a sum of line bundles $\xi = \gamma_1 \oplus \cdots \oplus \gamma_n \in \mathrm{Vect}_{\mathbb{F}}(X)$, it follows from the properties of Whitney sum and tensor product that the properties of the theorem are satisfied if

$$\Psi^k_{\mathbb{F}}(\gamma_1 \oplus \cdots \oplus \gamma_n) = \gamma_1^k \oplus \cdots \oplus \gamma_n^k$$

We wish to define a general formula that specialises to this for sums of line bundles.

Recall the properties of exterior products for vector bundles from p. 12, clearly these also hold for isomorphism classes of vector bundles. For a sum of line bundles, $\xi = \gamma_1 \oplus \cdots \oplus \gamma_n \in \mathrm{Vect}_{\mathbb{F}}(X)$, we define a polynomial with coefficients in $\mathrm{Vect}_{\mathbb{F}}(X)$ by

$$\lambda_t(\xi) = \sum_i \lambda^i(\xi) t^i$$

By property iv) of exterior products this sum is finite, and generalising property i) of exterior products we obtain,

$$\lambda_t(\xi) = \prod_i \lambda_t(\gamma_i)$$

Since $\lambda^0(\gamma) = 1$, $\lambda^1(\gamma) = \gamma$ and $\lambda^k(\gamma) = 0$ for $k > 1$ this formula can be rewritten,

$$\lambda_t(\xi) = \prod_i (1 + \gamma_i t)$$

Thus we see that the coefficient $\lambda^j(\xi)$ of the t^j'th term is the j'th elementary symmetric function σ_j of the γ_i's, namely the sum of all products of j distinct line bundles,

$$\lambda^j(\xi) = \sigma_j(\gamma_1, \ldots, \gamma_n)$$

The fundamental theorem of symmetric polynomials assert that any symmetric polynomial of degree k in n variables can be expressed uniquely as a polynomial in $\sigma_1, \ldots, \sigma_k$, where σ_i is the i'th elementary symmetric polynomial of n variables. In particular viewing $\gamma_1^k \oplus \cdots \oplus \gamma_n^k$ as a polynomial of degree k in n this can be expressed as a polynomial $s_k(\sigma_1, \ldots, \sigma_k)$ called the *Newton polynomial* of degree k. The Newton polynomials are independent of the number of variables, we can pass from n to $n-1$ variables by letting terms with the n'th variable vanish.

Now we define for arbitrary classes of vector bundles the operation $\Psi^k_{\mathbb{F}} : \mathrm{Vect}_{\mathbb{F}}(X) \to \mathrm{Vect}_{\mathbb{F}}(X)$ by the formula

$$\Psi^k_{\mathbb{F}}(\xi) = s_k(\lambda^1(\xi), \dots, \lambda^k(\xi))$$

Clearly, it follows from the properties of exterior products that the naturality property i) is satisfied for arbitrary classes of vector bundles, and from the discussion above property ii) is clear.

We next verify that the operations are additive and multiplicative. This follows using the splitting principle and the naturality of the operations proved above. Let $\xi \oplus \xi' \in \mathrm{Vect}_{\mathbb{F}}(X)$ be a class of bundles and let $f : Y \to X$ be a splitting such that $f^*(\xi) = \bigoplus_i \gamma_i$ and $f^*(\xi') = \bigoplus_j \gamma'_j$. Then we have

$$\begin{aligned} f^\# \Psi^k_{\mathbb{F}}(\xi \oplus \xi') &= \Psi^k_{\mathbb{F}} f^\#(\xi \oplus \xi') \\ &= \Psi_{\mathbb{F}}(\gamma_1 \oplus \cdots \oplus \gamma_m \oplus \gamma'_1 \oplus \cdots \oplus \gamma'_n) \\ &= \gamma_1^k \oplus \cdots \oplus \gamma_m^k \oplus \gamma_1'^k \oplus \cdots \oplus \gamma_n'^k \\ &= \Psi^k_{\mathbb{F}}(\gamma_1 \oplus \cdots \oplus \gamma_m) \oplus \Psi^k_{\mathbb{F}}(\gamma'_1 \oplus \cdots \oplus \gamma'_n) \\ &= \Psi^k_{\mathbb{F}} f^\#(\xi) \oplus \Psi^k_{\mathbb{F}} f^\#(\xi') \\ &= f^\#(\Psi^k_{\mathbb{F}}(\xi) \oplus \Psi^k_{\mathbb{F}}(\xi')) \end{aligned}$$

To show multiplicity, using the splitting principle on each factor, the product $\xi \otimes \xi'$ decomposes as a sum of terms of the form $\gamma_i \otimes \gamma'_j$. Thus we obtain

$$\begin{aligned} f^\#(\Psi^k_{\mathbb{F}}(\xi \otimes \xi')) &= \Psi^k_{\mathbb{F}} f^\#(\xi \otimes \xi') \\ &= \Psi^k_{\mathbb{F}}(f^\#(\xi) \otimes f^\#(\xi')) \\ &= \Psi^k_{\mathbb{F}}((\bigoplus_i \gamma_i) \otimes (\bigoplus_j \gamma'_j)) \\ &= \Psi^k_{\mathbb{F}}(\bigoplus_{i,j} \gamma_i \otimes \gamma'_j) \\ &= \bigoplus_{i,j} \Psi^k_{\mathbb{F}}(\gamma_i \otimes \gamma'_j) \\ &= \bigoplus_{i,j} \Psi^k_{\mathbb{F}}(\gamma_i) \otimes \Psi^k_{\mathbb{F}}(\gamma'_j) \\ &= \left(\bigoplus_i \Psi^k_{\mathbb{F}}(\gamma_i) \right) \otimes \left(\bigoplus_j \Psi^k_{\mathbb{F}}(\gamma'_j) \right) \\ &= \Psi^k_{\mathbb{F}}(\bigoplus_j \gamma_i) \otimes \Psi^k_{\mathbb{F}}(\bigoplus_j \gamma'_j) \\ &= \Psi^k_{\mathbb{F}} f^\#(\xi) \otimes \Psi^k_{\mathbb{F}} f^\#(\xi') \\ &= f^\#(\Psi^k_{\mathbb{F}}(\xi) \otimes \Psi^k_{\mathbb{F}}(\xi')) \end{aligned}$$

Since the operations $\Psi^k_{\mathbb{F}}$ preserves sums and products of classes of vector bundles, they induce ring homomorphisms $\Psi^k_{\mathbb{F}} : K_{\mathbb{F}}(X) \to K_{\mathbb{F}}(X)$ by $\Psi^k_{\mathbb{F}}(\xi - \eta) = \Psi^k_{\mathbb{F}}(\xi) - \Psi^k_{\mathbb{F}}(\eta)$. Additivity is clear and multiplicativity follows by simple calculations using the multiplication rules on p. 22. □

5.1.4 Definition. The operations $\Psi^k_{\mathbb{F}} : K_{\mathbb{F}}(X) \to K_{\mathbb{F}}(X)$ introduced in theorem 5.1.1 are called the *Adams operations.*

5.1.5 Theorem. *The Adams operations satisfy the following equation*

$$\Psi^k_{\mathbb{F}} \circ \Psi^l_{\mathbb{F}} = \Psi^{kl}_{\mathbb{F}}$$

Proof: Using the splitting principle, we only need to prove the theorem for a line bundle $\gamma \in \mathrm{Vect}_{\mathbb{F}}(X)$. For a line bundle γ we have,

$$\begin{aligned}
\Psi^k_{\mathbb{F}} \circ \Psi^l_{\mathbb{F}}(\gamma) &= \Psi^k_{\mathbb{F}}(\gamma^l) \\
&= \Psi^k_{\mathbb{F}}(\gamma) \otimes \cdots \otimes \Psi^k_{\mathbb{F}}(\gamma) \\
&= \gamma^k \otimes \cdots \otimes \gamma^k \\
&= \gamma^{kl}
\end{aligned}$$

□

5.1.6 Theorem. *The following diagram is commutative,*

$$\begin{array}{ccc}
K_{\mathbb{R}}(X) & \xrightarrow{\Psi^k_{\mathbb{R}}} & K_{\mathbb{R}}(X) \\
{\scriptstyle C}\downarrow & & \downarrow{\scriptstyle C} \\
K_{\mathbb{C}}(X) & \xrightarrow{\Psi^k_{\mathbb{C}}} & K_{\mathbb{C}}(X)
\end{array}$$

Proof: Complexification commutes with sums and products, and so with exterior products. The result now follows. □

The Adams operations can be extended to negative powers consistent with the results above as follows:

5.1.7 Definition. For $K_{\mathbb{C}}(X)$ we define

$$\Psi^{-1}_{\mathbb{C}} = T$$

and for $K_{\mathbb{R}}(X)$ we define

$$\Psi^{-1}_{\mathbb{R}} = \Psi^1_{\mathbb{R}}$$

This definition extends the Adams operations to negative powers by the formula $\Psi^{-k}_{\mathbb{F}} = \Psi^{-1}_{\mathbb{F}} \Psi^k_{\mathbb{F}}$.

The Adams operations restricts to similar operations in reduced K-theory, $\Psi^k_{\mathbb{F}} : \tilde{K}_{\mathbb{F}}(X) \to \tilde{K}_{\mathbb{F}}(X)$ by property i) since $\tilde{K}_{\mathbb{F}}(X)$ is the kernel of the map $K_{\mathbb{F}}(X) \to K_{\mathbb{F}}(x_0)$ for some $x_0 \in X$. For the reduced external K-cup product we then have,

$$\begin{aligned}
\Psi^k_{\mathbb{F}}(\xi \hat{\otimes} \eta) &= \Psi^k_{\mathbb{F}}(p^{\#}_X(\xi) \otimes p^{\#}_Y(\eta)) \\
&= \Psi^k_{\mathbb{F}}(p^{\#}_X(\xi)) \otimes \Psi^k_{\mathbb{F}}(p^{\#}_Y(\eta)) \\
&= p^{\#}_X(\Psi^k_{\mathbb{F}}(\xi)) \otimes p^{\#}_Y(\Psi^k_{\mathbb{F}}(\eta)) \\
&= \Psi^k_{\mathbb{F}}(\xi) \hat{\otimes} \Psi^k_{\mathbb{F}}(\eta)
\end{aligned}$$

5.1.8 Proposition. *The Adams operations*

$$\Psi^k_{\mathbb{F}} : \tilde{K}_{\mathbb{F}}(S^{2n}) \to \tilde{K}_{\mathbb{F}}(S^{2n})$$

where n is even for $\mathbb{F} = \mathbb{R}$, are given by

$$\Psi^k_{\mathbb{F}}(\xi) = k^n \xi$$

Proof: Since $\Psi^k_{\mathbb{C}}$ is additive it suffices to show the theorem for generators. For $\tilde{K}_{\mathbb{C}}(S^2)$ we have from example 4.4.3 the generator $\mu = \gamma - 1$, where γ is the canonical line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$. Since $(\gamma - 1)^2 = 0$ we have,

$$\begin{aligned}\Psi^k_{\mathbb{C}}(\alpha) &= \Psi^k_{\mathbb{C}}(\gamma - 1) \\ &= \gamma^k - 1 \\ &= (\alpha + 1)^k - 1 \\ &= 1 + k\alpha - 1 \\ &= k\alpha\end{aligned}$$

Since $S^{2n} = S^2 \wedge \cdots \wedge S^2$, $\tilde{K}_{\mathbb{C}}(S^{2n})$ is generated by $\mu \otimes \cdots \otimes \mu$ and the result follows.

For $\mathbb{F} = \mathbb{R}$ and n even it follows from the work of Bott[9] that the complexification

$$C : \mathbb{Z} = \tilde{K}_{\mathbb{R}}(S^{2n}) \to \tilde{K}_{\mathbb{C}}(S^{2n}) = \mathbb{Z}$$

is monomorphic, and so the result follows from the complex case. □

5.1.9 Remark. In fact, it follows from the work of Bott[9] that $C : \tilde{K}_{\mathbb{R}}(S^{2n}) \to \tilde{K}_{\mathbb{C}}(S^{2n})$ is an isomorphism for $n \equiv 0 \mod 4$ and $\mathrm{im} C = 2\mathbb{Z}$ for $n \equiv 2 \mod 4$.

5.2 Characteristic classes

In this section we provide a connection to ordinary cohomology, but the exposition given here will be limited to definitions and stating the properties we need for our further studies, no proofs will be given. For a more extensive expositions we refer to [19, chp. 18] and [15, chp. 17].

For a space X there is a sequence of morphisms $c_k : \mathrm{Vect}_{\mathbb{C}}(X) \to H^{2k}(X;\mathbb{Z})$ assigning to each class $\xi \in \mathrm{Vect}_{\mathbb{C}}(X)$ a class $c_k(\xi) \in H^{2k}(X;\mathbb{Z})$ satisfying the following properties:

i) $c_0(\xi) = 1$ and $c_k(\xi) = 0$ for $\dim(\xi) < k$

ii) If $f : Y \to X$ is a map, then the diagram

$$\begin{array}{ccc}\mathrm{Vect}_{\mathbb{C}}(X) & \xrightarrow{c_k} & H^{2k}(X;\mathbb{Z}) \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{Vect}_{\mathbb{C}}(Y) & \xrightarrow{c_k} & H^{2k}(Y;\mathbb{Z})\end{array}$$

commutes for all k, where the vertical arrows are the maps induced by f.

iii) For the canonical line bundle γ over $S^2 = \mathbb{C}\mathrm{P}^1$, $c_1(\gamma)$ is a generator of $H^2(S^2;\mathbb{Z})$.

5.2.1 Theorem. *For vector bundles $\xi, \eta \in \mathrm{Vect}_{\mathbb{C}}(X)$,*

$$c_k(\xi \oplus \eta) = \sum_{i=0}^{k} c_i(\xi) c_{k-i}(\eta)$$

5.2.2 Definition. For $\xi \in \mathrm{Vect}_{\mathbb{C}}(X)$ the class $c_k(\xi) \in H^{2k}(X;\mathbb{Z})$ is called the *k'th Chern class* of ξ and we define the *total Chern class* $c : \mathrm{Vect}_{\mathbb{C}}(X) \to H^{2*}(X;\mathbb{Z})$ by the formula

$$c(\xi) = \sum_{k=0}^{\infty} c_k(\xi)$$

For a space X there is a sequence of morphisms $\mathrm{ch}_k : \mathrm{Vect}_{\mathbb{C}}(X) \to H^{2k}(X;\mathbb{Q})$ such that the following properties are satisfied:

i) $\mathrm{ch}_0(\xi) = \dim(\xi)$ and $\mathrm{ch}_1(\xi) = c_1(\xi)$.

ii) For a map $f : Y \to X$ with the induced morphisms $f^* : \mathrm{Vect}_{\mathbb{C}}(X) \to \mathrm{Vect}_{\mathbb{C}}(Y)$ and $f^* : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$, the diagram

$$\begin{array}{ccc} \mathrm{Vect}_{\mathbb{C}}(X) & \xrightarrow{\mathrm{ch}_k} & H^{2k}(X,\mathbb{Q}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Vect}_{\mathbb{C}}(Y) & \xrightarrow{\mathrm{ch}_k} & H^{2k}(Y,\mathbb{Q}) \end{array}$$

commutes for all k.

5.2.3 Theorem. *For vector bundles $\xi, \eta \in \mathrm{Vect}_{\mathbb{C}}(X)$ the morphism ch_k satisfies*

$$\mathrm{ch}_k(\xi \oplus \eta) = \mathrm{ch}_k(\xi) + \mathrm{ch}_k(\eta)$$

and

$$\mathrm{ch}_k(\xi \otimes \eta) = \sum_{i=0}^{k} \mathrm{ch}_i(\xi)\mathrm{ch}_{k-i}(\eta)$$

for all k.

5.2.4 Definition. For $\xi \in \mathrm{Vect}_{\mathbb{C}}(X)$ the class $\mathrm{ch}_k(\xi)$ is called the *k'th Chern character* of ξ and we define the *total Chern character*, $\mathrm{ch} : \mathrm{Vect}_{\mathbb{C}}(X) \to H^{2*}(X;\mathbb{Q})$, by the formula

$$\mathrm{ch}(\xi) = \sum_{k=0}^{\infty} \mathrm{ch}_k(\xi)$$

For a line bundle γ over X

$$\mathrm{ch}_k(\gamma) = \frac{1}{k!}\mathrm{ch}_1(\gamma)^k$$

which is the k'th term of the power expansion of the exponential function and so for the total Chern character we have

$$\mathrm{ch}(\gamma) = e^{\mathrm{ch}_1(\gamma)}$$

For arbitrary vector bundles we obtain, using the splitting principle,

$$\mathrm{ch}(\xi) = e^{\mathrm{ch}_1(\gamma_1)} + \cdots + e^{\mathrm{ch}_1(\gamma_n)}$$

where $\xi \approx \gamma_1 \oplus \cdots \oplus \gamma_n$.

By the universal property of the construction of $K_{\mathbb{C}}(X)$ and application of theorem 5.2.3, the Chern character induces a ring homomorphism $\mathrm{ch} : K_{\mathbb{C}}(X) \to H^{2*}(X;\mathbb{Q})$ and we will refer to this morphism as the Chern character in the following.

To end the section, we remark that the Chern class has an analog for real vector bundles, namely the *Stiefel-Whitney class*, $w : \mathrm{Vect}_{\mathbb{R}}(X) \to H^*(X;\mathbb{Z}_2)$ satisfying properties analogous to those for the Chern class.

5.3 Spectral sequences of K-theory

In this section we state one important result due to Atiyah and Hirzebruch, namely the existence of a certain spectral sequence. No proof will be given, instead we refer to [6, §2], [15, Sec. 9.5] and [9, p. 315].

Let X be a CW-complex and X^p the p-skeleton of X. The natural inclusion $X^{p-1} \to X$ then induces a morphism $K^q(X) \to K^q(X^{p-1})$. We define

$$K_p^q(X) = \ker[K_{\mathbb{F}}^q(X) \to K_{\mathbb{F}}^q(X^{p-1})]$$

When working with spectral sequences we drop $\mathbb{F}$ from notation and let it be determined from the context whether to use real or complex K-theory.

5.3.1 Theorem. *Let X be a finite CW-complex, and $x_0 \in X$ a point. Then there exists a spectral sequence, $E_r^{p,q}$, where $r \geq 1$ and $-\infty < p, q < \infty$ with*

$$\begin{aligned} E_1^{p,q} &\approx C^p(X, K^q(x_0)) \\ E_2^{p,q} &\approx H^p(X, K^q(x_0)) \\ E_\infty^{p,q} &\approx K_p^{p+q}(X)/K_{p+1}^{p+q}(X) \end{aligned}$$

where ∂_1 is the ordinary coboundary operator. In the complex case if r is even, the differential $\partial_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ vanish since $E_r^{p,q} = 0$ for q odd.

The spectral sequence does not determine $K_{\mathbb{F}}(X)$ explicitly, but it does reveal some structure of $K_{\mathbb{F}}(X)$. It follows from the theorem that $\tilde{K}_{\mathbb{F}}(X)$ is filtered by the groups $\tilde{K}_{\mathbb{F}}(X/X^{p-1})$ and the successive quotients are the entries of $E_\infty^{p,-p}$. Given an element κ in $\tilde{K}_{\mathbb{F}}(X/X^{p-1})$ this maps to some element in $\tilde{K}_{\mathbb{F}}(X^{p-1+r}/X^{p-1})$ where $1 \leq r \leq \infty$. Passing to quotients we then obtain an element κ_r of $E_r^{p,-p}$ such that $\partial_r \kappa_r = 0$.

In particular, for $\mathbb{F} = \mathbb{C}$ and $r = 1$, an element $\kappa \in \tilde{K}_{\mathbb{C}}(X^p/X^{p-1})$ defines an element κ_1 of $C^p(X, K^{-p}(x_0))$. Since X^p/X^{p-1} is a wedge of spheres, we obtain an isomorphism $\tilde{K}_{\mathbb{C}}(X^p/X^{p-1}) \approx C^p(X, K^{-p}(x_0))$. Hence, we can determine if κ_1 is a generator by examining the Chern character of κ.

6

Thom spaces

This chapter provides the first part of the proof of the non-existence theorem. We give a reduction of the problem to a problem of coreducibility of certain stunted projective spaces. This reduction was first proved by James (1958) and Atiyah later proved the theorem using Thom complexes and S-theory [4], we will follow his approach.

The first section defines Thom complexes and we show that these may be viewed as generalised suspensions. In section 2, we introduce S-theory and state two fundamental theorems due to Spanier, Whitehead and Atiyah. Section 3 is concerned with fibre homotopy type, and we relate this with the results of the previous sections in theorem 6.3.3. In section 4, we give an account of reducibility which provide the basis for the reduction of the non-existence theorem. Finally in section 5, we are ready to prove the reduction theorem.

6.1 Thom spaces and vector bundles

To each real vector bundle ξ, $p : E \to X$, we associate the *projective bundle* $P(\xi) \to X$ with total space $P(\xi)$ defined as follows: Let $s : X \to E$ be the zero section, then the total space of the projective bundle is given by $P(\xi) = E - s(X)/\sim$ where we factor out scalar multiplication on the fibres of ξ. The projection $P(\xi) \to X$ is defined from the projection of ξ. Thus if for $x \in X$ the fibre $p^{-1}(x)$ is $\mathbb{R}^n$ in E then the fibre over x in $P(\xi)$ is $\mathbb{RP}^{n-1}$, that is the fibres of the projective bundle are homeomorphic to projective spaces.

Similarly we define the *sphere bundle* $S(\xi) \to X$ as the bundle over X with total space $S(\xi) = E - s(X)/\sim$ where we factor out positive scalars, and we obtain the projection map $S(\xi) \to X$ from the projection map of ξ. Thus, if the fibre $p^{-1}(x)$ over $x \in X$ is $\mathbb{R}^n$ in E, then the fibre over x in $S(\xi)$ is S^{n-1}. It is clear from the construction that we can also obtain the projective bundle as a quotient space of the sphere bundle.

From the sphere bundle we now define the *disk bundle* $D(\xi) \to X$ where total space $D(\xi)$ is the mapping cylinder of the projection $S(\xi) \to X$.

A similar construction can be carried out for complex vector bundles with only few modifications. In this chapter we will only consider *real* vector bundles.

6.1.1 Definition. For a vector bundle ξ over X we define the *Thom space* as the quotient space $T(\xi) = D(\xi)/S(\xi)$.

6.1.2 Proposition. *For a vector bundle ξ with total space E, $T(\xi)$ is homeomorphic to the one point compactification of E.*

Proof: Note that $D(\xi) - S(\xi)$ is homeomorphic to E. Since $D(\xi)/S(\xi)$ is the one point compactification of $D(\xi) - S(\xi)$, it follows that $T(\xi)$ is homeomorphic to the one point compactification of E. □

We define the Cartesian product of bundles, $\xi \times \eta = (E(\xi) \times E(\eta), p_\xi \times p_\eta, B(\xi) \times B(\eta))$. We then have,

6.1.3 Proposition. *Let ξ and η be real vector bundles over a compact space. Then the Thom space $T(\xi \times \eta)$ is homeomorphic to the smash product $T(\xi) \wedge T(\eta)$.*

Proof: From 6.1.2, we have that the Thom space $T(\xi \times \eta)$ is the one point compactification of $E(\xi \times \eta)$ and $T(\xi) \wedge T(\eta)$ is the one point compactification of $E(\xi) \times E(\eta)$. The proposition now follows from the identity $E(\xi) \times E(\eta) = E(\xi \times \eta)$. □

6.1.4 Corollary. *The Thom space $T(\xi \oplus \varepsilon^n)$ is homeomorphic to the n-fold suspension $S^n(T(\xi))$.*

Proof: First note that the vector bundle $\xi \oplus \varepsilon^n$ is isomorphic to $\xi \times \mathbb{R}^n$, where we regard $\mathbb{R}^n$ as the n-dimensional vector bundle over a point. Since $T(\mathbb{R}^n) = S^n$ we have from proposition 6.1.3 that $T(\xi \oplus \varepsilon^n)$ is homeomorphic to $S^n \wedge T(\xi) = S^n(T(\xi))$. □

From the proof, we see that the Thom space may be viewed as a generalisation of suspension to vector bundles.

6.1.5 Proposition. *Let ξ_k be the canonical line bundle over $\mathbb{R}\mathrm{P}^k$. Then the Thom space $T(m\xi_k \oplus \varepsilon^n)$ and the n-fold suspension of the stunted projective space $S^n(\mathbb{R}\mathrm{P}^{m+k}/\mathbb{R}\mathrm{P}^{m-1})$ are homeomorphic.*

Proof: We only need to prove that $T(m\xi_k)$ is homeomorphic to $\mathbb{R}\mathrm{P}^{m+k}/\mathbb{R}\mathrm{P}^{m-1}$. First note that the total space of ξ_k is homeomorphic to $S^k \times \mathbb{R}/\sim$, where $(x, y) \sim (-x, -y)$. Then the m-fold Whitney sum of ξ_k has total space $S^k \times \mathbb{R}^m/\sim$, where $(x, y) \sim (-x, -y)$ for $x \in S^k$ and $y \in \mathbb{R}^m$. Hence we obtain the disk bundle of $m\xi_k$ with total space $D(m\xi_k) = S^k \times D^m/\sim$ and sphere bundle with total space $S(m\xi_k) = S^k \times S^{m-1}/\sim$.

Define the map, $f : S^k \times D^m \to S^{k+m}$ by the relation $f(x, y) = (y, (1 - ||y||)x)$. Then $f(S^k \times S^{m-1}) = S^{m-1} \subset S^{k+m}$. Since f is odd, f defines a map $g : D(m\xi_k) \to \mathbb{R}\mathrm{P}^{k+m}$ such that the restrictions $g : S(m\xi_k) \to \mathbb{R}\mathrm{P}^{m-1}$, and $g : D(m\xi_k) - S(m\xi_k) \to \mathbb{R}\mathrm{P}^{m+k} - \mathbb{R}\mathrm{P}^{m-1}$ are homeomorphisms. Hence, the quotient map,

$$h : T(m\xi_k) \to \mathbb{R}\mathrm{P}^{k+m}/\mathbb{R}\mathrm{P}^{m-1}$$

is a homeomorphism. □

6.2 Duality

Let X and Y be pointed spaces and let $[X, Y]$ denote the set of base point preserving maps. Then the suspension $S : X \mapsto SX$ induces a map $S : [X, Y] \to [SX, SY]$. The direct limit of the sequence

$$[X, Y] \longrightarrow [SX, SY] \longrightarrow \cdots \longrightarrow [S^{k+1}X, S^{k+1}Y] \longrightarrow \cdots$$

denoted $\{X, Y\}$ is stable under the suspension map, $\{X, Y\} = \{SX, SY\}$. The elements of $\{X, Y\}$, denoted $\{f\}$ are called *S-maps* and hence we obtain the *suspension category* or the *S-category* with pointed spaces as objects and S-maps as maps.

For $k \geq 1$ the set $[S^kX, S^kY]$ can be regarded as a homotopy group and $S : [S^kX, S^kY] \to [S^{k+1}X, S^{k+1}Y]$ as a group homomorphism, the *suspension homomorphism*. Passing to the limit we obtain a group isomorphism $\{X, Y\} \to \{SX, SY\}$.

We state the following theorem due to Spanier and Whitehead. It goes beyond the scope of this thesis to give a full and satisfying account of the proof, instead we refer to [21, p. 458].

6.2.1 Theorem (Spanier & Whitehead). *Let X and Y be pointed spaces where X is a CW-complex with dimension m and Y is n-connected. Then the suspension map*

$$S : [X, Y] \to [SX, SY]$$

is a bijection if $m \leq 2n$.

6.2.2 Corollary. *If X is an m-dimensional CW-complex and Y is n-connected with $2n \geq m$, then there is a bijection,*

$$[X, Y] \to \{X, Y\}$$

Proof: If X has dimension m then $\dim S^kX = m+k$, and if Y is n-connected then by theorem 6.2.1, S^kY is $n+k$-connected. Hence for every $k \geq 0$, the map $[S^kX, S^kY] \to [S^{k+1}X, S^{k+1}Y]$ is a bijection by theorem 6.2.1. We now obtain the result from the sequence

$$[X, Y] \longrightarrow [SX, SY] \longrightarrow \cdots \longrightarrow [S^{k+1}X, S^{k+1}Y] \longrightarrow \cdots$$

passing to the limit. □

For pointed spaces X and X' a map $u : X \wedge X' \to S^n$ is called an *n-pairing*. For a pointed space Z, u then defines two group homomorphisms $u_Z : \{Z, X'\} \to \{X \wedge Z, S^n\}$ and $u^Z : \{X, Z\} \to \{Z \wedge X', S^n\}$ by the relations, $u_Z(\{f\}) = \{u(1 \wedge f)\}$ and $u^Z(\{g\}) = \{u(g \wedge 1)\}$. If $Z = S^k$ we write u_k and u^k respectively.

6.2.3 Definition. A space X is called an *n-dual* of the space X' if there exists an n-pairing $u : X \wedge X' \to S^n$, such that $u_k : \{S^k, X'\} \to \{X \wedge S^k, S^n\}$ and $u^k : \{X', S^k\} \to \{S^k \wedge X, S^n\}$ are group isomorphisms for all $k \geq 1$. The map u is called an *n-duality map*.

If for some integers, k, m and n, the suspension S^kX is n-dual of the suspension S^mX', we say that X is *S-dual* of X'.

In section 6.1 we saw that Thom complexes may be viewed as generalised suspensions. The following theorem due to Atiyah [4, thm. 3.3] gives an important relation between S-duality and Thom spaces.

6.2.4 Theorem (Atiyah). *Let ξ and η be vector bundles over a closed differentiable manifold X such that $\xi \oplus \eta \oplus \tau(X)$ is stably trivial, where $\tau(X)$ is the tangent bundle of X. Then $T(\xi)$ and $T(\eta)$ are S-duals.*

6.3 Stable fibre homotopy equivalence

Recall that two spaces X, Y are said to have same homotopy type if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq 1_X$ and $gf \simeq 1_Y$. We write $X \simeq Y$. In the following we will generalise this to fibre bundles, we will be particularly interested in the fibre homotopy type of the associated sphere bundles.

Let ξ and ξ' be bundles over X. A homotopy $h_t : E \to E'$ is a *fibre homotopy* if for every $t \in I$, $p'h_t = p$, i.e. the homotopy preserves fibres.

Let $u_0, u_1 : E \to E'$ be bundle morphisms, then u_0 and u_1 are *fibre homotopic*, if there exists a fibre homotopy $h_t : E \to E'$ with $h_0 = u_0$ and $h_1 = u_1$. A bundle morphism $u : E \to E'$ is a *fibre homotopy equivalence* if there exists a bundle morphism $u' : E' \to E$ such that uu' and $u'u$ are fibre homotopic to the identity maps. Finally, two bundles ξ and ξ' have same *fibre homotopy type* if there exists a fibre homotopy equivalence $u : E \to E'$.

6.3.1 Definition. Let ξ and η be vector bundles over X. The associated sphere bundles are *stable fibre homotopically equivalent* if for some integers m, n, the bundles $S(\xi \oplus \varepsilon^m)$ and $S(\eta \oplus \varepsilon^n)$ have same fibre homotopy type.

Clearly stable fibre homotopy equivalence is an equivalence relation. For a vector bundle ξ over X, we denote by $J(\xi)$ the stable fibre homotopy equivalence class defined by ξ and $J(X)$ denotes the set of all stable fibre homotopy equivalence classes over X.

6.3.2 Proposition. *If X is a finite CW-complex, then Whitney sum of vector bundles induces the structure of an abelian group on $J(X)$.*

Proof: The Whitney sum of vector bundles is commutative and associative up to isomorphism, and hence it is sufficient to show that $J(\xi \oplus \eta) = J(\xi' \oplus \eta)$ for vector bundles ξ, ξ' and η over X with $J(\xi) = J(\xi')$.

Replacing ξ by $\xi \oplus \varepsilon^m$ and ξ' by $\xi' \oplus \varepsilon^n$ for suitable choices of m and n, we may assume that $S(\xi)$ and $S(\xi')$ are fibre homotopy equivalent, hence it is sufficient to prove that $S(\xi \oplus \eta)$ is fibre homotopy equivalent to $S(\xi' \oplus \eta)$.

Let $f : S(\xi) \to S(\xi')$ and $f' : S(\xi') \to S(\xi)$ be fibre homotopy equivalences and let $h_t : S(\xi) \to S(\xi)$ and $h'_t : S(\xi') \to S(\xi')$ be fibre homotopies such that $h_0 = f'f$, $h_1 = 1$, $h'_0 = ff'$ and $h'_1 = 1$.

We then define maps $\tilde{f} : S(\xi \oplus \eta) \to S(\xi' \oplus \eta)$ and $\tilde{f}' : S(\xi' \oplus \eta) \to S(\xi \oplus \eta)$ by the relations

$$\tilde{f}(u \cos\theta, v \sin\theta) = (f(u) \cos\theta, v \sin\theta)$$
$$\tilde{f}'(u' \cos\theta, v \sin\theta) = (f'(u') \cos\theta, v \sin\theta)$$

where $u \in S(\xi)$, $u' \in S(\xi')$, $v \in S(\eta)$ and $\theta \in [0, \frac{1}{2}\pi]$. From these maps we then obtain the desired fibre homotopies $\tilde{h}_t : S(\xi \oplus \eta) \to S(\xi \oplus \eta)$ and $\tilde{h}'_t : S(\xi' \oplus \eta) \to S(\xi' \oplus \eta)$ defined by the equations,

$$\tilde{h}_t(u \cos\theta, v \sin\theta) = (h_t(u) \cos\theta, v \sin\theta)$$
$$\tilde{h}'_t(u' \cos\theta, v \sin\theta) = (h'_t(u') \cos\theta, v \sin\theta)$$

Clearly, $\tilde{h}_0 = \tilde{f}'\tilde{f}$, $\tilde{h}_1 = 1$, $\tilde{h}'_0 = \tilde{f}\tilde{f}'$ and $\tilde{h}'_1 = 1$. □

It follows, that if X is a finite CW-complex, then there is a natural epimorphism

$$\tilde{K}_{\mathbb{R}}(X) \to J(X)$$

where $\tilde{K}_{\mathbb{R}}(X)$ is regarded as an abelian group with addition defined by Whitney sum. If X is a finite CW-complex, then $J(X)$ is a finite group [4, prop. 1.5].

6.3.3 Proposition. *Let ξ and η be vector bundles over X. If $S(\xi)$ and $S(\eta)$ have same fibre homotopy type, then $T(\xi)$ and $T(\eta)$ have the same homotopy type. If $J(\xi) = J(\eta)$ then $T(\xi)$ and $T(\eta)$ have the same S-type, i.e. there exists integers m, n, such that $S^m T(\xi)$ and $S^n T(\eta)$ have same homotopy type.*

Proof: Let $f : S(\xi) \to S(\eta)$ and $f' : S(\eta) \to S(\xi)$ be fibre homotopy maps that are inverses of each other. Then f and f' prolong to maps $g : D(\xi) \to D(\eta)$ and $g' : D(\eta) \to D(\xi)$. Hence, the homotopy from ff' to the identity map then prolongs radially to a homotopy $D(\xi) \to D(\xi)$ between gg' and the identity on $D(\xi)$. Similarly for $f'f$ and $g'g$. Using the quotient map we now obtain maps $h : T(\xi) \to T(\eta)$ and $h' : T(\eta) \to T(\xi)$ which are homotopy inverses of each other.

For the second statement, we have by definition, that if $J(\xi) = J(\eta)$, then there exists integers m, n such that $S(\xi \oplus \varepsilon^m)$ and $S(\eta \oplus \varepsilon^n)$ have same fibre homotopy type. By the first statement, it follows that $T(\xi \oplus \varepsilon^m)$ and $T(\eta \oplus \varepsilon^n)$ have same homotopy type, and from corollary 6.1.4 it follows that $T(\xi \oplus \varepsilon^m) = S^m(T(\xi))$ and $T(\eta \oplus \varepsilon^n) = S^n(T(\eta))$ have same homotopy type. Hence, $T(\xi)$ and $T(\eta)$ have same S-type. □

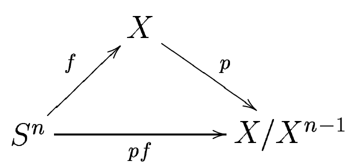
6.4 Reducibility

6.4.1 Definition. A pointed space X is *reducible* if for some integer n there exists a map $f : S^n \to X$ such that $f_* : \tilde{H}_i(S^n) \to \tilde{H}_i(X)$ is an isomorphism for $i \geq n$. A space X is called *S-reducible* if for some integer k, $S^k X$ is reducible.

6.4.2 Definition. A pointed space X is called *coreducible* if for some integer n there exists a map $g : X \to S^n$ such that $g^* : \tilde{H}^i(S^n) \to \tilde{H}^i(X)$ is an isomorphism for $i \leq n$. A space X is *S-coreducible* if for some integer k, $S^k X$ is coreducible.

For the following, for an n-dimensional CW-complex X, we denote the subcomplex of X consisting of cells up to dimension k by X^k.

6.4.3 Proposition. *Let X be a n-dimensional CW-complex with one n-cell and let $p : X \to X/X^{n-1} = S^n$ be the natural projection map. The space X is reducible if and only if there is a map $f : S^n \to X$ such that pf is homotopic to the identity.*



Proof: From the sequence of spaces,

$$X^{n-1} \longrightarrow X \longrightarrow X/X^{n-1}$$

we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & \tilde{H}_n(S^n) & & & & & & \\ & & \downarrow f_* & \searrow^{(pf)_*} & & & & & \\ 0 = \tilde{H}_n(X^{n-1}) & \longrightarrow & \tilde{H}_n(X) & \xrightarrow{p_*} & \tilde{H}_n(X/X^{n-1}) & \xrightarrow{\partial} & \tilde{H}_{n-1}(X^{n-1}) & \longrightarrow & \cdots \end{array}$$

First assume X is reducible, let $f : S^n \to X$ be a map that satisfies the condition on the homology groups described above. We need to prove that pf is homotopic to the identity, i.e. $(pf)_* : H_n(S^n) \to H_n(X/X^{n-1})$ is an isomorphism. p_* is monomorphic by exactness and since f_* is assumed an isomorphism, we only need to show that p_* is an epimorphism.

Since f_* is an isomorphism we have $H_n(X) = \mathbb{Z}$ and hence p_* is multiplication by an integer p, that is $p_*(H_n(X/X^{n-1})) = p\mathbb{Z}$. From exactness, $\text{im}\partial \approx \mathbb{Z}/p\mathbb{Z}$. Since $\tilde{H}_{n-1}(X^{n-1})$ is free abelian, it follows that ∂ is zero, hence $p = 1$, and we conclude that p_* is an epimorphism.

Conversely, assume that there exists a map $f : S^n \to X$ such that pf is homotopic to the identity, hence $(pf)_*$ is an isomorphism. Since X has dimension n and only one n-cell, it follows from the cellular chain complex that $\tilde{H}_n(X) = \mathbb{Z}$ or 0. Thus, $(pf)_*$ has degree 1, and hence f_* gives a reduction of $\tilde{H}_n(X)$. □

6.4.4 Proposition. *Let X be a CW-complex with one n-cell, and no cells of lower dimension, i.e. X^n is homeomorphic to S^n. Let $\imath : X^n \to X$ be the natural inclusion map of the n-cell of X into X. The space X is coreducible if and only if there is a map $g : X \to S^n$ such that $g\imath$ is homotopic to the identity.*

$$\begin{array}{ccc} & X & \\ {}^{\imath}\nearrow & & \searrow^{g} \\ X^n & \xrightarrow[g\imath]{} & S^n \end{array}$$

Proof: From the sequence of spaces,

$$X^n \longrightarrow X \longrightarrow X/X^n$$

we have the commutative diagram in cohomology,

$$\begin{array}{ccccccccc} 0 = \tilde{H}^n(X/X^n) & \longrightarrow & \tilde{H}^n(X) & \xrightarrow{\imath^*} & \tilde{H}^n(X^n) & \longrightarrow & \tilde{H}^{n+1}(X/X^n) & \longrightarrow & \cdots \\ & & \uparrow g^* & \nearrow_{(g\imath)^*} & & & & & \\ & & \tilde{H}^n(S^n) & & & & & & \end{array}$$

If X is coreducible, then there exists a map $g : X \to S^n$ with $g^* : \tilde{H}^i(S^n) \to \tilde{H}^i(X)$ an isomorphism for $i \leq n$. With the same arguments as the previous proof, $\imath^*$ is an isomorphism, hence, $(g\imath)^*$ is an isomorphism, i.e. $g\imath$ is homotopic to the identity.

Conversely, if there exists a map $g : X \to S^n$ such that $g\imath$ is homotopic to the identity, then $(g\imath)^*$ has degree 1, and hence, g gives a reduction of $\tilde{H}^n(X)$. □

6.4.5 Theorem. *Let X be a CW-complex with one n-cell, and no cells of lower dimension. Then X is coreducible if and only if X is S-coreducible.*

Proof: The direct implication is clear from the definition. For the converse implication assume that X is S-coreducible. Then by proposition 6.4.4 for some integer $k \geq 0$ there exists a map $g : S^k X \to S^{k+n}$ such that composition with the natural inclusion $\imath : S^{k+n} \to S^k X$ is homotopic to the identity.

Since X has only one n-cell and no cells of lower dimension, X is n-connected. Since S^n has dimension n, it follows from corollary 6.2.2 that there is a bijection $[X, S^n] \to \{X, S^n\}$. In particular from the construction there is a bijection $[X, S^n] \to [S^k X, S^{k+n}]$.

Hence the bijection defines a map $g : X \to S^n$ such that g composed with the natural inclusion $\imath : S^n \to X$ is homotopic to the identity. By proposition 6.4.4 it follows that X is coreducible. □

6.5 Vector fields on the sphere and reducibility

In the following consider $\mathbb{R}^n$ as the n-dimensional real vector space with basis $\{e_1, \ldots, e_n\}$ and inner product $\langle \cdot | \cdot \rangle$. S^{n-1} is the subset of unit vectors and we obtain the real projective space $\mathbb{R}\mathrm{P}^n$ by identifying antipodal points in S^{n-1}.

For the real projective space $\mathbb{R}\mathrm{P}^{n-1}$ we then define the map $\Theta : \mathbb{R}\mathrm{P}^{n-1} \to O(n)$ as follows. An element $x \in \mathbb{R}\mathrm{P}^{n-1}$ spans a line through the origin in $\mathbb{R}^n$, then $\Theta(x)$ is the reflection through the hyperplane perpendicular to this line, that is for $x \in \mathbb{R}\mathrm{P}^{n-1}$ and $y \in \mathbb{R}^n$ $\Theta(x)y = y - 2\langle x|y\rangle x$.

With this map we then obtain the diagram

$$\begin{array}{ccc}
\mathbb{R}\mathrm{P}^{n-k-1} & \xrightarrow{\Theta} & O(n-k) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{\Theta} & O(n) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1} & \xrightarrow{\Theta} & O(n)/O(n-k)
\end{array}$$

where the vertical maps are the natural inclusions and projections respectively. The bottom row is defined by commutativity of the diagram, hence we have $\Theta(x) = (v_1, \ldots, v_k)$ where $v_i = \Theta(x)e_{n-k+1}$ for $1 \leq i \leq k$.

6.5.1 Proposition. *The map $\Theta : S^{n-1} = \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} \to O(n)/O(n-1) = S^{n-1}$ defined as above is a homeomorphism.*

Proof: Observe that $\Theta(x)e_n = e_n - 2\langle x|e_n\rangle x = (-2x_1x_n, \ldots, -2x_{n-1}x - n, 1 - x_n^2)$. For $y \in S^{n-1}$ with $y_n \neq 1$ there exists an unique $x \in \mathbb{R}\mathrm{P}^{n-1}$ such that $x_n > 0$ and $\Theta(x)e_n = y$. For $x_n = 0$, we have $\Theta(x)e_n = e_n$. □

Since $O(n)$ consists of $n \times n$-matrices with orthonormal column vectors, the map $\Theta : \mathbb{R}\mathrm{P}^{n-1} \to O(n)$ can be regarded as a map $\Theta : \mathbb{R}\mathrm{P}^{n-1} \to V_n(\mathbb{R}^n)$. Factoring out the last $n-k$ vectors we now obtain a homeomorphism $\Theta : \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1} \to V_k(\mathbb{R}^n)$ [10, p. 464].

6.5.2 Proposition. *For $i < 2n-2k-1$, Θ induces an isomorphism $\pi_i(\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1}) \to \pi_i(V_k(\mathbb{R}^n))$.*

Proof: The proof is by induction on k. For $k = 1$ the result follows from proposition 6.5.1. Assume the statement holds for $k-1$, then the map Θ induces the commutative diagram,

$$\begin{array}{ccc} \mathbb{R}\mathrm{P}^{n-2}/\mathbb{R}\mathrm{P}^{n-k-1} & \xrightarrow{\Theta} & V_{k-1}(\mathbb{R}^{n-1}) \\ \downarrow u & & \downarrow \\ \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1} & \xrightarrow{\Theta} & V_k(\mathbb{R}^n) \\ \downarrow & & \downarrow v \\ \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} & \xrightarrow{\Theta} & V_1(\mathbb{R}^n) \end{array}$$

By the induction hypothesis Θ induces isomorphisms of homotopy groups on the top row, and on the bottom row Θ induces isomorphisms by the induction start.

Using the Wang sequence [21, cor. 6 p. 456] of homology on the right column we obtain a commutative "ladder" in homology with exact columns. Since Θ in the top and bottom row of the diagram induces isomorphisms of homotopy groups, Θ induces isomorphisms of the corresponding homology groups by the Whitehead theorem [21, thm. 9 p. 399].

It then follows by the five lemma that the homology homomorphisms induced by $\Theta : \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1} \to V_k(\mathbb{R}^n)$ are isomorphisms. Applying the Whitehead theorem, we conclude that $\Theta : \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1} \to V_k(\mathbb{R}^n)$ induces isomorphisms of homotopy groups. □

For an integer n we write $n = (2a+1)2^{c+4d}$ where a, c and d are integers and $0 \leq c \leq 3$. Define $\rho(n) = 2^c + 8d$. Then if $n-1 < 2n - (2\rho(n)+2) - 1$, that is for $2\rho(n)+2 < n$, we have an isomorphism $\pi_{n-1}(\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}) \to \pi_{n-1}(V_{\rho(n)+1}(\mathbb{R}^n))$. This leads to the following proposition,

6.5.3 Theorem. *If $2\rho(n)+2 < n$, then the bundle $p : V_{\rho(n)+1}(\mathbb{R}^n) \to S^{n-1}$ has a cross section if and only if there exists a map*

$$f : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$$

such that composition with the natural projection $\pi : \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} = S^{n-1}$ has degree 1, i.e. $\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ is reducible.

Proof: With $k = \rho(n)+1$ we obtain as in the proof above the commutative diagram,

$$\begin{array}{ccc} \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2} & \xrightarrow{\Theta} & V_{\rho(n)+1}(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ S^{n-1} = \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} & \xrightarrow{\Theta} & V_1(\mathbb{R}^n) = S^{n-1} \end{array}$$

Let $s : S^{n-1} \to V_{\rho(n)+1}(\mathbb{R}^n)$ be a cross section. Since Θ induces isomorphisms by proposition 6.5.2, there exists a map $f : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$, which composed with Θ is homotopic to s. Hence composing with s we obtain a map homotopic to the identity.

Conversely assume there is a map $f : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ such that composition with π has degree 1. Since Θ defines an isomorphism on the top row by 6.5.2, composing f with Θ defines a map $s' : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ such that ps' is homotopic to the identity on S^{n-1}. Since p is a fibre map this homotopy lifts to $V_{\rho(n)+1}(\mathbb{R}^n)$ to define a cross section. □

6.5.4 Proposition. *Let ξ_{k-1} be the canonical line bundle over $\mathbb{R}\mathrm{P}^{k-1}$ and let r denote the order of $J(\xi_{k-1})$ in $J(\mathbb{R}\mathrm{P}^{k-1})$. Then for all positive integers p, $\mathbb{R}\mathrm{P}^{n+rp}/\mathbb{R}\mathrm{P}^{n+rp-k}$ and $\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{n-k}$ have the same S-type. For $rp > n+1$ the space $\mathbb{R}\mathrm{P}^{rp+k-n}/\mathbb{R}\mathrm{P}^{rp-n-2}$ is an S-dual of $\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{n-k}$.*

Proof: For the first part, we have by proposition 6.1.5 that the space $\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{n-k}$ is homeomorphic to $T((n-k+1)\xi_{k-1})$, and $\mathbb{R}\mathrm{P}^{n+rp}/\mathbb{R}\mathrm{P}^{n-k+rp}$ is homeomorphic to $T((n-k+1+rp)\xi_{k-1}) = T((n-k+1)\xi_{k-1} \oplus rp\xi_{k-1})$. Since r is the order of $J(\xi_{k-1})$, $J(rp\xi_{k-1})$ is trivial as sphere bundle. The Thom space is completely determined by the sphere bundle, hence $T((n-k+1)\xi_{k-1} \oplus rp\xi_{k-1})$ is homeomorphic to $T((n-k+1)\xi_{k-1} \oplus \varepsilon^{rp})$. It follows that $\mathbb{R}\mathrm{P}^{n+rp}/\mathbb{R}\mathrm{P}^{n-k+rp}$ is homeomorphic to $T((n-k+1)\xi_{k-1} \oplus \varepsilon^{rp})$. From proposition 6.1.5 we have that $T((n-k+1)\xi_{k-1} \oplus \varepsilon^{rp})$ and $S^{rp}(T((n-k+1)\xi_{k-1}))$ are homeomorphic. Hence, we conclude that $\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{n-k}$ and $\mathbb{R}\mathrm{P}^{n+rp}/\mathbb{R}\mathrm{P}^{n-k+rp}$ have same S-type.

We prove the second statement by applying the Atiyah theorem, 6.2.4. First, recall from example 1.4.4 that $\tau(\mathbb{R}\mathrm{P}^{k-1}) \oplus \varepsilon^1$ is isomorphic to $k\xi_{k-1}$. From proposition 6.1.5 we have that $\mathbb{R}\mathrm{P}^{n}/\mathbb{R}\mathrm{P}^{n-k}$ is homeomorphic to $T((n-k+1)\xi^{k-1})$ and $\mathbb{R}\mathrm{P}^{rp+k-n}/\mathbb{R}\mathrm{P}^{rp-n-2}$ is homeomorphic to $T((rp-n-1)\xi_{k-1})$. From the first statement we have that $T((rp-n-1)\xi_{k-1})$ and $T((-n-1)\xi_{k-1})$ have same S-type. Hence, we obtain $(n-k+1)\xi_{k-1} \oplus (-n-1)\xi_{k-1} \oplus k\xi_{k-1} \approx_s \varepsilon^0$, and the result follows from theorem 6.2.4. □

6.5.5 Lemma. *If S^{n-1} admits k orthonormal tangent vector fields $v_1, \ldots, v_k$, then for any integer $q \geq 1$, S^{qn-1} admits k orthonormal tangent vector fields $v'_1, \ldots, v'_k$.*

Proof: We may view the vector fields as a set of continuous maps, $v_i : S^{n-1} \to \mathbb{R}^n$ satisfying $\langle x | v_i(x) \rangle = 0$ and $\langle v_i(x) | v_j(x) \rangle = \delta_{ij}$ for $1 \leq i, j \leq k$.

The sphere S^{qn-1} can be regarded as a q-fold join of spheres, S^{n-1}: For $x \in S^{qn-1}$ we can write $x = (\alpha(1)x(1), \ldots, \alpha(q)x(q))$ where $x_i \in S^{n-1}$ and $\sum_i \alpha_i^2 = 1$ with $\alpha_i \geq 0$. We then define the vector fields, $v'_i : S^{qn-1} \to \mathbb{R}^{qn}$ by the formula $v'_i(\alpha(1)x(1), \ldots, \alpha(q)x(q)) = \sum_i \alpha_i v_i(x_i)$. It follows by easy computations that $\langle x | v'_i(x) \rangle = 0$ and $\langle v'_i(x) | v'_j(x) \rangle = \delta_{ij}$ for $1 \leq i, j \leq k$. □

6.5.6 Theorem. *If there exists $\rho(n)$ orthonormal vector fields on S^{n-1}, then there exists an integer $m \geq 1$ with $\rho(m) = \rho(n)$ such that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is coreducible.*

Proof: Given $\rho(n)$ vector fields on S^{n-1}, then from lemma 6.5.5 it is possible to construct $\rho(n)$ vector fields on S^{qn-1} for any integer $q \geq 1$.

For q odd, $\rho(qn) = \rho(n)$ and if $qn \geq 2(\rho(n)+1)$, it follows from theorem 6.5.3 that the stunted projective space $\mathbb{R}\mathrm{P}^{qn-1}/\mathbb{R}\mathrm{P}^{qn-\rho(n)-2}$ is reducible.

Let $\xi_{\rho(n)}$ denote the canonical line bundle over $\mathbb{R}\mathrm{P}^{\rho(n)}$ and let r denote the order of $J(\xi_{\rho(n)})$ in $J(\mathbb{R}\mathrm{P}^{\rho(n)})$. Let p be a multiple of $2n$ such that $m = rp - qn \geq 1$. From proposition 6.5.4 we have that the stunted projective space, $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$, is an S-dual of $\mathbb{R}\mathrm{P}^{qn-1}/\mathbb{R}\mathrm{P}^{qn-\rho(n)-2}$. Hence, since $\mathbb{R}\mathrm{P}^{qn-1}/\mathbb{R}\mathrm{P}^{qn-\rho(n)-2}$ is reducible, it follows that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is S-coreducible.

The result now follows from theorem 6.4.5, since we have that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is S-coreducible if and only if $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is coreducible. □

7

Vector fields on the sphere

With our preliminary work, we are now able to prove the key theorem to the problem of vector fields on the sphere, and thus deduce the main result of the thesis. The results of the previous chapter reduced the problem of existence of vector fields on the sphere to a problem of reducibility of certain projective spaces, and so motivates further studies of projective spaces.

7.1 Computations on projective spaces

In this section we will calculate the rings $\tilde{K}_{\mathbb{F}}(X)$ for stunted real and complex projective spaces, together with the Adams operations on these rings. Our interest is to determine $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ and the Adams operations on this ring. We follow the procedure presented by J. F. Adams [2] and compute $\tilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$, $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ and $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ together with the Adams operations, in that order. The following lemma provides the connection between the K-rings of our interest.

7.1.1 Lemma. *Let ξ be the canonical real line bundle over $\mathbb{R}\mathrm{P}^{2k+1}$ and let η be the canonical complex line bundle over $\mathbb{C}\mathrm{P}^k$. Let $\pi : \mathbb{R}\mathrm{P}^{2k+1} \to \mathbb{C}\mathrm{P}^k$ be the standard projection. Then*

$$C\xi = \pi^*\eta$$

Proof: The complex line bundles are classified by their first Chern class, c_1. For $k > 0$ we have $H^2(\mathbb{R}\mathrm{P}^{2k+1}; \mathbb{Z}) \approx \mathbb{Z}_2$. Since $c_1\pi^*\eta = \pi^*c_1\eta \neq 0$, we need to prove that $C\xi$ is non-trivial. Let w be the total Stiefel-Whitney class, and let $x \in H^1(\mathbb{R}\mathrm{P}^{2k+1}; \mathbb{Z}_2)$ be a generator. Then $RC\xi = \xi \oplus \xi$ and so $w(RC\xi) = 1 + x^2$. Since $x^2 \in H^2(\mathbb{R}\mathrm{P}^{2k+1}; \mathbb{Z}_2) = \mathbb{Z}_2$ is non-trivial we have that $C\xi$ is non-trivial. □

Let ξ be the canonical real line bundle over $\mathbb{R}\mathrm{P}^n$ and η the canonical complex line bundle over $\mathbb{C}\mathrm{P}^n$. We then define,

$$\begin{aligned}
\lambda &= \xi - 1 \in \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) \\
\mu &= \eta - 1 \in \tilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n) \\
\nu &= C\lambda = \pi^{\#}\mu \in \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)
\end{aligned}$$

7.1.2 Remark. For the last identity, if $n = 2k+1$ then $\mu \in \tilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^k)$ defines the element $\nu \in \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$. If $n = 2k$ then $\nu \in \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ is obtained by composition with $\imath^\# : \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{2k+1}) \to \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{2k})$ where $\imath^\#$ is induced by the standard inclusion.

The remaining part of the section will be devoted to show that these elements are generators of the corresponding rings, and to calculate the Adams operations on these elements.

7.1.3 Theorem. *For the complex projective space $\mathbb{C}\mathrm{P}^n$, the ring $K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ is a truncated polynomial ring with integer coefficients and one generator μ with one relation $\mu^{n+1} = 0$.*

The projection $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m$ maps $\tilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^m)$ isomorphically onto the subgroup of $K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ generated by the elements $\mu^{m+1}, \ldots, \mu^n$.

Proof: We prove the first statement using induction on n. For $n = 0$, $\mathbb{C}\mathrm{P}^n$ is a point and hence the statement is clear. Now assume the statement holds for $n-1$. To do the inductive step we first note that if $x \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is the canonical generator, then $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[x]/x^{n+1}$ [14, thm. 3.12, p. 210]. Identifying $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ with its image under the natural inclusion map $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \hookrightarrow H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Q})$ [14, cor. 3A.6, p. 264] we have $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Q}) = \mathbb{Q}[x]/x^{n+1}$.

From the sequence of spaces $\mathbb{C}\mathrm{P}^{n-1} \longrightarrow \mathbb{C}\mathrm{P}^n \longrightarrow \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}$ we obtain the commutative diagram,

$$\begin{array}{ccccccccc} & & \tilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}) & \longrightarrow & K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n) & \longrightarrow & K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^{n-1}) & & \\ & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \\ 0 & \longrightarrow & H^*(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1};\mathbb{Q}) & \longrightarrow & H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Q}) & \longrightarrow & H^*(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Q}) & \longrightarrow & 0 \end{array}$$

Since the groups on the bottom row are torsion free, ch is a monomorphism [7, prop. 2.3], and since $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1} \approx S^{2n}$ we have $\tilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}) = \mathbb{Z}$. Hence the Chern character maps $\tilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1})$ isomorphically onto the image of $H^{2n}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})$ in $H^{2n}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1};\mathbb{Q})$ [7, prop. 2.2].

We know that $x^n \in H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Q})$ maps to zero in $H^*(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Q})$ and by exactness x^n is the image of the generator of $H^*(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1};\mathbb{Q})$. It now follows from the commutativity of the diagram that the generator $\mu \in K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ is mapped to the generator of $K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^{n-1})$ and $\mu^n \in K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ is mapped to zero.

By definition, for $\mu \in K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ and x the canonical generator of $H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$, we have

$$\mathrm{ch}(\mu) = x + \frac{x^2}{2} + \ldots$$

and the sum terminates since $x^{n+1} = 0$. Hence,

$$\mathrm{ch}(\mu^n) = x^n \qquad \text{and} \qquad \mathrm{ch}(\mu^{n+1}) = 0$$

Since $\mathrm{ch} : K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n) \to H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Q})$ is a monomorphism, $\mu^{n+1} = 0$ in $K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$. The algebraic structure of $K_{\mathbb{C}}(\mathbb{C}\mathrm{P}^n)$ now follows from the algebraic structure of $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Q})$.

For the second statement, we have the case $m = n - 1$ from the proof above. For $m < n - 1$ the result follows from the sequence of the pair $(\mathbb{C}P^n, \mathbb{C}P^m)$ since the sequence of spaces induces the exact sequence of rings,

$$\tilde{K}_{\mathbb{C}}(\mathbb{C}P^n/\mathbb{C}P^m) \longrightarrow K_{\mathbb{C}}(\mathbb{C}P^n) \longrightarrow K_{\mathbb{C}}(\mathbb{C}P^m)$$

and the generator of $K_{\mathbb{C}}(\mathbb{C}P^n)$ is mapped to the generator of $K_{\mathbb{C}}(\mathbb{C}P^m)$. □

7.1.4 Proposition. *For $K_{\mathbb{C}}(\mathbb{C}P^n)$ the Adams operations are given by*

$$\Psi^i_{\mathbb{C}}\mu^j = ((1+\mu)^i - 1)^j$$

Proof: From theorem 5.1.1 we have $\Psi^i_{\mathbb{C}}(\eta) = \eta^i$, thus we obtain,

$$\Psi^i_{\mathbb{C}}(1+\mu) = (1+\mu)^i$$

Using additivity we have,

$$\Psi^i_{\mathbb{C}}\mu = (1+\mu)^i - 1$$

It now follows from multiplicity,

$$\Psi^i_{\mathbb{C}}\mu^j = ((1+\mu)^i - 1)^j$$

□

7.1.5 Notation. To establish the next theorem we introduce the following notation. Let $n = 2k + \delta_n$ and $m = 2\ell + \delta_m$, where $\delta_n = 0$ for n even and $\delta_n = 1$ for n odd. We then write $\mu^{(\ell+1)}$ for the element of $\tilde{K}_{\mathbb{C}}(\mathbb{C}P^k/\mathbb{C}P^\ell)$ which maps to $\mu^{\ell+1} \in K_{\mathbb{C}}(\mathbb{C}P^k)$. Evidently this is independent of k which justifies the omission of k from the notation.

The standard projection $\pi : \mathbb{R}P^{2k+1} \to \mathbb{C}P^k$ factors to give projections

$$\bar{\varpi} : \mathbb{R}P^{2k+1}/\mathbb{R}P^{2\ell+1} \to \mathbb{C}P^k/\mathbb{C}P^\ell$$
$$\varpi : \mathbb{R}P^{2k+1}/\mathbb{R}P^{2\ell} \to \mathbb{C}P^k/\mathbb{C}P^\ell$$

Where the map ϖ is factors through $\bar{\varpi} : \mathbb{R}P^{2k+1}/\mathbb{R}P^{2\ell+1} \to \mathbb{C}P^k/\mathbb{C}P^\ell$. We write $\bar{\nu}^{(\ell+1)} = \bar{\varpi}^{\#}\mu^{(\ell+1)}$ and $\nu^{(\ell+1)} = \varpi^{\#}\mu^{(\ell+1)}$. Then $\bar{\nu}^{(\ell+1)}$ maps to $\nu^{(\ell+1)}$ which maps to $\nu^{\ell+1}$ in $K_{\mathbb{C}}(\mathbb{R}P^{2k+1})$.

For integers m, n we define the $\sigma(m,n)$ to be the number of even integers p, such that $m + 1 < p \leq n$, i.e. if $m = 2\ell + \delta_m$ and $n = 2k + \delta_n$, then

$$\sigma(m,n) = \begin{cases} k - \ell - 1 & \text{for } m \text{ odd} \\ k - \ell & \text{for } m \text{ even} \end{cases}$$

Note that $\sigma(0,n) = k$. In the following, to simplify notation, we just write σ for $\sigma(m,n)$. This should not give rise to confusion.

7.1.6 Theorem. *For the real projective space $\mathbb{R}\mathrm{P}^n$, $K_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ is a polynomial ring over the integers with one generator, ν, modulo the relations,*

$$\nu^2 = -2\nu \qquad \text{and} \qquad \nu^{k+1} = 0$$

where k is given by the equation, $n = 2k + \delta_n$.

For $m > 0$, we write $m = 2\ell + \delta_m$. If $\delta_m = 0$ then $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ is generated by $\nu^{(\ell+1)}$ and the projection $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$ induces a group isomorphism of $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ onto the subgroup of $K_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ generated by $\nu^{\ell+1}$. If $\delta_m = 1$, then

$$\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m) = \mathbb{Z} \oplus \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{m+1})$$

where the first term is generated by $\bar{\nu}^{(\ell+1)}$ and the second term is embedded by the projection map $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{m+1}$.

Proof: We first apply the spectral sequence to the space $X = \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$. Recall from theorem 5.3.1 that for a finite CW-complex X we have

$$E_2^{p,q} \approx H^p(X, K_{\mathbb{C}}^q(x_0))$$

and from the periodicity we have,

$$K_{\mathbb{C}}^q(x_0) = \begin{cases} \mathbb{Z} & \text{for } q \text{ even} \\ 0 & \text{for } q \text{ odd} \end{cases}$$

We can then determine $E_2^{p,q}$ of the spectral sequence for the space $X = \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$.

To obtain the spectral sequence we need to determine the cohomology groups for different values of p. For even p with $m+1 < p \leq n$, $H^p(X, \mathbb{Z}) = \mathbb{Z}_2$. If n is odd, $H^n(X, \mathbb{Z}) = \mathbb{Z}$, and if m is odd, $H^{m+1}(X, \mathbb{Z}) = \mathbb{Z}$. All other groups are zero. Clearly the sequence converges since $E_2^{p,q} = 0$ for all $p < m$ and $p > n$.

First consider the space $X = \mathbb{R}\mathrm{P}^n$. Then all differentials vanish since for n even only entries with even coordinates are non-trivial, hence differentials either maps to zero or comes from zero. For n odd, differentials mapping onto $E_2^{n,q} = \mathbb{Z}$ will originate from 0 or $\mathbb{Z}_2$, and by exactness, these vanish. Hence, all terms of E_2 survive to E_∞.

Examining the $E_\infty^{p,-p}$ diagonal we find k copies of $\mathbb{Z}_2$, every other term is zero and hence we immediately conclude that $K_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{2k}) \approx K_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{2k+1})$. From theorem 5.3.1 we have that $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ is filtered by $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{p-1})$, and with the notation introduced above, we have an element $\bar{\nu}^{(i)} \in \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{p-1})$ for $p = 2i$ which maps to $\nu^i \in \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$. Since this element is independent of n, we obtain an element $\bar{\nu}^{(i)} \in \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^p/\mathbb{R}\mathrm{P}^{p-1}) = E_1^{p,-p}$. By the naturality of the Chern character, we have the commutative diagram,

$$\begin{array}{ccc} \tilde{K}_{\mathbb{C}}(\mathbb{C}\mathrm{P}^i/\mathbb{C}\mathrm{P}^{i-1}) & \xrightarrow{\bar{\varpi}^{\#}} & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{2i}/\mathbb{R}\mathrm{P}^{2i-1}) \\ \downarrow \mathrm{ch}_i & & \downarrow \mathrm{ch}_i \\ H^{2i}(\mathbb{C}\mathrm{P}^i/\mathbb{C}\mathrm{P}^{i-1};\mathbb{Q}) & \xrightarrow{\bar{\varpi}^{*}} & H^{2i}(\mathbb{R}\mathrm{P}^{2i}/\mathbb{R}\mathrm{P}^{2i-1};\mathbb{Q}) \end{array}$$

By construction $\bar{\nu}^{(i)} = \bar{\varpi}^{\#}\mu^{(i)}$, and since $p = 2i$ we have $\mathbb{R}\mathrm{P}^p/\mathbb{R}\mathrm{P}^{p-1} = S^{2i} = \mathbb{C}\mathrm{P}^i/\mathbb{C}\mathrm{P}^{i-1}$. As in the proof of theorem 7.1.3 the Chern character $\mathrm{ch}_i : \tilde{K}_{\mathbb{C}}(S^{2i}) \to H^{2i}(S^{2i};\mathbb{Q})$ maps $\tilde{K}_{\mathbb{C}}(S^{2i})$

isomorphically onto the image of $H^{2i}(S^{2i};\mathbb{Z})$ in $H^{2i}(S^{2i};\mathbb{Q})$. Hence, by commutativity ch_i maps $\bar{\nu}^{(i)}$ to a generator of $H^{2i}(\mathbb{R}P^{2i}/\mathbb{R}P^{2i-1};\mathbb{Q})$. Thus, we conclude that $\nu,\ldots,\nu^k$ are the generators of the successive quotients of the filtration and so $K_{\mathbb{C}}(\mathbb{R}P^n) = \mathbb{Z}_{2^k}$.

Recall that $\nu = \pi^{\#}\mu = \pi^{\#}(\eta - 1)$. We then get the relation $\nu^{k+1} = 0$ from the fact that $\mu^{k+1} = 0$ and the naturality of $\pi^{\#}$. Since $\pi^*\eta$ is a complex line bundle over $\mathbb{R}P^n$, it is completely determined by its first Chern class in $H^2(\mathbb{R}P^n,\mathbb{Z}) = \mathbb{Z}_2$, hence $(\pi^*\eta)^2 = 1$. We now have $\nu^2 = (\pi^*\eta - 1)^2 = (\pi^*\eta)^2 - 2\pi^*\eta + 1 = -2\pi^*\eta + 2 = -2\nu$. Combining these relations, we have $\nu^{k+1} = (-2)^k\nu = 0$. This establishes the theorem for $m = 0$.

If m is even all differentials vanish, by the same arguments as above. Hence if m is even, the entries of E_2 all survive to E_∞. Since $\tilde{K}_{\mathbb{C}}(X)$ is filtered by the diagonal $E_\infty^{p,-p}$ we obtain

$$\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^m) = \mathbb{Z}_{2^\sigma}$$

If m is odd, consider the sequence of spaces,

$$\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1} \longrightarrow \mathbb{R}P^n/\mathbb{R}P^{2\ell+1} \longrightarrow \mathbb{R}P^n/\mathbb{R}P^{2\ell+2}$$

Since $\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1} = S^{2\ell+2}$ we have $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1}) = \mathbb{Z}$ and by periodicity that $\tilde{K}_{\mathbb{C}}^{-1}(\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1}) = \tilde{K}_{\mathbb{C}}(S(\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1})) = \tilde{K}_{\mathbb{C}}(S^{2\ell+3}) = 0$. Since $\mathrm{im}\,\imath^{\#}$ is a subring of $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1})$, $\mathrm{im}\,\imath^{\#} = 0$ or $\mathbb{Z}$, but by construction $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^{2\ell+1})$ is mapped non-trivially onto $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1})$. Hence, $\mathrm{im}\,\imath^{\#} = \mathbb{Z}$, and we obtain the split exact sequence,

$$0 \longrightarrow \tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^{2\ell+2}) \longrightarrow \tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^{2\ell+1}) \longrightarrow \tilde{K}_{\mathbb{C}}(\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1}) \longrightarrow 0$$

From the splitting it then follows that

$$\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^m) = \mathbb{Z} \oplus \tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^{m+1})$$

Next, consider the space $\mathbb{R}P^n/\mathbb{R}P^m$ where m is even, $m = 2\ell$. The sequence of the pair $(\mathbb{R}P^n, \mathbb{R}P^m)$ yields

$$\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^m) \longrightarrow K_{\mathbb{C}}(\mathbb{R}P^n) \longrightarrow K_{\mathbb{C}}(\mathbb{R}P^m)$$

Since the generator of $K_{\mathbb{C}}(\mathbb{R}P^n)$ is mapped to the generator of $K_{\mathbb{C}}(\mathbb{R}P^m)$, it follows by exactness that $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^m)$ is generated by $\nu^{(\ell+1)}$.

If m is odd, $m = 2\ell + 1$, we have the split exact sequence

$$0 \longrightarrow \tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^{2\ell+2}) \longrightarrow \tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^{2\ell+1}) \longrightarrow \tilde{K}_{\mathbb{C}}(\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1}) \longrightarrow 0$$

which maps $\bar{\nu}^{(\ell+1)}$ to a generator of $\mathbb{Z}$. It follows that $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^{2\ell+2})$ generated by $\nu^{(\ell+2)}$ is mapped monomorphically into $K_{\mathbb{C}}(\mathbb{R}P^n)$. This concludes the proof. □

7.1.7 Proposition. *For $K_{\mathbb{C}}(\mathbb{R}P^n)$ the Adams operations are given by the relations,*

$$\Psi^i_{\mathbb{C}}\nu^j = \begin{cases} 0 & i \text{ even}, \\ \nu^j & i \text{ odd} \end{cases}$$

For $K_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^m)$, write $m = 2\ell + \delta_m$. If $\delta_m = 0$ then the Adams operations are given by

$$\Psi^i_{\mathbb{C}}\nu^{(\ell+1)} = \begin{cases} 0 & i \textit{ even,} \\ \nu^{(\ell+1)} & i \textit{ odd} \end{cases}$$

and for $\delta_m = 1$ we have

$$\Psi^i_{\mathbb{C}}\nu^{(\ell+2)} = \begin{cases} 0 & i \textit{ even,} \\ \nu^{(\ell+2)} & i \textit{ odd} \end{cases}$$

$$\Psi^i_{\mathbb{C}}\bar{\nu}^{(\ell+1)} = i^{\ell+1}\bar{\nu}^{(\ell+1)} + \tfrac{1}{2}(i^{\ell+1} - \delta_i)\nu^{(\ell+2)}$$

Proof: From theorem 5.1.1 we have $\Psi^i_{\mathbb{C}}(\gamma) = \gamma^i$, for a complex line bundle γ and since we have $(\pi^{\#}\eta)^2 = 1$ for the line bundle $\pi^*\eta$ over $\mathbb{R}\mathrm{P}^n$ we obtain,

$$\Psi^i_{\mathbb{C}}(\pi^*\eta) = \begin{cases} 1 & i \text{ even,} \\ \pi^*\eta & i \text{ odd} \end{cases}$$

Using $\pi^*\eta = \nu \oplus 1$ we have,

$$\Psi^i_{\mathbb{C}}(\nu \oplus 1) = \begin{cases} 1 & i \text{ even,} \\ \nu \oplus 1 & i \text{ odd} \end{cases}$$

Additivity of the Adams operations gives,

$$\Psi^i_{\mathbb{C}}(\nu) = \begin{cases} 0 & i \text{ even,} \\ \nu & i \text{ odd} \end{cases}$$

And so, using multiplicity of the Adams operations we have

$$\Psi^i_{\mathbb{C}}\nu^j = \begin{cases} 0 & i \text{ even,} \\ \nu^j & i \text{ odd} \end{cases}$$

For the case $K_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^m)$, recall that if $m = 2\ell$ then $K_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^m)$ is mapped monomorphically into $K_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$, hence we immediately have,

$$\Psi^i_{\mathbb{C}}\nu^{(\ell+1)} = \begin{cases} 0 & i \text{ even,} \\ \nu^{(\ell+1)} & i \text{ odd} \end{cases}$$

If $m = 2\ell + 1$, then

$$\Psi^i_{\mathbb{C}}\nu^{(\ell+2)} = \begin{cases} 0 & i \text{ even,} \\ \nu^{(\ell+2)} & i \text{ odd} \end{cases}$$

follows as above.

To establish the result for $\bar{\nu}^{(\ell+1)}$, we may express $\Psi^k_{\mathbb{C}}\bar{\nu}^{(\ell+1)}$ in terms of the generators, i.e.

$$\Psi^k_{\mathbb{C}}\bar{\nu}^{(\ell+1)} = a\bar{\nu}^{(\ell+1)} + b\nu^{(\ell+2)}$$

for some integer coefficients a, b.

Using the natural inclusion $\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1} \to \mathbb{R}P^n/\mathbb{R}P^{2\ell+1}$ it follows from lemma 5.1.8 that $a = i^{\ell+1}$ since $\mathbb{R}P^{2\ell+2}/\mathbb{R}P^{2\ell+1} = S^{2\ell+2}$.

To determine b recall that $\bar{\nu}^{(\ell+1)}$ maps into $\nu^{(\ell+1)}$ and $\nu^{(\ell+2)}$ maps to $-2\nu^{(\ell+1)}$ under the projection $\mathbb{R}P^n/\mathbb{R}P^{2\ell+1} \to \mathbb{R}P^n/\mathbb{R}P^{2\ell+2}$. It follows that

$$\Psi^i_{\mathbb{C}}\nu^{(\ell+1)} \equiv a\nu^{(\ell+1)} - 2b\nu^{(\ell+1)} \mod 2^{\sigma+1}$$

and so we obtain

$$b \equiv \tfrac{1}{2}(i^{\ell+1} - \delta_i) \mod 2^{\sigma}$$

□

To state the next theorem we define the function $\omega(m,n)$ for integers m, n with $0 \le m < n$, as the number of integers $m < k \le n$ such that $k \equiv 0, 1, 2$ or $4 \mod 8$. Note that $\omega(m,n) = \omega(0,n) - \omega(0,m)$. To ease notation we will write ω for $\omega(m,n)$, ω_m for $\omega(0,m)$ and ω_n for $\omega(0,n)$.

7.1.8 Theorem. *For the real projective space $\mathbb{R}P^n$, $K_{\mathbb{R}}(\mathbb{R}P^n)$ is a polynomial ring over the integers generated by λ with two relations,*

$$\lambda^2 = -2\lambda \qquad \textit{and} \qquad \lambda^{\omega_n+1} = 0$$

For $m \not\equiv 3 \mod 4$, $\tilde{K}_{\mathbb{R}}(\mathbb{R}P^n/\mathbb{R}P^m) = \mathbb{Z}_{2^\omega}$ and the projection $\mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^m$ induces an isomorphism of $\tilde{K}_{\mathbb{R}}(\mathbb{R}P^n/\mathbb{R}P^m)$ onto the subgroup of $K_{\mathbb{R}}(\mathbb{R}P^n)$ generated by λ^{ω_m+1}. We will write $\lambda^{(\omega_m+1)}$ for the element which maps to $\lambda^{\omega_m+1} \in K_{\mathbb{R}}(\mathbb{R}P^n)$.

For $m \equiv 3 \mod 4$ we write $m = 4\ell + 3$. Then

$$\tilde{K}_{\mathbb{R}}(\mathbb{R}P^n/\mathbb{R}P^{4\ell+3}) = \mathbb{Z} \oplus \tilde{K}_{\mathbb{R}}(\mathbb{R}P^n/\mathbb{R}P^{4\ell+4})$$

The first summand is generated by an element $\bar{\lambda}^{(\omega_m+1)}$ yet to be defined and the second is embedded by the projection $\mathbb{R}P^n/\mathbb{R}P^{4\ell+3} \to \mathbb{R}P^n/\mathbb{R}P^{4\ell+4}$.

The proof of the theorem relies on a number of lemmas which we will prove along the way as we need them.

Proof: As in the proof of theorem 7.1.6 we obtain the spectral sequence from the cohomology groups for different values of p using the formula of theorem 5.3.1 for the finite CW-complex X,

$$E_2^{p,q} \approx H^p(X, K^q_{\mathbb{R}}(x_0))$$

Recall that

$$K^q_{\mathbb{R}}(x_0) = \begin{cases} \mathbb{Z}_2 & \text{for } q \equiv 6, 7 \mod 8 \\ \mathbb{Z} & \text{for } q \equiv 0, 4 \mod 8 \\ 0 & \text{otherwise} \end{cases}$$

We can now determine $E_2^{p,q}$ of the spectral sequence for the space $X = \mathbb{R}P^n/\mathbb{R}P^m$. If $K^q_{\mathbb{R}}(x_0) = \mathbb{Z}$, then we have as in the proof of theorem 7.1.6, that for even p with $m+1 < p \le n$, $H^p(X, \mathbb{Z}) = \mathbb{Z}_2$. If n is odd, $H^n(X, \mathbb{Z}) = \mathbb{Z}$, and if m is odd, $H^{m+1}(X, \mathbb{Z}) = \mathbb{Z}$. If

$K^q_{\mathbb{R}}(x_0) = \mathbb{Z}_2$, then for all p where $m < p \leq n$, the group $H^p(X, \mathbb{Z}_2) = \mathbb{Z}_2$. All other groups are zero, and hence the sequence converges since $E_2^{p,q} = 0$ for all $p \leq m$ and $p > n$.

Looking at the non-trivial terms with total degree zero, i.e. the terms $E_2^{p,-p}$, we find that ω terms are $\mathbb{Z}_2$ for $m \not\equiv 3 \mod 4$ and for $m \equiv 3 \mod 4$ we find one term is $\mathbb{Z}$ and the remaining $\omega - 1$ terms are $\mathbb{Z}_2$.

We first consider the case $m = 0$, that is $X = \mathbb{R}\mathrm{P}^n$. We then have the following lemma,

7.1.9 Lemma. *For $n \equiv 6, 7$ or $8 \mod 8$ the complexification*

$$C : \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) \to \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$$

is an isomorphism.

Proof: The complexification homomorphism $C : \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) \to \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ is an epimorphism since $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ is generated by $\nu = C\lambda$.

If $n = 8k + 6$ or $n = 8k + 7$ for some k then $\omega(n) = 4k + 3$. Thus $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ contains at most 2^{4k+3} elements. By theorem 7.1.6 $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ contains 2^{4k+3} elements and it follows that C is an isomorphism.

If $n = 8(k + 1)$ for some k, then $\omega(n) = 4k + 4$ and so $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ contains at most 2^{4k+4} elements, but again by theorem 7.1.6 $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ contains 2^{4k+4} elements and it follows that C is an isomorphism. □

Hence, since $\nu = C\lambda$ generates $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$, it follows that $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_{2^{\omega_n}}$ is generated by λ for $n \equiv 6, 7$ or $8 \mod 8$. Thus, all the terms on the diagonal $(p, -p)$ of E_2 survive to E_∞.

For $n \not\equiv 6, 7$ or $8 \mod 8$ there is an $N > n$ such that $N \equiv 6, 7$ or $8 \mod 8$. Hence, using the inclusion $\mathbb{R}\mathrm{P}^n \hookrightarrow \mathbb{R}\mathrm{P}^N$ it follows that the terms on the diagonal $(p, -p)$ of E_2 survive to infinity also for smaller values of n. Since $K_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ is filtered by the elements of the diagonal we conclude that $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_{2^{\omega_n}}$ and $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ is generated by λ.

We next establish the two relations for λ. Recall that $\lambda = \xi - 1$ where ξ is the canonical real vector bundle over $\mathbb{R}\mathrm{P}^n$. ξ is characterised by the first Stiefel-Whitney class, $w_1(\xi) \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2$, hence $\xi^2 = 1$. We now have $\lambda^2 = (\xi - 1)^2 = \xi^2 - 2\xi + 1 = -2\xi + 2 = -2\lambda$. The relation $\lambda^{\omega_n+1} = 0$ now follows from the fact that $\lambda^{\omega_n+1} = 2^{\omega_n}\lambda = 0$. This completes the theorem for $X = \mathbb{R}\mathrm{P}^n$.

We now turn to determine the generators for the space $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$ with $m > 0$. For $m \not\equiv 3 \mod 4$, consider exact the sequence

$$\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m) \xrightarrow{\pi^\#} \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) \xrightarrow{\imath^\#} \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^m)$$

Since $\imath^\#$ maps the generator of $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ to the generator of $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^m)$, the kernel of $\imath^\#$ has 2^ω elements. Since $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ has at most 2^ω elements, hence $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ is mapped isomorphically onto the subgroup of $K_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ generated by the elements, $\pm 2^{\omega_m}\lambda = \pm\lambda^{\omega_m+1}$. Following Adams, we denote the element which maps to λ^{ω_m+1} under the projection by $\lambda^{(\omega_m+1)}$.

Now consider the case $m \equiv 3 \mod 4$. We have the following lemma,

7.1.10 Lemma. *For $n > m$ and $m = 4\ell + 3$ the sequence*

$$\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{m+1}) \xrightarrow{\pi^{\#}} \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{m}) \xrightarrow{\imath^{\#}} \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{m+1}/\mathbb{R}\mathrm{P}^{m})$$

splits, where $\pi^{\#}$ is induced by the standard projection and $\imath^{\#}$ is induced by the standard inclusion.

Proof: Above we have shown that the composite

$$\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{m+1}) \xrightarrow{\pi^{\#}} \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{m}) \longrightarrow \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n})$$

is monomorphic, hence $\pi^{\#}$ is monomorphic. We need to prove that $\imath^{\#}$ is epimorphic. To prove this consider the commutative diagram, with exact rows and columns,

$$\begin{array}{ccccc}
\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3}) & \xrightarrow{\imath^{\#}} & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3}) & \xrightarrow{\partial} & \tilde{K}^1_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4}) \\
\Big\downarrow{\scriptstyle \pi_1^{\#}} & & \Big\downarrow{\scriptstyle \pi_2^{\#}} & & \\
\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+2}) & \xrightarrow{\imath_1^{\#}} & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+2}) & & \\
\Big\downarrow & & \Big\downarrow & & \\
\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+3}/\mathbb{R}\mathrm{P}^{4\ell+2}) & \xrightarrow{=} & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+3}/\mathbb{R}\mathrm{P}^{4\ell+2}) & &
\end{array}$$

Since $\mathbb{R}\mathrm{P}^{4\ell+3}/\mathbb{R}\mathrm{P}^{4\ell+2} = S^{4\ell+3}$, we have that $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+3}/\mathbb{R}\mathrm{P}^{4\ell+2}) = 0$, hence $\pi_1^{\#}$ and $\pi_2^{\#}$ are epimorphic. From the computations above we have that $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+2}) = \mathbb{Z}_2$, hence $\imath_1^{\#}$ is epimorphic, and thus the composite $\imath_1^{\#}\pi_1^{\#}$ is epimorphic. By commutativity of the diagram, it follows that $\pi_2^{\#}\imath^{\#}$ is epimorphic.

Since $\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3} = S^{4\ell+4}$, $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3}) = \mathbb{Z}$, thus $\pi_2^{\#}$ is an epimorphism from $\mathbb{Z}$ onto $\mathbb{Z}_2$, hence since $\pi_2^{\#}\imath^{\#}$ is an epimorphism, $\imath^{\#}$ must be multiplication by an odd integer α, i.e. $\mathrm{im}\imath^{\#} = \mathbb{Z}\alpha$. By exactness of the top row of the diagram, $\mathrm{im}\imath^{\#} = \ker\partial$, hence $\mathrm{im}\partial = \mathbb{Z}_{\alpha}$.

Since $\tilde{K}^1_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4})$ is filtered by the elements $E_{\infty}^{p,-p+1}$ of the spectral sequence, it follows that $\tilde{K}^1_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4})$ contains no elements of odd order. Hence $\alpha = 1$, and $\imath^{\#}$ is epimorphic. □

Since the sequence splits, it follows that

$$\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3}) = \mathbb{Z} \oplus \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4})$$

For $m = 4\ell + 3$, $\omega(0, m+1) = \omega_m + 1$. Hence from the embedding of $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4})$ into $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3})$ we have that $\lambda^{(\omega_m+2)}$ is a generator.

To complete the proof we need to determine $\bar{\lambda}^{(\omega_m+1)}$. We need the following lemma,

7.1.11 Lemma. *For $n \equiv 6, 7$ or $8 \mod 8$ and $m = 4\ell + 3$ the complexification*

$$C : \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3}) \to \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3})$$

is a monomorphism. Further, C is an isomorphism for ℓ odd.

Proof: Consider the commutative diagram,

$$\begin{array}{ccccc} \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4}) & \xrightarrow{\pi^\#} & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3}) & \xrightarrow{\imath^\#} & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3}) \\ \downarrow C_1 & & \downarrow C & & \downarrow C_2 \\ \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4}) & \longrightarrow & \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3}) & \longrightarrow & \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3}) \end{array}$$

where the rows are split exact by lemma 7.1.10 and theorem 7.1.6 and C_1, C and C_2 are the complexification homomorphisms.

Since $\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3} = S^{4\ell+4}$, we have $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3}) = \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3}) = \mathbb{Z}$. As remarked in section 5.1, we have that $C_2 : \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3}) \to \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3})$ is a monomorphism for ℓ even and an isomorphism for ℓ odd.

Next, consider the complexification homomorphism on the left. Note that if ℓ is odd then $\sigma = \omega$ and $\sigma(4\ell+4) = \omega(4\ell+4)$, if ℓ is even then $\sigma = \omega+1$ and $\sigma(4\ell+4) = \omega(4\ell+4)+1$. Hence for ℓ odd $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4})$ is generated by $\lambda^{(\omega_{m+1}+1)}$ and $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4})$ is generated by $\nu^{(\omega_{m+1}+1)}$. Thus,

$$C_1\lambda^{(\omega_{m+1}+1)} = \nu^{(\omega_{m+1}+1)}$$

and so C_1 is an isomorphism. For ℓ even, $\tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+4})$ is generated by $\nu^{(\omega_{m+1})}$, hence

$$C_1\lambda^{(\omega_{m+1}+1)} = \nu^{(\omega_{m+1}+1)} = \nu^{\omega_{m+1}}$$

Thus, C_1 is monomorphic for ℓ even. The lemma now follows from the five lemma. □

Conclusion of proof of theorem 7.1.8: For $m = 4\ell+3$ let $\ell = 2s + \delta_\ell$. If $\delta_\ell = 0$ then

$$\omega(0, 4\ell+3) = \omega(0, 8s+3) = 4s+2 = 2\ell+2$$

and if $\delta_\ell = 1$ then

$$\omega(0, 4\ell+3) = \omega(0, 8s+7) = 4s+3 = 2\ell+1$$

Recall that for $\ell = 2s+1$, $\omega(0, 4\ell+3) = \sigma(0, 4\ell+3)$. Hence, the $\mathbb{Z}$-term of $K_C(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ is generated by $\bar{\nu}^{(\omega_m+1)}$. We then choose $\bar{\lambda}^{(\omega_m+1)}$ such that $C\bar{\lambda}^{(\omega_m+1)} = \bar{\nu}^{(\omega_m+1)}$.

For $\ell = 2s$, $\omega(0, 4\ell+3) = \sigma(0, 4\ell+3)+1$. Hence the $\mathbb{Z}$-term of $K_C(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ is generated by $\bar{\nu}^{(\omega_m)}$. If we choose $\bar{\lambda}^{(\omega_m+1)}$ such that $\bar{\lambda}^{(\omega_m+1)} = -R\bar{\nu}^{(\omega_m)}$, then

$$\begin{aligned} C\bar{\lambda}^{(\omega_m+1)} &= -CR\bar{\nu}^{(\omega_m)} \\ &= -(1+\Psi_{\mathbb{C}}^{-1})\bar{\nu}^{(\omega_m)} \\ &= -2\bar{\nu}^{(\omega_m)} \end{aligned}$$

Since $\mathrm{im} C_2 = 2\mathbb{Z}$, we have that $\imath^\#\bar{\lambda}^{(\omega_m+1)}$ is a generator of $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3}) = \mathbb{Z}$ and we may choose $\bar{\lambda}^{(\omega_m+1)}$ as our generator of the $\mathbb{Z}$-term of $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3})$.

The above argument only holds for $n \equiv 6, 7$ or $8 \mod 8$. For other values of n choose integers n_1, n_2 such that $m < n_1 < n < n_2$ and $n_1, n_2 \equiv 6, 7$ or $8 \mod 8$. Then the inclusions $\mathbb{R}\mathrm{P}^{n_1} \longrightarrow \mathbb{R}\mathrm{P}^n \longrightarrow \mathbb{R}\mathrm{P}^{n_2}$ factors to induce the commutative diagram,

$$\begin{array}{ccccc} \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n_2}/\mathbb{R}\mathrm{P}^m) & \longrightarrow & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m) & \longrightarrow & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n_1}/\mathbb{R}\mathrm{P}^m) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n_2}) & \longrightarrow & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) & \longrightarrow & \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n_1}) \end{array}$$

The composition map of the top row maps the generator $\bar{\lambda}^{(\omega_m+1)}$ in $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n_2}/\mathbb{R}\mathrm{P}^m)$ to the generator $\bar{\lambda}^{(\omega_m+1)}$ in $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{n_1}/\mathbb{R}\mathrm{P}^m)$, and so defines an element $\bar{\lambda}^{(\omega_m+1)} \in \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ with the same properties.

We note that $\bar{\lambda}^{(\omega_m+1)}$ is mapped to λ^{ω_m+1}. For $n \equiv 6, 7$ or $8 \mod 8$ this follows since $\bar{\lambda}^{(\omega_m+1)}$ is mapped to $\bar{\nu}^{(\omega_m+1)}$ which maps to $\nu^{\omega_m+1} \in \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ by theorem 7.1.6, and from lemma 7.1.9 the complexification $C : \tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n) \to \tilde{K}_{\mathbb{C}}(\mathbb{R}\mathrm{P}^n)$ is an isomorphism. For other values of n the result then follows from the commutativity of the diagram above. This justifies the notation. □

7.1.12 Remark. The above construction does not consider the case when $m \equiv 3 \mod 8$ and $n = m + 2$. We will only be interested in $K_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1})$ when $\rho(m) > 1$, i.e. when m is even. Hence, the case excluded is of no relevance for the following.

7.1.13 Proposition. *For $K_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ the Adams operations are given by,*

$$\Psi^i_{\mathbb{R}}\lambda^j = \begin{cases} 0 & i \text{ even} \\ \lambda^j & i \text{ odd} \end{cases}$$

If $m \not\equiv 3 \mod 4$, then the Adams operations for $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ are given by,

$$\Psi^i_{\mathbb{R}}\lambda^{(\omega_m+1)} = \begin{cases} 0 & i \text{ even,} \\ \lambda^{(\omega_m+1)} & i \text{ odd} \end{cases}$$

For $m \equiv 3 \mod 4$, the Adams operations are given by,

$$\Psi^i_{\mathbb{R}}\lambda^{(\omega_m+2)} = \begin{cases} 0 & i \text{ even,} \\ \lambda^{(\omega_m+2)} & i \text{ odd} \end{cases}$$
$$\Psi^i_{\mathbb{R}}\bar{\lambda}^{(\omega_m+1)} = i^{2\ell+2}\bar{\lambda}^{(\omega_m+1)} + \tfrac{1}{2}(i^{2\ell+2} - \delta_i)\lambda^{(\omega_m+2)}$$

Proof: This proof is analogous to that of proposition 7.1.7. For the first equation, recall that $\lambda = \xi - 1$ and ξ is a real line bundle over $\mathbb{R}\mathrm{P}^n$. Hence ξ is determined by its first Whitney class, $w_1(\xi) \in H^1(\mathbb{R}\mathrm{P}^n, \mathbb{Z}_2) = \mathbb{Z}_2$. Thus we have $\xi^2 = 1$. As in the proof of proposition 7.1.7 we now obtain

$$\Psi^i_{\mathbb{R}}\lambda^j = \begin{cases} 0 & i \text{ even} \\ \lambda^j & i \text{ odd} \end{cases}$$

For $m \not\equiv 3 \mod 4$ recall from above that $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ is mapped isomorphically onto the subgroup of $K_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n)$ generated by $\pm\lambda^{\omega_m+1}$. Hence,

$$\Psi^i_{\mathbb{R}}\lambda^{(\omega_m+1)} = \begin{cases} 0 & i \text{ even} \\ \lambda^{(\omega_m+1)} & i \text{ odd} \end{cases}$$

For $m \equiv 3 \mod 4$ let $m = 4\ell + 3$. Then

$$\Psi^i_{\mathbb{R}}\lambda^{(\omega_m+2)} = \begin{cases} 0 & i \text{ even} \\ \lambda^{(\omega_m+2)} & i \text{ odd} \end{cases}$$

as in proposition 7.1.7, since $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{m+1})$ is embedded in $\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m)$ by the projection $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{m+1}$. For the element $\bar{\lambda}^{(\omega_m+1)}$ we have

$$\Psi^i_{\mathbb{R}}\bar{\lambda}^{(\omega_m+1)} = a\bar{\lambda}^{(\omega_m+1)} + b\lambda^{(\omega_m+2)}$$

From the inclusion $\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3} \to \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{4\ell+3}$ it follows by proposition 5.1.8 that $a = i^{2\ell+2}$ since $\mathbb{R}\mathrm{P}^{4\ell+4}/\mathbb{R}\mathrm{P}^{4\ell+3} = S^{2(2\ell+2)}$. By the same arguments as for proposition 7.1.7 it follows that

$$\Psi^i_{\mathbb{R}}\bar{\lambda}^{(\omega_m+1)} = i^{2\ell+2}\bar{\lambda}^{(\omega_m+1)} + \tfrac{1}{2}(i^{2\ell+2} - \delta_i)\lambda^{(\omega_m+2)}$$

□

7.2 The non-existence theorem

For $n \in \mathbb{N}$ we may write $n = (2a+1)2^b$ with $b = c + 4d$ and $0 \le c \le 3$ where a, b, c and d are integers. We define $\rho(n) = 2^c + 8d$.

7.2.1 Theorem. *With $\rho(n)$ defined as above, there do not exists $\rho(n)$ linear independent tangent vector fields on the $n-1$ dimensional sphere, S^{n-1}.*

With the reduction theorem of section 6.5 the theorem of non-existence of vector vector fields on the sphere is reduced to the following theorem:

7.2.2 Theorem. *$\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is not coreducible, that is, there is no map*

$$f : \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1} \longrightarrow S^m$$

such that the composite,

$$S^m = \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-1} \xrightarrow{\imath} \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1} \xrightarrow{f} S^m$$

has degree 1, where $\imath : \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-1} \to \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is the natural inclusion.

For the proof we will need the following lemma,

7.2.3 Lemma. *If $n = (2a+1)2^b$ then $3^n - 1 \equiv 2^{b+2}$ mod 2^{b+3}.*

Proof: We first prove by induction that for $k \ge 1$

$$3^{(2^k)} - 1 \equiv 2^{k+2} \mod 2^{k+4}$$

For $k = 1$ we have $3^2 - 1 = 8$. Now assume that the result holds for some $k \ge 1$. Then,

$$\begin{aligned}
3^{(2^{k+1})} - 1 &= (3^{(2^k)} - 1)(3^{(2^k)} + 1) \\
&= (2^{k+2} + x2^{k+4})(2^3(2^{k-1} + x2^{k+1}) + 2) \\
&= (2^{k+2} + x2^{k+4})(2 + y2^3) \\
&= (2^{k+3} + x2^{k+5})(1 + y2^2) \\
&\equiv 2^{k+3} \mod 2^{k+5}
\end{aligned}$$

proving the induction step. It now follows, since $3^{(2^{k+1})} - 1 \equiv 2^{k+3}$ that,

$$3^{(2^{k+1})} \equiv 1 \mod 2^{k+3}$$

and

$$3^{(2a2^k)} = 3^{(a2^{k+1})} \equiv 1 \mod 2^{k+3}$$

From this we obtain

$$\begin{aligned} 3^{(2a+1)2^k} &\equiv 3^{(2^k)} - 1 \mod 2^{k+3} \\ &\equiv 2^{k+2} \mod 2^{k+3} \end{aligned}$$

□

Proof of theorem 7.2.2: In the following we write $m = (2a+1)2^b$, where a, b, c and d are integers with $b = c + 4d$ and $0 \leq c \leq 3$. We define $\rho(m) = 2^c + 8d$. The theorem was proved for $d = 0$ by Steenrod and Whitehead using Steenrod squares [24, thm. 1.1], so we will assume that $m \equiv 0 \mod 8$. The proof is by contradiction. Since $d > 0$ we have $\omega(m, m+\rho(m)) = b+1$. Following the notation from theorem 7.1.8 we write $m-1 = 4\ell+3$. Since $m - 1 \equiv 3 \mod 4$, we have $\omega(0, m-1) = \omega_m - 1$ and $\omega_m = 2\ell + 2$. From theorem 7.1.8 we then have

$$\tilde{K}_{\mathbb{R}}(\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}) = \mathbb{Z} \oplus \mathbb{Z}_{2^{b+1}}$$

where the summands are generated by elements $\bar{\lambda}^{(\omega_m)}$ and $\lambda^{(\omega_m+1)}$ respectively. For the inclusion map, $\imath : \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-1} \to \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ we have $\imath^\# \lambda^{(\omega_m+1)} = 0$ and $\imath^\# \bar{\lambda}^{(\omega_m)}$ maps to a generator γ of $\tilde{K}_{\mathbb{R}}(S^m) = \mathbb{Z}$.

For the map f we have $f^\# \gamma = \bar{\lambda}^{(\omega_m)} + a\lambda^{(\omega_m+1)}$ for some integer a. We now have from proposition 5.1.3 that

$$f^\# \Psi^k_{\mathbb{R}} \gamma = \Psi^k_{\mathbb{R}} f^\# \gamma$$

hence, using proposition 5.1.8 we obtain

$$f^\#(k^{2\ell+2}\gamma) = \Psi^k_{\mathbb{R}}(\bar{\lambda}^{(\omega_m)} + a\lambda^{(\omega_m+1)})$$

It follows from theorem 7.1.8 that

$$k^{2\ell+2}(\bar{\lambda}^{(\omega_m)} + a\lambda^{(\omega_m+1)}) = k^{2\ell+2}\bar{\lambda}^{(\omega_m)} + \tfrac{1}{2}(k^{2\ell+2} - \delta_k)\lambda^{(\omega_m+1)} + \delta_k a\lambda^{(\omega_m+1)}$$

Thus we obtain

$$(a - \tfrac{1}{2})(k^{2\ell+2} - \delta_k)\lambda^{(\omega_m+1)} = 0$$

or

$$(a - \tfrac{1}{2})(k^{2\ell+2} - \delta_k) \equiv 0 \mod 2^{b+1}$$

We then obtain a contradiction if

$$k^{2\ell+2} - \delta_k \equiv 2^{b+1} \mod 2^{b+2}$$

for some k. But we have shown in lemma 7.2.3 that for $k = 3$ we have $k^{2\ell+2} - \delta_k \equiv 2^{b+1} \mod 2^{b+2}$. This completes the proof. □

In theorem 6.5.3 we assumed $2\rho(n) + 2 < n$, hence theorem 7.2.2 above excludes the cases $n = 1, 2, 3, 4, 6, 8$ and 16. The case $n = 1$ is both extraordinary and trivial, and we will exclude this. For $n = 2, 4$ or 8, the theorem follows since $\rho(n) = n$. The case $n = 3$ is a classical result from algebraic topology as explained in the introduction. Finally we obtain the result for $n = 6$ or 16 from lemma 6.5.5 using that $\rho(3n) = \rho(n)$.

Theorem 7.2.1 resolves a long development in mathematics, particular in homotopy theory. The implications of theorem 7.2.1 is summarised in the following corollary:

7.2.4 Corollary. *The following statements are true if and only if* $n = 1, 2, 4$ *or 8.*

i) $\mathbb{R}^n$ *is a division algebra over* $\mathbb{R}$.

ii) S^{n-1} *is parallelisable with the usual differential structure.*

iii) S^{n-1} *is an H-space.*

iv) There exists an element of Hopf invariant one in $\pi_{2n-1}(S^n)$.

Proof: According to Adams [1] the statements are related as follows, i) ⇒ ii), i) ⇒ iii), ii) ⇒ iii) and iii) ⇔ iv). We know that i) holds for $n = 1, 2, 4$ or 8 and ii) is simply a less strong version of theorem 7.2.1. Hence, the corollary follows. □

7.2.5 Remark. A proof of i) ⇒ ii) can be found in [12, p. 289], for proof of i) ⇒ iii) and ii) ⇒ iii) confer [13, lem. 2.15 p. 48].

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Master Thesis

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The existence or non-existence of vector fields on the sphere is a classical problem arising in the theory of fibre bundles and homotopy theory with implications in many areas of mathematics.
In this thesis the negative statement is proved, that is, there exists at most $\rho(n) - 1$ linearly independent tangent vector fields on the $n-1$-sphere, where $\rho(n) = 2^c + 8d$ for $n = (2a+1)2^{c+4d}$. This was first proved by J. F. Adams in his article from 1962 using K-theory. The thesis follows his approach.

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