Master's thesis

Integration on Manifolds, Volume of Geodesic Balls and Tubes, and Invariant Measure of $Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R})$

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Abstract

In this Master's thesis I will generalize integration from $\mathbb{R}^n$ to smooth finite-dimensional manifolds. This generalization will be used to calculate the volume of some classical manifolds and give a proof of Weyl's tube formula [We].

In chapter 1 a classical example for the *volume* of the n-dimensional Euclidean sphere and ball in $\mathbb{R}^n$ is given. A few concepts from differential geometry is mentioned without definitions.

Chapter 2 is a theoretical chapter which summarizes the basics in the theory of differential and Riemannian geometry. Many theorems are stated without proofs, but detailed discussion is made when integration is extended from $\mathbb{R}^n$ to manifolds. The proof of theorem 2.82 is my contribution, although a better proof can without a doubt be found.

Chapter 3 goes through the proof of *Weyl's Tube formula*, which was first published in 1939 [We]. For a q-dimensional submanifold P of $\mathbb{R}^n$, the volume of a tube around P with small radius is deduced. The result is independent of the embedding of P. Moreover, the volume of geodesic balls is calculated for some special kinds of manifolds. The main source for this part of my thesis are chapters 2–4 and a part of chapter 6 in a book called *Tubes* by Alfred Gray. Many theorems will be stated without proofs, either because they are well known or I do not find the proofs instructive. In many cases proofs are written in much more detail than Gray does. The last section deals with Kähler manifolds.

In chapter 4 both number theory and differential geometry are used to calculate an invariant meassure of the quotient space $Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R})$ with respect to an invariant density on $Sl(n,\mathbb{R})$. My main source for this part are a couple of pages taken from [Vos]. The source [Vos] is extremely compact and many statements are stated without proof and without citation. My search for better proofs were unsuccessful in most cases so the majority of the proofs in this part of this thesis are my personal deductions. At the end of the chapter I will discuss some speculations which materialized during my writing and could easily be sufficient material for a small project or another Master's thesis.

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Contents

Chapter 1

Introduction

1.1 The meaning of area and volume in relations to manifolds

A *real n-dimensional manifold* is a topological space which locally is "similar" to an open subset in $\mathbb{R}^n$. An $(n-1)$-dimensional *embedded submanifold* in another n-dimensional manifold is called a hypersurface. The precise definitions of the these concepts will be given in chapter 2.

Curves and surfaces are examples of a 1-dimensional and 2-dimensional manifolds respectively. $\mathbb{R}^n$ is an example of an n-dimensional manifold. A curve in $\mathbb{R}^2$ and a surface in $\mathbb{R}^3$ are examples of hypersurfaces. In the following definition an example of a manifold and a hypersurface is given. We will then deduce formulas for the *volume* of the manifold and for the *area* of the hypersurface.

1.1 Definition

(1) Let $R > 0$. The set

$$B_p^n(R) := \{x \in \mathbb{R}^n : |x - p| < R\}$$

is called the n-dimensional **open ball** *with center p and radius R. If $R = 1$ and $p = 0$ we write B^n instead of $B_p^n(R)$. B^n is called the n-dimensional* **open unit ball**. *The n-dimensional open ball is an example of an n-dimensional manifold.*

(2) Let $r > 0$. The set

$$S_p^n(r) := \{x \in \mathbb{R}^{n+1} : |x - p| = r\}$$

is called the n-dimensional **sphere** *with center p and radius r. If $r = 1$ we write S^n instead of $S_p^n(r)$. S^n is called the n-dimensional* **unit sphere**. *The n-dimensional sphere is an example of an n-dimensional manifold. Furthermore $S_p^n(r)$ is a hypersurface in $B_p^{n+1}(R)$ when $0 < r < R$. $S_p^n(r)$ is also a hypersurface in $\mathbb{R}^{n+1}$.*

1.2 Example

(1) $S_p^1(r)$ is a closed curve in $\mathbb{R}^2$ and its *length* is $2\pi r$.

(2) $S_p^2(r)$ is a closed surface in $\mathbb{R}^3$ and its *area* is $4\pi r^2$.

(3) $B_p^1(R)$ is an open interval and it's *length* is $2R$.

(4) $B_p^2(R)$ is an open surface and it's *area* is πR^2.

(5) $B_p^3(R)$ is an open ball and it's *volume* is $4\pi R^3/3$.

Instead of the common words *length*, *area* and *volume* in the above definition we could have used *Lebesgue measure* using concepts from measure theory. In this thesis however the word *volume* will in general be used independent of n. Thus the first line in the above example would change to

(1) $S_p^1(r)$ is a closed curve in $\mathbb{R}^2$ and its *volume* is $2\pi r$.

It's convenient to use the word *area* instead of *volume* when we are working with manifolds as hypersurfaces in another manifold. This convention will be used in chapter 3. So if we for example are working with $S_p^1(r)$ as a hypersurface in $B_p^2(R)$ for $0 < r < R$ we wold say that the *area* of $S_p^1(r)$ as a hypersurface in $B_p^2(R)$ is $2\pi r$. On the other hand the *volume* of $B_p^2(R)$ is πR^2.

We denote by $\mu(S_p^n(r))$ and $\mu(B_p^n(R))$ the Lebesgue measure of $S_p^n(r)$ and $B_p^n(R)$ respectively. So here we'd call $\mu(S_p^n(r))$ the *n-dimensional area* of $S_p^n(r)$ (as a hypersurface in $B_p^n(R)$) and $\mu(B_p^{n+1}(R))$ the *(n+1)-dimensional volume* of $B_p^{n+1}(R)$ independent of n. In the following section we'll deduce formulas for $\mu(S_p^n(r))$ and $\mu(B_p^{n+1}(R))$ [Cox].

1.2 Formulas for $\mu(S_p^n(r))$ and $\mu(B_p^{n+1}(R))$

We first observe that $\mu(S_p^n(r))$ and $\mu(B_p^{n+1}(R))$ are independent of p. We therefore can assume that $p = 0$. We have that $\mu(S_p^n(r)) = \mu(S^n)r^n$ and the relation

$$\mu(B_p^{n+1}(R)) = \int_0^R \mu(S^n)r^n\,dr = \frac{\mu(S^n)}{n+1}R^{n+1}. \tag{1.1}$$

If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a function of $r = \left(\sum_{i=1}^{n+1} x_i^2\right)^{1/2}$, we have

$$\int_0^\infty f(r)\mu(S^n)r^n\,dr = \int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(x_1, \ldots, x_{n+1})\,dx_1 \cdots dx_{n+1}.$$

This is a well known fact from functional analysis. We apply this formula to the function $f(r) = e^{-r^2}$. Then

$$\begin{aligned}\int_0^\infty e^{-r^2}\mu(S^n)r^n\,dr &= \int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty e^{-x_1^2 - \cdots - x_{n+1}^2}\,dx_1 \cdots dx_{n+1} \\ &= \left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)^{n+1} = \pi^{(n+1)/2}.\end{aligned}$$

The substitution $t = r^2$ yields

$$\int_0^\infty e^{-r^2} r^n \, dr = \frac{1}{2} \int_0^\infty e^{-t} t^{(n+1)/2-1} \, dt = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right).$$

By combining these last two equations we obtain the *area* of $S_p^n(r)$, namely

$$\mu(S_p^n(r)) = \mu(S^n) r^n = \frac{2\pi^{(n+1)/2} r^n}{\Gamma\left(\frac{n+1}{2}\right)}. \tag{1.2}$$

From this and (1.1), we deduce that

$$\mu(B_p^{n+1}(R)) = \frac{2\pi^{(n+1)/2} R^{n+1}}{(n+1)\Gamma\left(\frac{n+1}{2}\right)} = \frac{\pi^{(n+1)/2} R^{n+1}}{\Gamma\left(\frac{n+3}{2}\right)}. \tag{1.3}$$

Naturally we get the values in example 1.2 when we put a suitable value of n in the above formulas.

1.3 Remark

We have used many undefined concepts in the last couple of pages. In the following chapters precise definitions of the terms mentioned are given and I'll explain explicitly what is meant by integration on manifolds.

Chapter 2

Differential and Riemannian geometry

In this chapter many basic definitions and theorems from differential and Riemannian geometry are stated. The material presented is often taught in an one-year graduate course. Most of the theorems will be stated without proofs. For more details and proofs see for example [Ax, Bre, Lee, War] or other books about differential geometry.

2.1 Notation

2.1 Definition

(1) *Let* $\{A_i | i \in I\}$ *be a collection of sets. Then*

$$\coprod_{i\in I} A_i \tag{2.1}$$

denotes the **disjoint union** *of the sets* A_i *for* $i \in I$.

(2) *For* $i, j \in \mathbb{Z}$ *we define*

$$\delta_{i,j} := \begin{cases} 1 & i = j; \\ 0 & i \neq j. \end{cases} \tag{2.2}$$

$\delta_{i,j}$ *is called the* **Kronecker symbol**.

2.1.1 The Permutation group

2.2 Definition

(1) *Let* $i, j \in \mathbb{N}$ *where* $i \leqslant j$. *Then we let*

$$[\![i, j]\!] := \mathbb{N} \cap [i, j].$$

(2) *Let* $s \in \mathbb{N}$. *A bijective map* $\sigma : [\![1, s]\!] \to [\![1, s]\!]$ *is called a* **permutation** *of* s *elements.*

(3) *The set of all permutations of s elements form a group. It is non-commutative for $s > 2$. It is called the* **permutation group** *of s elements and is denoted by*

$$\mathfrak{S}_s.$$

(4) *Let $i, j \in [\![1, s]\!]$. We denote by $(i, j) \in \mathfrak{S}_s$ the element which satisfies the following conditions:*

   (a) *$(i, j)(i) = j$ and $(i, j)(j) = i$;*

   (b) *$(i, j)(k) = k$ if $k \neq i, j$.*

   (i, j) is called a **2-cycle**.

(5) *For $\sigma_1, \ldots, \sigma_r \in \mathfrak{S}_s$ the element $\sigma_1 \cdots \sigma_r := \sigma_1 \circ \cdots \circ \sigma_r \in \mathfrak{S}_s$ is called the* **product** *of $\sigma_1, \ldots, \sigma_r$.*

(6) *Note that $(i, j)^1 = (i, j)$ and $(i, j)^0 = \mathrm{id}_{\mathfrak{S}_s}$.*

2.3 Remark
It is well known that every permutation can be written as a product of 2-cycles, though of course not in a unique way. However, the numbers of two-cycles which product is the same as a given element $\sigma \in \mathfrak{S}_s$ is either alway even or always odd. Furthermore it's straightforward to show that $\mathrm{card}(\mathfrak{S}_s) = s!$

2.4 Definition
Let $\sigma \in \mathfrak{S}_s$. Then we define

$$\varepsilon_\sigma := \begin{cases} 1 & \text{if } \sigma \text{ is a product of even numbers of 2-cycles;} \\ -1 & \text{if } \sigma \text{ is a product of odd numbers of 2-cycles.} \end{cases}$$

We call ε_σ the **sign** *of σ.*

2.2 Manifolds and tangent spaces

2.5 Definition
Let X be a topological space.

(1) *Let $\mathcal{U}$ be a collection of subsets of X. If, for each $x \in X$ there exists an open neighborhood of x which meets, non-trivially, only finitely many elements of $\mathcal{U}$ then $\mathcal{U}$ is said to be* **locally finite**.

(2) *Let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of X. If*

$$A \in \mathcal{U} \Rightarrow \exists\, B \in \mathcal{V} : A \subset B$$

holds for every $A \in \mathcal{U}$, then we say that $\mathcal{U}$ is a **refinement** *of $\mathcal{V}$.*

(3) If every open covering of X has an open locally finite refinement then X is said to be **paracompact**.

2.6 Definition

(1) Let $\mathbb{F}$ stand for either $\mathbb{R}$ or $\mathbb{C}$ and let M be a Hausdorff space which has a countable basis. M, together with a collection of maps, called **charts**, *is said to be an n-dimensional* **smooth manifold** *over $\mathbb{F}$ if the following conditions holds:*

(a) A chart is a homeomorphism $\phi : U \to U' \subset \mathbb{F}^n$ where U is open in M and U' is open in $\mathbb{F}^n$;

(b) Each point $x \in M$ is in the domain of some chart;

(c) For charts $\phi : U \to U' \subset \mathbb{F}^n$ and $\psi : V \to V' \subset \mathbb{F}^n$ the function $\phi\psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is infinitely often $\mathbb{F}$-differentiable;

(d) The collection of charts is maximal with properties (a), (b) and (c).

A set of charts which satisfies the first three conditions above is called an **atlas**. *If $\mathbb{F} = \mathbb{R}$ we say that M is a* **real manifold**. *If $\mathbb{F} = \mathbb{C}$ we say that M is a* **complex manifold**.

(2) Let M and N be smooth manifolds over $\mathbb{F}$. Then a map $f : M \to N$ is said to be **smooth** *if, for all charts ϕ on M and ψ on N, the function $\psi \circ f \circ \phi^{-1}$ is infinitely often $\mathbb{F}$-differentiable where it is defined. If $N = \mathbb{R}$ and $f : M \to N$ is a smooth map then f is said to be a* **smooth function**. *We denote by $C^\infty(M)$ the algebra of all smooth functions on M.*

2.7 Definition

If we let $\mathbb{F} = \mathbb{R}$ in definition 2.6(1) and let M satisfy the given conditions where we have replaced $\mathbb{F}^n$ by $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 \leqslant 0\}$ where it occurs, we say that M along with its collection of charts is a **manifold with boundary**. *The $(n-1)$-dimensional manifold which consist of all points in M which get mapped to $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 = 0\}$ by a chart, is called the* **boundary** *of M.*

General agreement

> We'll always let the letter M stand for a real smooth manifold of dimension n unless otherwise stated.

2.8 Remark

(1) There are other definitions of manifolds than given in definition 2.6 with weaker conditions. The countable basis axiom in the definition of a manifold guarantees that M is paracompact. Theorem 2.10 below shows how this can be useful.

(2) It can be shown that an atlas on a manifold has a unique maximal extension. This maximal extension is called a ***maximal atlas***. If $\mathcal{A}$ is a maximal atlas for M, an atlas on M which has $\mathcal{A}$ as it's maximal extension is called a ***subatlas*** of $\mathcal{A}$ on M.

2.9 Definition
Let $\{U_\alpha |\, \alpha \in A\}$ be an open covering of a space X. A collection of continuous maps $\{f_\beta : X \to [0,1] |\, \beta \in B\}$ which satisfies:

(1) *There exists a locally finite open refinement $\{V_\beta |\, \beta \in B\}$ of $\{U_\alpha |\, \alpha \in A\}$ such that $\operatorname{supp}(f_\beta) \subset V_\beta$ for all $\beta \in B$;*

(2) *$\sum_\beta f_\beta(x) = 1$ for all $x \in X$;*

is called a **partition of unity** *subordinate to $\{U_\alpha |\, \alpha \in A\}$.*

2.10 Theorem
Let $\{U_\alpha |\, \alpha \in A\}$ be an open covering of a paracompact space X. Then there exists a partition of unity subordinate to $\{U_\alpha |\, \alpha \in A\}$. Since M is paracompact then for any open covering of M there exist a partition of unity subordinate to that covering. It can actually be shown that we can find a partition of unity for M such that the functions f_β in definition 2.9 are smooth. For proof see [Bre]

□

2.11 Definition
Let M have a maximal atlas $\mathcal{A}$. If there exist a subatlas $\mathcal{B}$ of $\mathcal{A}$ on M such that

$$\det\left(\frac{\partial \phi_i}{\partial \psi_j}\right) > 0 \text{ on } V \cap U$$

for all charts $\phi : U \to U' \subset \mathbb{R}^n$ and $\psi : V \to V' \subset \mathbb{R}^n$ in $\mathcal{B}$ then M is said to be **orientable** *and M along with $\mathcal{B}$ is said to be an* **oriented manifold**.

2.12 Remark
Note that an oriented manifold is not a manifold in general since we don't assume that it's atlas is maximal. There are more equivalent definitions of an oriented manifold. See for example [War].

2.13 Definition

(1) *We denote by $\mathcal{E}(M,p)$ the set of all smooth functions which are defined in a neighborhood of $p \in M$. If $f, g \in \mathcal{E}(M,p)$ then we regard $f = g$ if f agrees with g on some neighborhood of p.*

(2) *A map $\delta : \mathcal{E}(M,p) \to \mathbb{R}$ is called a* **differential operator** *if it is $\mathbb{R}$-linear and satisfies*

$$\delta(fg) = f(p)\delta(g) + g(p)\delta(f) \tag{2.3}$$

for all $f, g \in \mathcal{E}(M,p)$.

(3) *Let γ be a smooth curve on M with $\gamma(0) = p$. A map $D_\gamma : \mathcal{E}(M,p) \to \mathbb{R}$ defined by*

$$D_\gamma(f) = \frac{d}{dt} f(\gamma(t))|_{t=0} \tag{2.4}$$

is called the **directional derivative of f along γ at** *p and D_γ is called the* **tangent vector to γ at** *p. We regard $D_\gamma = D_{\gamma'}$ if they have the same value at p for every f.*

2.14 Definition
Let $p \in M$.

(1) *The set of all differential operators $\delta : \mathcal{E}(M,p) \to \mathbb{R}$ is called the* **tangent space** *of M at p.*

(2) *The set of all tangent vectors $D_\gamma : \mathcal{E}(M,p) \to \mathbb{R}$ at p is called the* **tangent space** *of M at p.*

2.15 Remark
The two definitions of the tangent space are isomorphic, and we will not make a distinction between them. The tangent space at $p \in M$ is a vector space over $\mathbb{R}$ of same dimension as M.

□

2.16 Definition
We will denote the tangent space of M at p by

$$T_pM.$$

2.17 Example
Let V be an n-dimensional manifold which also is a vector space over $\mathbb{R}$. Then for a vector $v \in V$ we let $\gamma_v : I \to V$ be a curve such that

$$\gamma_v(0) = p \text{ and } \left.\frac{d}{dt}\right|_{t=0} \gamma_v(t) = v.$$

An obvious choice for such curve is $\gamma_v(t) = p + tv$. Then the map $\phi : V \to T_pV$ defined by

$$\phi(v) = D_{\gamma_v}$$

is a isomorphism, and as before the precise choice of γ_v is not important as long as it satisfies the required conditions. γ_v is called a ***representation curve for the tangent vector*** v. We call this the ***natural isomorphism*** of T_pV with V. Note that if $U \subset V$ is open in V, then $T_pU = T_pV$ for all $p \in U$. The most classical examples are $V = \mathbb{R}^n$ and $V = T_pM$.

2.18 Definition
Let $\phi : M \to N$ be a smooth map between manifolds. Then the function $\phi_* : T_pM \to T_{\phi(p)}N$ defined by

$$\phi_*(D_\gamma) = D_{\phi\circ\gamma} \tag{2.5}$$

is called the **differential** *of ϕ at $p \in M$.*

2.19 Theorem
ϕ_* defined above is well defined and linear and satisfies

$$\phi_*(D_\gamma)(g) = D_\gamma(g \circ \phi). \tag{2.6}$$

for all $g \in \mathcal{E}(N, \phi(p))$. Furthermore if $\psi : N \to P$ is also a smooth map between manifolds we have

$$\psi_*\phi_* = (\psi \circ \phi)_*. \tag{2.7}$$

Equation (2.7) is known as the **chain rule**.

□

2.20 Remark
Let V and W be n-dimensional manifolds, which are also vector spaces, and $\phi : V \to W$ be a smooth map. Then by the natural identifications in example 2.17 we have

$$\phi_*(v) = \left.\frac{d}{dt}\right|_{t=0} \phi \circ \gamma_v(t).$$

2.21 Definition
Let $\phi : M \to N$ be a smooth map between manifolds.

(1) We say that ϕ is:

  (a) an **immersion** *if ϕ_* is a monomorphism at all $p \in M$.*

  (b) a **submersion** *if ϕ_* is onto at all $p \in M$.*

(2) (M, ϕ) is said to be a **submanifold** *if ϕ is an injective immersion.*

(3) We say that ϕ is an **embedding** *if (M, ϕ) is a submanifold and ϕ is homeomorphic to it's range for the relative topology on $\phi(M)$. If ϕ is an embedding, then we say that $\phi(M)$ is an* **embedded submanifold** *of N.*

(4) If $\phi(M)$ is an embedded submanifold of N and $\dim(M) = \dim(N) - 1$ then we say that $\phi(M)$ is a **hypersurface** *of N.*

2.3 Tensors, vector bundles and differential forms

2.22 Definition
Let V be an n-dimensional vector space over $\mathbb{R}$, and let V^* denote the dual space of V.

*(1) A (k,l)-***tensor** *over V is an $\mathbb{R}$-multilinear mapping*

$$F : \underbrace{V^* \times \cdots \times V^*}_{\text{l times}} \times \underbrace{V \times \cdots \times V}_{\text{k times}} \to \mathbb{R}.$$

(2) A (k,l)-tensor is said to be **contravariant** *if $k = 0$, and* **covariant** *if $l = 0$.*

(3) The vector space of all (k,l)-tensors over V is denoted by

$$T^{(k,l)}(V).$$

(4) A covariant $(k,0)$-tensor F which satisfies

$$F(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \varepsilon_\sigma F(v_1, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$ and every $\sigma \in \mathfrak{S}_k$ is called an **alternating covariant tensor***.*

(5) We denote by

$$\Lambda_k(V)$$

*the vector space of all alternating covariant $(k,0)$-tensors on V. An element in $\Lambda_k(V)$ is called a k-***form***.*

(6) We define

$$\Lambda(V) := \bigoplus_{k=0}^{\infty} \Lambda_k(V)$$

We regard $\Lambda_k(V)$ as a subspace of $\Lambda(V)$ by identify it with the subspace

$$\bigoplus_{l=0}^{k-1}\{0\} \oplus \Lambda_k(V) \oplus \bigoplus_{l=k+1}^{\infty}\{0\}.$$

When we take an element $\eta \in \Lambda(V)$ we generally assume that there exist a non-negative integer k such that $\eta \in \Lambda_k(V)$.

2.23 Remark
It can be shown that M is orientable if and only if there exist a non-vanishing n-form on M. (See for example [War]).

2.24 Definition
Let E be smooth manifold, and $\pi : E \to M$ be a surjective map. The triple (E, M, π) is said to be a k-dimensional **smooth vector bundle** *if the following conditions are satisfied:*

(1) $E_p := \pi^{-1}(p)$ is endowed with the structure of an k-dimensional vector space, for every $p \in M$. E_p is called the **fiber** *of E over p.*

(2) For every $p \in M$ there exists a neighborhood $U \subset M$ of p and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times \mathbb{R}^k \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ U & = & U \end{array}$$

Here π_1 is the projection onto the first factor. Moreover $\phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$ is a linear isomorphism for all $q \in U$. ϕ is called a **local trivilization** *of E.*

We say that E is the **total space**, *M is the* **base** *and π is the* **projection**.

2.25 Definition

(1) $TM := \coprod_{p \in M} T_pM$ is called the **tangent bundle** *of M.*

(2) $T^*M := \coprod_{p \in M} (T_pM)^*$ is called the **cotangent bundle** *of M.*

(3) $T^{(k,l)}M := \coprod_{p \in M} T^{(k,l)}(T_pM)$ is called the **bundle of (k,l)-tensor** *of M.*

(4) $\Lambda^k M := \coprod_{p \in M} \Lambda_k(T_pM)$ is called the **bundle of k-forms** *of M.*

2.26 Example
We define mappings

(1) $\pi_1 : TM \to M$ such that $(TM)_p = \pi_1^{-1}(p) = T_pM$;

(2) $\pi_2 : T^*M \to M$ such that $(T^*M)_p = \pi_2^{-1}(p) = (T_pM)^*$;

(3) $\pi_3 : T^{(k,l)}M \to M$ such that $(T^{(k,l)}M)_p = \pi_3^{-1}(p) = T^{(k,l)}(T_pM)$;

(4) $\pi_4 : \Lambda^k M \to M$ such that $(\Lambda^k M)_p = \pi_4^{-1}(p) = \Lambda_k(T_pM)$.

Then (TM, M, π_1), (T^*M, M, π_2), $(T^{(k,l)}M, M, \pi_3)$ and $(\Lambda^k M, M, \pi_4)$ are all examples of vector bundles. The first two are n-dimensional, and the other ones are of dimension $\binom{n}{k}$.

2.27 Remark
The only non-trivial thing we have to do to show that the triples in the above example are indeed vector bundles, is to show that the sets we propose are the total spaces are indeed manifolds which satisfies the desired conditions. The following lemma, proved for example in [Lee], gives a very helpful criterium which often makes this easy. It can for example be used in the above example.

2.28 Lemma
Let E be a set and $\pi : E \to M$ be a surjective map. Let's assume that we have a collection $\mathfrak{A}$ of bijective maps $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ where $\{U_\alpha\}$ is an open covering on M which satisfies the following conditions:

(1) $\pi_1 \circ \phi_\alpha = \pi$;

(2) *If $U_\alpha \cap U_\beta \neq 0$, where $\alpha, \beta \in \mathfrak{A}$, then the composite map $\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is of the form*

$$\phi_\alpha \circ \phi_\beta^{-1}(p, V) = (p, \tau(p)V)$$

where $\tau : U_\alpha \cap U_\beta \to Gl(k, \mathbb{R})$ is a smooth map where $Gl(k, \mathbb{R})$ denotes the group of invertible $(k \times k)$-matrices (see definition 2.56).

Then E can be made into a smooth k-dimensional manifold in a unique way such that (E, M, π) is a smooth k-dimensional vector bundle and ϕ_α are local trivializations for all $\alpha \in \mathfrak{A}$.

□

2.29 Definition
Let (E, M, π) be a vector bundle. A map $F : M \to E$ such that $F(p) \in E_p$ for all p, is said to be a **section** *of E. If $F(p) = 0 \in E_p$ for all $p \in M$ then F is called the* **zero section** *of E. We say that a section F is smooth if F is a smooth map between the manifolds M and E. We denote the set of all smooth sections on (E, M, π) by*

$$\mathcal{E}(M).$$

2.30 Definition

(1) *A smooth section of TM is called a* **smooth vector field** *on M. The set of all smooth vector fields on M is denoted by*

$$\mathcal{X}(M)$$

(2) *A smooth section of $T^{(k,l)}M$ is called a* **smooth (k,l)-tensor field** *on M. The set of all smooth (k,l)-tensor fields on M is denoted by*

$$\mathcal{T}^{(k,l)}(M)$$

(3) *A smooth section of $\Lambda^k M$ is called a* **differential k-form** *on M. The vector space of all differential k-form on M is denoted by*

$$\Omega^k(M).$$

2.31 Remark

(1) For $X \in \mathcal{X}(M)$ and $f : M \to \mathbb{R}$ we define $X(f) : M \to \mathbb{R}$ by

$$X(f)(p) := X(p)(f).$$

(2) By definition we have that $\Omega^k(M) \subset \Lambda^k M$.

2.32 Definition
We define

$$\Omega(M) := \bigoplus_{k=0}^{\infty} \Omega^k(M).$$

We make the same assumptions for $\Omega(M)$ as was done for $\Lambda(V)$ in definition 2.22(6).

A well known lemma called the ***Tensor Characterization Lemma*** states the following:

2.33 Lemma
(1) A map

$$\tau : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{\text{l times}} \times \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{\text{k times}} \to C^\infty(M)$$

is induced by a smooth (k,l)-tensor field in a natural way if and only if it is multilinear over $C^\infty(M)$. By abuse of language we say that the map τ is also a smooth (k,l)-tensor field if it is multilinear over $C^\infty(M)$.

(2) A map

$$\tau' : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{\text{l times}} \times \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{\text{k times}} \to \mathcal{X}(M)$$

is induced by a smooth $(k,l+1)$-tensor field in a natural way if and only if it is multilinear over $C^\infty(M)$. We also say that the map τ' is a smooth $(k,l+1)$-tensor field if it is multilinear over $C^\infty(M)$.
For more details see for example [Lee].

□

2.34 Definition

The map $\wedge : \Lambda(V) \times \Lambda(V) \to \Lambda(V)$, $(\omega,\eta) \mapsto \omega \wedge \eta$ defined by

$$\omega \wedge \eta(X_1, \ldots, X_{k+k'}) = \frac{1}{k!k'!} \sum_{\sigma \in \mathfrak{S}_{k+k'}} \varepsilon_\sigma \, \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \ldots, X_{\sigma_(k+k')}), \tag{2.8}$$

is called the **exterior** *(or* **wedge***)* **product***. Here we have assumed that $\omega \in \Lambda_k(V)$ and $\eta \in \Lambda_{k'}(V)$.*

2.35 Remark
We can define a wedge product $\wedge : \Omega(M) \times \Omega(M) \to \Omega(M)$ by (2.8) where X_i are now smooth vector fields, instead of vectors.
We have that $\wedge(\Lambda_k(V) \times \Lambda_{k'}(V)) \subset \Lambda_{k+k'}(V)$ and $\wedge(\Omega^k(M) \times \Omega^{k'}(M)) \subset \Omega^{k+k'}(M)$.
If $f \in \Omega^0(M) = C^\infty(M)$ and $\omega \in \Omega^k(M)$ we write $f\omega$ instead of $f \wedge \omega$.

2.36 Theorem

(1) *The wedge product is bilinear and associative.*

(2) *If $\omega_1, \ldots, \omega_k$ are 1-forms then*

$$(\omega_1 \wedge \cdots \wedge \omega_p)(X_1, \ldots, X_k) = \det\big(\omega_i(X_j)\big). \tag{2.9}$$

(3) *If $\omega_1, \ldots, \omega_n$ is a basis for V^* then $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} | \, i_1 < \cdots < i_k\}$ is a basis of $\Lambda_k(V)$ over $\mathbb{R}$.*

□

2.37 Corollary
Let V be an n-dimensional vector space. Then

$$\dim \Lambda_k(V) = \binom{n}{k}; \tag{2.10}$$

$$\omega \in \Lambda_k(V),\ \eta \in \Lambda_{k'}(V) \Rightarrow \omega \wedge \eta = (-1)^{kk'} \eta \wedge \omega; \tag{2.11}$$

$$\omega \in \Lambda_k(V),\ k \text{ odd} \Rightarrow \omega \wedge \omega = 0. \tag{2.12}$$

□

2.38 Remark
ω is a differential k-form if and only if for every local coordinates $x = (x_1, \ldots, x_n)$, ω can be written as

$$\omega = \sum_{i_1, \ldots, i_k = 1}^{n} f_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{2.13}$$

where all the functions $f_{i_1, \ldots, i_k}$ are smooth.

2.39 Definition
The operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$, defined by

(1) *If $k = 0$, and $X \in \mathcal{X}(M)$ then $df(X) := X(f)$;*

(2) *$d\omega = d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) := df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for $k > 0$ and then we extend by linearity to all forms;*

is called the **exterior derivative**.

The following properties for the exterior derivative are easy to prove:

2.40 Theorem

(1) d is independent of the choice of coordinates, so d is well defined.

(2) $df = \sum_i \frac{\partial f}{\partial x_i}\, dx_i$.

(3) $d^2 = 0$.

(4) For $\omega \in \Omega(M), \eta \in \Omega^k(M)$ *we have*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

2.41 Definition

Let $\phi : M \to N$ *be a smooth map between manifolds. We define an operator* $\phi^* : \Omega(N) \to \Omega(M)$ *by letting*

$$\phi^*(\omega)(X_1, \ldots, X_k) = \omega(\phi_*(X_1), \ldots, \phi_*(X_k)), \tag{2.14}$$

where $X_1, \ldots, X_k \in \mathcal{X}(M)$ *and then extend by linearity. Here we have assumed that* $\omega \in \Omega^k(N)$. *Note that* $\phi^*(\Omega^k(N)) \subset \Omega^k(M)$ *for all* k. *If* $k = 0$ *we set*

$$\phi^*(f) = f \circ \phi. \tag{2.15}$$

In the following theorem many basic features of ϕ^* are stated.

2.42 Theorem

Let θ *and* ϕ *be smooth mapping between manifolds. Then*

(1) $(\theta \circ \phi)^* = \phi^* \circ \theta^*$;

(2) $\theta^*(\omega \wedge \eta) = \theta^*(\omega) \wedge \theta^*(\eta)$;

(3) $\theta^*(f\, dy_1 \wedge \cdots \wedge dy_k) = (f \circ \theta)\, d(y_1 \circ \theta) \wedge \cdots \wedge d(y_k \circ \theta)$;

(4) $\theta^*(d\omega) = d(\theta^*(\omega))$.

2.43 Definition

Let V *be an* n*-dimensional vector space. By (2.10) we have that* $\Lambda_n(V)$ *is 1-dimensional and thus* $\Lambda_n(V)\backslash\{0\}$ *has two components.*

(1) An **orientation** *of* V *is a choice of a component of* $\Lambda_n(V)\backslash\{0\}$.

(2) V *along with an orientation of* V *is said to be an* **oriented vector space**.

(3) Let V *be an oriented vector space. An element which is in the orientation of* V *is said to* **lie in the orientation**.

2.44 Remark

There are close relation between orientable manifolds and oriented vector spaces. For details see for example [War]

The following lemma is obvious.

2.45 Lemma

Let V *be an oriented vector space, and let* λ, μ *lie in the orientation of* V. *Then there exist* $c > 0$ *such that* $\lambda = c\mu$.

□

2.4 Integration of forms

2.46 Definition
Let X be a topological space, and $f : X \to \mathbb{R}$ be a function. Then we define the **support of** f *to be*

$$\operatorname{supp}(f) = \overline{f^{-1}(\mathbb{R}\backslash\{0\})}.$$

Let $x_1, \ldots, x_n$ be the coordinates in $\mathbb{R}^n$ and ω be an n form on $\mathbb{R}^n$ whose support is compact an lies in in some open set $U \subset \mathbb{R}^n$. We can thus write

$$\omega = f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n,$$

where $\operatorname{supp}(f) \subset U$ is compact. We define the ***integral*** of ω over U to be

$$\int_U \omega := \iint \cdots \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

where the right hand side is the ordinary Riemannian (or Lebesgue) integral. By this definition we see that

$$\int_{\mathbb{R}^n} \omega = \int_U \omega.$$

If $W \subset \mathbb{R}^n$ is another open set and $\theta : W \to U$ is a diffeomorphism, we have

$$\theta^*(\omega) = (f \circ \theta) d(x_1 \circ \theta) \wedge \cdots \wedge d(x_n \circ \theta)$$

by theorem 2.42(3). Furthermore by theorem 2.40(2), we have

$$d(x_i \circ \theta) = \sum_{j=1}^{n} \frac{\partial(x_i \circ \theta)}{\partial x_j}\, dx_j = \sum_{j=1}^{n} J_{i,j}(\theta)\, dx_j,$$

where $J_{i,j}(\theta)$ is the (i,j)-th entry of the Jacobian matrix of θ. By simple computation we have (see for example [Bre, Ru] for details):

$$d(x_1 \circ \theta) \wedge \cdots \wedge d(x_n \circ \theta) = \det(J_{i,j}(\theta))\, dx_1 \wedge \cdots \wedge dx_n. \tag{2.16}$$

Hence we deduce that

$$\theta^*(\omega) = (f \circ \theta) \det(J_{i,j}(\theta))\, dx_1 \wedge \cdots \wedge dx_n \tag{2.17}$$

and thus by the standard Riemann change of variables we have (see for example [Ru])

$$\begin{aligned} \int_W \theta^*(\omega) = \int_{\mathbb{R}^n} \theta^*(\omega) &= \iint \cdots \int_{\mathbb{R}^n} f(\theta(x_1, \ldots, x_n)) \det(J_{i,j}(\theta))\, dx_1 \cdots dx_n \\ &= \pm \iint \cdots \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = \pm \int_U \omega. \end{aligned} \tag{2.18}$$

The sign is determined by the sign of $\det(J_{i,j}(\theta))$ where we assume the same sign on all components of U if U is not connected. We say that θ is ***orient preserving*** if $\det(J_{i,j}(\theta)) > 0$.

Let's assume that M along with an atlas $\mathcal{A}$ is an oriented manifold. We let $\phi : U \to U' \subset \mathbb{R}^n$ be a chart in $\mathcal{A}$ and ω be an n-form on M such that $\operatorname{supp}(\omega)$ is compact and contained in U. Then $(\phi^{-1})^*\omega$ is an n-form on U' which we extend by 0 to the whole $\mathbb{R}^n$. Thus we can look at $(\phi^{-1})^*\omega$ as a n-form in $\mathbb{R}^n$ which has compact support in U'. We define the ***integral*** of ω over U to be

$$\int_U \omega := \int_{U'} (\phi^{-1})^*\omega. \tag{2.19}$$

This is clearly an extension of integrals of forms defined on $U \subset \mathbb{R}^n$. We'll have to show though that our definition is independent of which chart in $\mathcal{A}$ we choose. Let $\psi : V \to V' \subset \mathbb{R}^n$ be another chart in $\mathcal{A}$ such that $\operatorname{supp}(\omega) \subset V$, and let $\theta = \psi \circ \phi^{-1}$. Since $\psi^{-1} \circ \theta = \phi^{-1}$ we have $(\phi^{-1})^* = \theta^* \circ (\psi^{-1})^*$ by theorem 2.42(1) and therefore

$$\int_{U'} (\phi^{-1})^*\omega = \int_{\mathbb{R}^n} (\phi^{-1})^*\omega = \int_{\mathbb{R}^n} \theta^*(\psi^{-1})^*\omega = \int_{\mathbb{R}^n} (\psi^{-1})^*\omega = \int_{V'} (\psi^{-1})^*\omega$$

by (2.18) so (2.19) is well defined. Note that we have used that $(M, \mathcal{A})$ is an oriented manifold.

Now let ω be an arbitrary n-form on M with compact support K. Let $\{f_i\}$ be a smooth partition of unity subordinate to a locally finite covering $\{U_i\}$ on M such that U_i is in the domain of some chart on $(M, \mathcal{A})$ for all $i \in I$. Let $I' = \{i \in I | \, U_i$ intersects $K\}$. Since K is compact, K intersects only finitely many U_α so I' is a finite set. Now we have that $\operatorname{supp}(f_i\omega) \subset U_i$ for all $i \in I'$ and we define

$$\int_M \omega = \sum_{i \in I} \int_{U_i} f_i\omega. \tag{2.20}$$

This is also clearly an extension of integrals of forms defined on $U \subset M$. Let $\{g_j\}$ be another smooth partition of unity subordinate to another locally finite covering $\{V_j\}$ on M such that V_j is in the domain of some chart on $(M, \mathcal{A})$ for all $j \in J$. Let $J' = \{j \in J | \, V_j$ intersects $K\}$. Since $\sum_i \sum_j f_i g_j = 1$ we have

$$\int_{U_i} f_i\omega = \int_{U_i} f_i \sum_{j \in J} g_j\omega = \int_{U_i} \sum_{j \in J'} f_i g_j\omega = \sum_{j \in J'} \int_{U_i} f_i g_j\omega = \sum_{j \in J'} \int_{V_j} g_j f_i\omega \tag{2.21}$$

where we have changed the integral and the finite sum. The last equality follows from (2.19). By combining (2.20) and (2.21) we get

$$\sum_{i \in I} \int_{U_i} f_i\omega = \sum_{i \in I'} \sum_{j \in J'} \int_{V_j} g_j f_i\omega = \sum_{j \in J'} \int_{V_j} g_j \sum_{i \in I} f_i\omega = \sum_{j \in J} \int_{V_j} g_j\omega.$$

So we've shown that the definition (2.20) is independent of which smooth partition of unity we take.

Thus if $\theta : M \to N$ is an orient preserving diffeomorphism between oriented n-dimensional manifolds and ω is an n-form on N which has a compact support we have that

$$\int_N \omega := \int_M \theta^* \omega, \tag{2.22}$$

by the above definitions and observations. The above definitions are very useful and can be used to give a rather simple proof of the general ***Stoke's Theorem***. The theorem states the following:

2.47 Theorem (Stoke's Theorem)
Let N be an n-dimensional oriented manifold with boundary ∂N and assume that $\omega \in \Omega^{n-1}(N)$ has a compact support. Then

$$\int_N d\omega = \int_{\partial N} \omega. \tag{2.23}$$

2.48 Remark
Many classical theorems in 2- and 3- dimensional calculus are obtained as a corollary from the above theorem (see for example [Bre, Fl]).

2.5 Lie groups and related topics

2.5.1 Topological groups

2.49 Definition
A **topological group** *is a Hausdorff space G together with a group structure on G such that*

(1) group multiplication $\mathrm{M} : G \times G \to G$, $(g, h) \mapsto gh$ *is continuous;*

(2) group inversion $\mathrm{INV} : G \to G$, $g \mapsto g^{-1}$ *is continuous.*

2.50 Definition

(1) Let G be a topological group, and X be a space. We denote the identity element of G by e. Then an **action** *of G on X is a map $A : G \times X \to X$, $(g, x) \mapsto g(x)$, which satisfies*

(a) $gh(x) = g(h(x))$;

(b) $e(x) = x$;

for all $g, h \in G$, and all $x \in X$.

(2) Let $x \in X$. Then $G(x) = \{g(x) \in X \mid g \in G\}$ is called the **orbit** *of x. $x, y \in X$ are in the same orbit with respect to G if and only if there exits an element $g \in G$ such that $x = gy$. Since G is a group we see that orbits define an equivalence relation on X. Sometimes we write $G \cdot x$ instead of $G(x)$.*

2.5.2 Lie groups and Lie algebras

2.51 Definition

(1) *A topological group G which also has a manifold structure is called a* **Lie group**.

(2) *Let $\mathfrak{g}$ be a real vector space over $\mathbb{R}$ and $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a bilinear form that satisfies*

$$\begin{aligned} [x,y] &= -[y,x]; \quad \text{(anti-commutativity)} \\ [[x,y],z] + [[y,z],x] + [[z,x],y] &= 0. \quad \text{(Jacobi identity)} \end{aligned} \tag{2.24}$$

Then the pair $(\mathfrak{g}, [\cdot,\cdot])$ is said to be a **Lie algebra**.

(3) *Let G be a Lie group and $\sigma \in G$. The function $l_\sigma : G \to G$ defined by*

$$l_\sigma(\tau) = \sigma\tau \tag{2.25}$$

is called the **left translation by** *σ.*

(4) *Let X be a vector field on G. If*

$$(l_\sigma)_* \circ X = X \circ l_\sigma \tag{2.26}$$

for every $\sigma \in G$ then X is said to be a **left invariant** *vector field on G.*

(5) *Let G be a Lie group. We denote by the letter $\mathfrak{g}$ the vector space of all left invariant vector fields on G.*

(6) *Let G be a n-dimensional Lie group. A form $\omega \in \Omega^k(G)$ such that*

$$l_g^*\omega = \omega. \tag{2.27}$$

for all $g \in G$ is said to be a **left invariant** *k-form on G.*

In the following theorem we get one of the main property of $\mathfrak{g}$. I'll go through the proof since I find it instructive.

2.52 Theorem

$\mathfrak{g}$ is a real vector space and the map $\alpha : \mathfrak{g} \to T_eG$ defined by $\alpha(X) = X(e)$ is an isomorphism. Hence $\dim \mathfrak{g} = \dim T_gG = \dim G$.

PROOF: It's easy to show the first part and α is linear by definition of a vector field.
Assume $\alpha(X) = \alpha(Y)$. For $\sigma \in G$ we have

$$\begin{aligned} X(\sigma) &= X \circ l_\sigma(e) = ((l_\sigma)_* \circ X)(e) \\ &= (l_\sigma)_*(X(e)) = (l_\sigma)_*(\alpha(X)) = (l_\sigma)_*(\alpha(Y)) \\ &= (l_\sigma)_*(Y(e)) = ((l_\sigma)_* \circ Y)(e) = (Y \circ l_\sigma)(e) = Y(\sigma), \end{aligned}$$

so α is injective.
Let $x \in T_eG$. We want to find $X \in \mathfrak{g}$ such that $\alpha(X) = x$. Define X on G by $X(\sigma) = (l_\sigma)_*(x)$. Now

$$\alpha(X) = X(e) = (l_e)_*(x) = x$$

and since by the chain rule (see (2.7)) we have

$$(X \circ l_\tau)\sigma = X(\tau\sigma) = (l_{\tau\sigma})_*(x) = (l_\tau l_\sigma)_*(x) = (l_\tau)_*(l_\sigma)_*(x) = (l_\tau)_*(X(\sigma)),$$

for all $\sigma \in G$ and thus X lies in $\mathfrak{g}$ and satisfies the required property. Hence α is surjective. The proof of the theorem is complete.

□

2.53 Remark
It can be shown that $\mathfrak{g}$ forms a Lie algebra under the Lie bracket operation (see definition 2.63(3)) on vector fields [War]. This explains the common notation of $\mathfrak{g}$ in definition 2.51(2) and 2.51(5).

2.54 Definition
The Lie algebra $\mathfrak{g}$ of left invariant vector fields on a Lie group G is said to be the **Lie algebra of the Lie group** *G.*

2.55 Remark
By theorem 2.52, we can identify the Lie algebra of the Lie group G with T_eG. We will make use of this in chapter 4.

2.5.3 Some classical Lie-groups and their Lie algebras

In this section I will give some examples of Lie groups and their Lie algebras. These examples are well known and more details can be found in books about Lie groups and differential geometry.

2.56 Definition
Let $\tilde{R}$ be a ring.

(1) *We denote by $M(n, \tilde{R})$ the vector space of all $(n \times n)$-matrices, with coefficients in $\tilde{R}$.*

(2) *We let $Gl(n, \tilde{R}) = \{A \in M(n, \tilde{R}) | \det(A) \neq 0\}$. $Gl(n, \tilde{R})$ is a Lie group.*

(3) *We let $M_m(n, \tilde{R}) = \{A \in M(n, \tilde{R}) | \det(A) = m\}$ where $m \in \tilde{R}$.*

(4) *We let $O(n, \tilde{R}) = \{A \in Gl(n, \tilde{R}) | A^t = A^{-1}\}$, where A^t denotes the* transpose *of A. $O(n, \tilde{R})$ is a Lie group and is called the* **orthogonal group**.

(5) *We let $SO(n, \tilde{R}) = \{A \in O(n, \tilde{R}) | \det(A) = 1\}$. $SO(n, \tilde{R})$ is a Lie group and is called the* **special orthogonal group**.

(6) *We let* $Sl(n, \tilde{R}) = M_1(n, \tilde{R})$. $Sl(n, \tilde{R})$ *is a Lie group called the* **special linear group**.

2.57 Theorem
Let $\mathbb{F}$ *stand for either* $\mathbb{R}$ *or* $\mathbb{C}$.

(1) *The Lie algebra of* $Gl(n, \mathbb{F})$ *is*

$$\mathfrak{gl}(n, \mathbb{F}) := M(n, \mathbb{F}) \tag{2.28}$$

(2) *The Lie algebra of* $O(n, \mathbb{F})$ *is*

$$\mathfrak{o}(n, \mathbb{F}) := \{A \in \mathfrak{gl}(n, \mathbb{F}) |\, A^t = -A\}. \tag{2.29}$$

Hence $\dim_{\mathbb{F}} \mathfrak{o}(n, \mathbb{F}) = n(n-1)/2$. $\mathfrak{o}(n, \mathbb{F})$ *is also the Lie algebra of* $SO(n, \mathbb{F})$.

(3) *The Lie algebra of* $Sl(n, \mathbb{F})$ *is*

$$\mathfrak{sl}(n, \mathbb{F}) := \{A \in \mathfrak{gl}(n, \mathbb{F}) |\, \mathrm{trace} A = 0\}. \tag{2.30}$$

Hence $\dim_{\mathbb{F}} \mathfrak{sl}(n, \mathbb{F}) = n^2 - 1$.

□

2.6 Riemannian manifolds

2.58 Definition
Let M *be a smooth manifold and* $g : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$ *be a smooth* $(2, 0)$*-tensor field.*

(1) g *is said to be* **symmetric** *if* $g(X, Y) = g(Y, X)$ *for all* $X, Y \in \mathcal{X}(M)$.

(2) g *is said to be* **positive definite** *if* $g(X, X) > 0$ *for all* $X \in \mathcal{X}(M)$, $X \neq 0$.

(3) g *is said to be a* **Riemannian metric** *if it is symmetric and positive definite, and then* M *is said to be a* **Riemannian manifold** *with respect to* g.

2.59 Remark
If g is a Riemannian metric on a manifold M then it induces an inner product $\langle \cdot, \cdot \rangle$ on each tangent space T_pM, by setting $\langle X(p), Y(p) \rangle := g(X, Y)|_p$ for all $X, Y \in \mathcal{X}(M)$. By using partition of unity it can be shown that there exist a Riemannian metric on every real manifold. We often write $\langle X, Y \rangle$ instead of $g(X, Y)$ for $X, Y \in \mathcal{X}(M)$.

2.60 Definition
Let $p \in M$ *and* $U \subset M$ *be an open neighborhood of* p *and* $(E_1, \ldots, E_n)$ *be* n *smooth vector fields defined on* U. *If* $(E_1|_q, \ldots, E_n|_q)$ *is a basis for* T_qM *for every* $q \in U$, *then* $(E_1, \ldots, E_n)$ *is said to be a* **local frame** *for* TM *on* U.

2.61 Remark
We often say that a local frame for TM defined on U is a local frame on M.

2.62 Definition

(1) Let $(E_1, \dots, E_n)$ be a local frame for TM. If we construct 1-forms $\phi_1, \dots, \phi_n$, defined by letting $\phi_i(E_j) = \delta_{i,j}$, then $(\phi_1, \dots, \phi_n)$ is a local frame for TM^*, called the **dual coframe**.

(2) Let $x : U \to U' \subset \mathbb{R}^n$ be a chart on M. If we write $x = (x_1, \dots, x_n)$ then

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$$

is a local frame on U for TM called a **coordinate frame**. *It's dual coframe is denoted by*

$$(dx_1, \dots, dx_n).$$

When it's clear which chart we're working with we often write (∂_i) for the coordinate frame and (dx_i) for it's dual coframe.

2.6.1 Connections

2.63 Definition

(1) Let (E, M, π) be a vector bundle. A map $\nabla : \mathcal{X}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$, $(X, Y) \mapsto \nabla_X Y$ such that for all $X, X_1, X_2, Y, Y_1, Y_2 \in \mathcal{X}(M)$ we have

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y, \quad \text{for all } f, g \in C^\infty(M); \tag{2.31}$$

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2, \quad \text{for all } a, b \in \mathbb{R}; \tag{2.32}$$

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y, \quad \text{for all } f \in C^\infty(M); \tag{2.33}$$

is called a **connection** *in E.*

(2) A connection in TM is called a **linear connection**.

(3) A linear connection ∇ which satisfies

$$[X, Y] := XY - YX = \nabla_X Y - \nabla_Y X \tag{2.34}$$

for all vector fields X, Y is said to be **symmetric**. *$[X, Y]$ is called the* **Lee brackets** *of $(X, Y) \in \mathcal{X}(M) \times \mathcal{X}(M)$. $[X, Y]$ acts on functions via*

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf).$$

where $p \in M$.

(4) If $\langle \cdot, \cdot \rangle$ is a Riemannian metric on M then a linear connection ∇ which satisfies

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \tag{2.35}$$

for all vector fields $X, Y, Z \in \mathcal{X}(M)$ is said to be **compatible with** $\langle \cdot, \cdot \rangle$.

2.64 Theorem (The fundamental theorem of Riemannian geometry)
Let (M, g) be a Riemannian manifold. There exist a unique linear connection that is compatible with g and symmetric.

□

2.65 Definition
The unique connection in the above theorem is called the **Levi-Civita connection** *of g. It is also known as the* **Riemannian connection** *of g.*

2.6.2 Geodesics and the exponential map

2.66 Definition
Let $\gamma : I \to M$, $t \mapsto \gamma(t)$ be a map, where $I \subset \mathbb{R}$ is an interval.

(1) *We define $\gamma'(t)$ as $\gamma_*(d/dt)$.*

(2) *If γ is smooth, then γ is called a* **curve**.

(3) *If $I = [a, b]$, and γ is a curve defined on I, then γ is said to be a* **curve segment**.

(4) *If γ is a curve and $\gamma'(t) \neq 0$ for all $t \in I$ then γ is said to be a* **regular curve**.

(5) *If γ is a regular curve such that $|\gamma'(t)| = 1$ for all $t \in I$ we say that γ is a* **unit speed curve**.

(6) *If $\gamma : [a, b] \to M$ is continuous and there exist real numbers $a_0, a_1, \ldots, a_k$ such that*

   (a) *$a_i < a_j$ if $i < j$;*

   (b) *$\gamma|_{[a_{i-1}, a_i]}$ is a regular curve for $i = 1, \ldots, k$;*

   then γ is said to be an **admissible curve**. *The numbers $a_0, a_1, \ldots, a_k$ are said to be a* **subdivision** *for the admissible curve γ.*

(7) *If γ is an admissible curve with subdivision $a_0, a_1, \ldots, a_k$, then we define the* **length** *of γ to be*

$$L(\gamma) := \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} |\gamma'(t)| \, dt.$$

   It's easy to show that this is well defined.

(8) *Let $\gamma : I \to M$ be an admissible curve such that, for every admissible curve $\tilde{\gamma} : [c, d] \to M$ with $\gamma(a) = \tilde{\gamma}(c)$ and $\gamma(b) = \tilde{\gamma}(d)$ we have that $L(\gamma) \leqslant L(\tilde{\gamma})$. Then γ is said to be a* **minimizing curve**.

(9) *If $\gamma : I \to M$ is an admissible curve such that for every t in the interior of I there exist an open neighborhood $]a, b[\subset I$ of t with the property that $\gamma|_{[a,b]}$ is a minimizing curve then γ is said to be a* **locally minimizing curve**.

2.67 Definition

(1) *Let $\gamma :]a, b[\to M$ be a curve. A* **vector field along the curve** *γ is a mapping Y, defined on $]a, b[$ which assigns every $t \in]a, b[$ a vector Y_t from the tangent space $T_{\gamma(t)}M$ such that, the mapping $]a, b[\to \mathbb{R}, t \mapsto Y_t f$ is smooth for every smooth function f on M.*

(2) *Let Y be a vector field along the curve γ and ∇ be a linear connection on M. We can define a new vector field*

$$\nabla_{\gamma'} Y$$

along the curve γ by the following:

(a) *If $\gamma'(t) = 0$, then we put $(\nabla_{\gamma'} Y)_t := 0$.*

(b) *If $\gamma'(t) \neq 0$, then there exist an open interval J around t, a neighborhood U around the point $\gamma(t)$ in M, such that $\gamma(J)$ is a 1-dimensional submanifold in U and $\gamma|_J : J \to \gamma(J)$ is a homeomorphism. Moreover the vector field Y may be extended from $\gamma(J)$ to a smooth tangent fields $\hat{Y}$ on U such that $\hat{Y}_{\gamma(t)} = Y_t$. We put*

$$(\nabla_{\gamma'} Y)_t := \nabla_{\gamma'(t)} \hat{Y}$$

2.68 Remark

(1) It can be shown that we can indeed find J, U and $\hat{Y}$ and that the definition does not depend on the choice we make. Furthermore if $\gamma : I \to M$ is a curve, then the derivative $\gamma' : I \to T_{\gamma(t)}(M)$, $t \mapsto \gamma'(t)$ is a smooth tangent field along the curve γ.

(2) Another equivalent method for defining vector field along curves can be found in [Lee].

2.69 Definition

Let ∇ be a linear connection on M. A **geodesic** *on M with respect to γ is a smooth curve $\gamma : I \to M$ on M, such that*

$$\nabla_{\gamma'} \gamma' = 0.$$

A geodesic with respect to the Riemannian connection is called a **Riemannian geodesic**.

2.70 Definition

(1) *We denote by $\Gamma_{p,q}$ the set of all admissible curves $\gamma : [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$.*

(2) *We define a map $d : M \times M \to \mathbb{R} \cup \{\infty\}$ by letting $d(p,q) = \infty$ if p and q lies in different connected components and by letting*

$$d(p,q) = \inf_{\gamma \in \Gamma_{p,q}} \{L(\gamma)\},$$

otherwise. If M is connected it can be shown that d is a metric on M.

(3) *If M is connected and (M,d) is a complete metric space then we say that M is a* **complete Riemannian manifold**.

(4) *Let M be connected and let $V, U \subset M$. Then we define*

$$\operatorname{dist}(V,U) := \inf_{p \in V, q \in U} d(p,q)$$

2.71 Remark
There are more equivalent definitions of a complete Riemannian manifold.

The following two theorems describe the importance of geodesics and gives of a intuitive way to think about them. We'll not go through the proofs but they can be found in most books about Riemannian geometry (see for example [Lee]).

2.72 Theorem
Let ∇ be a linear connection on M, $p \in M$, $V \in T_pM$ and $t_0 \in \mathbb{R}$. Then there exist an open interval around t_0 and a geodesic $\gamma : I \to M$ such that $\gamma(t_0) = p$ and $\gamma'(t_0) = V$. Every two such geodesics agree on their common domains.

□

2.73 Theorem

(1) *Every minimizing regular curve is a geodesic.*

(2) *All Riemannian geodesics are constant speed curves.*

(3) *Every Riemannian geodesic is locally minimizing.*

□

2.74 Definition
Let $v \in T_pM$. Then we denote by

$$\xi_v$$

the unique maximal unit speed geodesic which satisfies $\xi_v(0) = p$ and $\xi_v'(0) = v$.

2.75 Definition
Let $p \in M$.

(1) *The set*

$$\mathcal{E}_p := \{v \in T_pM \,|\, \xi_v \text{ is defined on an interval containing } [0,1]\}$$

is called the **domain for the exponential map** *at p.*

(2) The map $\exp_p : \mathcal{E}_p \to M$ *defined by*

$$\exp_p(v) = \xi_v(1) \tag{2.36}$$

is called the **exponential map** *at* p*.*

(3) Let $\varepsilon > 0$*. If*

$$\exp_p : B^n_\varepsilon(0) \to \exp_p(B^n_\varepsilon(0))$$

is a diffeomorphism then $\exp_p(B^n_\varepsilon(0))$ *is called a* **geodesic ball** *in* M *with center* p *and radius* ε*.*

In the following theorem some basic features of geodesics and the exponential map at p is stated.

2.76 Theorem

(1) $\exp_p$ *is smooth;*

(2) $\xi_v(t) = \exp_p(tv)$ *for all* $t \in \mathbb{R}$*;*

(3) $\xi_{cv}(t) = \xi_v(ct)$*, for all* $t \in \mathbb{R}$ *and all* $c \in \mathbb{R}\backslash\{0\}$*.*

□

In the following theorem we get one of the most essential property of the exponential map. I'll write down the proof since I find it instructive.

2.77 Theorem

Let $p \in M$*. Then there are neighborhoods* V *of* $0 \in T_pM$ *and* U *of* $p \in M$ *such that* $\exp_p : V \to U$ *is a diffeomorphism.*

PROOF: We have a natural isomorphism between $T_0(T_pM)$ and T_pM. Let $v \in T_pM$ and let $\tau : \mathbb{R} \to T_pM$ be defined by letting $\tau(t) = tv$. Then τ is a curve in T_pM such that $\tau(0) = 0 \in T_pM$ and $\tau'(0) = v \in T_0(T_pM) = T_pM$. By theorem 2.76(2) we have

$$\begin{aligned} (\exp_p)_*(v) &= \frac{d}{dt}\bigg|_{t=0} (\exp_p \circ \tau)(t) \\ &= \frac{d}{dt}\bigg|_{t=0} \exp_p(tv) = \frac{d}{dt}\bigg|_{t=0} \xi_v(t) = v. \end{aligned} \tag{2.37}$$

Hence $(\exp_p)_*$ is the identity map, which is invertible so $\exp_p$ is a diffeomorphism by the inverse function theorem.

□

2.6.3 Curvatures

2.78 Definition

Let $X, Y \in \mathcal{X}(M)$ and ∇ be the Riemannian connection on $\mathcal{X}(M) \times \mathcal{X}(M)$. The map $R_{XY} : \mathcal{X}(M) \to \mathcal{X}(M)$ defined by

$$R_{XY} := \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \tag{2.38}$$

is called the **curvature transformation** *with respect to (X, Y).*

2.79 Theorem

The curvature transformation of M satisfies

$$\begin{aligned} R_{XY} &= -R_{YX} && (2.39)\\ \langle R_{WX}Y, Z\rangle &= -\langle R_{WX}Z, Y\rangle && (2.40)\\ \langle R_{WX}Y, Z\rangle &= \langle R_{YZ}W, X\rangle && (2.41)\\ \underset{XYZ}{\mathfrak{S}} R_{XY}Z &= 0 && (2.42) \end{aligned}$$

where $\underset{XYZ}{\mathfrak{S}}$ denotes the cyclic sum. (2.42) is known as the **first Bianchi identity**.

□

2.80 Definition

We let $X, Y, Z, W \in \mathcal{X}(M)$.

(1) *The function $R_{WXYZ} := \langle R_{WX}Y, Z\rangle$ is called the* **curvature tensor**. *It is a $(4, 0)$-tensor field.*

(2) *Let $(E_1, \dots, E_n)$ be a local orthonormal frame for TM defined on $U \subset M$. The function $\rho : \mathcal{X}(U) \times \mathcal{X}(U) \to \mathbb{R}$ defined by*

$$\rho(X, Y) = \sum_{a=1}^{n} R_{XE_aYE_a} \tag{2.43}$$

is called the **Ricci curvature**. *Note that ρ is symmetric due to (2.41). By the* **scalar curvature** *we mean the function τ defined by*

$$\tau = \sum_{a=1}^{n} \rho(E_a, E_a) = \sum_{a,b=1}^{n} R_{E_aE_bE_aE_b}. \tag{2.44}$$

(3) *The map $K^M : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}$, $X, Y \mapsto K_{XY}$ defined for linearly independent $X, Y \in \mathcal{X}(M)$ by*

$$K_{XY} = \frac{R_{XYXY}}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}. \tag{2.45}$$

is called the **sectional curvature** *of M. If M is 2-dimensional there is only one sectional curvature which we call the* **Gaussian curvature**.

2.81 Remark

(1) It can be shown that both the Ricci and the scalar curvature are independent of the choice of orthonormal local frame. There are other equivalent definitions of the Gaussian curvature, which can be found in most books about differential geometry.

(2) We often use lowercase letters to indicate the value of a vector field at a given point. For example we would write R_{xyzw} for R_{XYZW} evaluated at some point $p \in M$.

2.7 The Riemannian volume element

2.82 Theorem

Let (M, g) be an oriented Riemannian manifold. Then there exist a unique differential n-form dV on M such that $dV(E_1, \ldots, E_n) = 1$ for every oriented orthonormal basis $(E_1, \ldots, E_n)$ for T_pM for all $p \in M$. Moreover dV can be written in terms of any oriented local frame $\{\hat{E}_i\}$ as

$$\sqrt{\det(\hat{g}_{ij})}\hat{\phi}_1 \wedge \cdots \wedge \hat{\phi}_n, \tag{2.46}$$

where $(\hat{g}_{ij}) = (\langle \hat{E}_i, \hat{E}_j \rangle)$ and (ϕ_i) is the dual coframe.

PROOF:

Uniqueness: In the proof we'll use ***Einstein summation convention***, namely

$$\alpha^i E_i := \sum_{i=1}^{n} \alpha_i E_i \tag{2.47}$$

where $\alpha_i \in \mathbb{R}$ and E_i is a smooth vector field on M, for $i = 1, \ldots, n$.
Assume that dV and dV' are n-forms that satisfies

$$dV(E_1, \ldots, E_n) = dV'(E_1, \ldots, E_n) = 1$$

for every oriented orthonormal basis $(E_1, \ldots, E_n)$ for T_pM. Now let $X_1, \ldots, X_n$ be vectors in T_pM. Now for every n-form dW we have

$$\begin{aligned} dW(X_1, \ldots, X_n) &= dW(\alpha_1^{i_1} E_{i_1}, \ldots, \alpha_n^{i_n} E_{i_n}) = \alpha_1^{i_1} \cdots \alpha_n^{i_n} dW(E_{i_1}, \ldots E_{i_n}) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \varepsilon_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)} \, dW(E_1, \ldots E_n) \end{aligned}$$

From this is follows that $dV = dV'$.

Existence: We show that 2.46 applied to oriented orthonormal basis $(E_1, \ldots, E_n)$ gives 1. Let

$$A := \begin{bmatrix} E_1 \\ \vdots \\ E_n \end{bmatrix} \begin{bmatrix} \hat{E}_1 & \cdots & \hat{E}_n \end{bmatrix} = \begin{bmatrix} \langle E_1, \hat{E}_1 \rangle & \cdots & \langle E_1, \hat{E}_n \rangle \\ \vdots & \ddots & \vdots \\ \langle E_n, \hat{E}_1 \rangle & \cdots & \langle E_n, \hat{E}_n \rangle \end{bmatrix}$$

and observe that

$$\begin{bmatrix} E_1 & \cdots & E_n \end{bmatrix} \cdot A = \begin{bmatrix} \langle E_1, \hat{E}_1\rangle E_1 + \cdots + \langle E_n, \hat{E}_1\rangle E_n \\ \vdots \\ \langle E_1, \hat{E}_n\rangle E_1 + \cdots + \langle E_n, \hat{E}_n\rangle E_n \end{bmatrix}' = \begin{bmatrix} \hat{E}_1 & \cdots & \hat{E}_n \end{bmatrix}. \quad (2.48)$$

We'll define $|A| := \det(A)$ for a $(n \times n)$-matrix A. Since for matrices A, B we have $|A \cdot B| = |A| \cdot |B|$ we see that

$$\begin{aligned} &\sqrt{\det(g_{ij})}\hat{\phi}_1 \wedge \cdots \wedge \hat{\phi}_n(E_1, \ldots, E_n) \\ &= \left| \begin{bmatrix} \hat{E}_1 \\ \vdots \\ \hat{E}_n \end{bmatrix} \begin{bmatrix} \hat{E}_1 & \cdots & \hat{E}_n \end{bmatrix} \right|^{1/2} \left| A^{-1} \cdot \begin{bmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\phi}_n \end{bmatrix} \begin{bmatrix} E_1 & \cdots & E_n \end{bmatrix} \cdot A \right| \\ &= \left| \begin{bmatrix} \hat{E}_1 \\ \vdots \\ \hat{E}_n \end{bmatrix} \begin{bmatrix} \hat{E}_1 & \cdots & \hat{E}_n \end{bmatrix} \right|^{1/2} |A^{-1}| \quad \text{(by 2.48)}. \end{aligned}$$

Since $A \cdot A$ is equal to

$$\begin{bmatrix} \langle E_1, \hat{E}_1\rangle\langle E_1, \hat{E}_1\rangle + \cdots + \langle E_1, \hat{E}_n\rangle\langle E_n, \hat{E}_1\rangle & \ldots & \langle E_1, \hat{E}_1\rangle\langle E_1, \hat{E}_n\rangle + \cdots + \langle E_1, \hat{E}_n\rangle\langle E_n, \hat{E}_n\rangle \\ \vdots & \ddots & \vdots \\ \langle E_n, \hat{E}_1\rangle\langle E_1, \hat{E}_1\rangle + \cdots + \langle E_n, \hat{E}_n\rangle\langle E_n, \hat{E}_1\rangle & \ldots & \langle E_n, \hat{E}_1\rangle\langle E_1, \hat{E}_n\rangle + \cdots + \langle E_n, \hat{E}_n\rangle\langle E_n, \hat{E}_n\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \hat{E}_1 \\ \vdots \\ \hat{E}_n \end{bmatrix} \begin{bmatrix} \hat{E}_1 & \ldots & \hat{E}_n \end{bmatrix}$$

we have that $\sqrt{\det(g_{ij})}\hat{\phi}_1 \wedge \ldots \wedge \hat{\phi}_n(E_1 \ldots E_n) = \sqrt{|A|^2}|A^{-1}| = 1$, due to the orientation of $\{\hat{E}_i\}$. So dV has the desired properties.

□

2.83 Definition
The unique form dV in the last theorem is called the **volume form** *of (M, g).*

2.84 Theorem
Let V be an oriented n-dimensional vector space. Then there exist a linear operator $* : \Lambda(V) \to \Lambda(V)$ called the **Hodge star operator** *which is well-defined by requiring that for any orthonormal basis $(e_1, \ldots, e_n)$ of V, it satisfies*

$$*(1) = \pm e_1 \wedge \cdots \wedge e_n, \qquad *\,(e_1 \wedge \cdots \wedge e_n) = \pm 1,$$
$$*(e_1 \wedge \cdots \wedge e_k) = \pm e_{k+1} \wedge \cdots \wedge e_n,$$

and then extend it linearly to whole $\Lambda(V)$. Here we choose $+$ if $e_1 \wedge \cdots \wedge e_n$ lies in the orientation and $-$ otherwise. Note that $*(\Lambda_k(V)) \subset \Lambda_{n-k}(V)$.

□

2.85 Remark
Like with the wedge product we have a natural extension of the Hodge star operators to differential forms. Let $(\phi_1, \ldots, \phi_n)$ be a local orthonormal frame for TM^*. Then we define the Hodge star operator $* : \Omega(M) \to \Omega(M)$ by

$$*(1) = \pm\phi_1 \wedge \cdots \wedge \phi_n, \qquad *\,(\phi_1 \wedge \cdots \wedge \phi_n) = \pm 1,$$
$$*(\phi_1 \wedge \cdots \wedge \phi_k) = \pm\phi_{k+1} \wedge \cdots \wedge \phi_n,$$

and then extend linearly to all differential forms. We choose the sign as before.

2.86 Remark
(1) Assume that M is oriented and ω is a non-vanishing differential k-form on M. Then $\omega \wedge *\omega$ is the volume form on M.

(2) For more details and proofs about the Hodge star operator see for example [Bis, War].

Chapter 3

Tubes

General agreement

> In this chapter we assume that M is Riemannian and P is a q-dimensional embedded submanifold of M unless otherwise stated.

To describe the geometry of a neighborhood around P in M, it is convenient to use the so-called *Fermi coordinates* and *Fermi fields*. We'll define these concepts in the following section and name some of their basic properties. When P is a point (0-dimensional manifold) Fermi coordinates and fields coincide with the better known *normal coordinates* and *normal fields*. Fermi fields are also closely related to *Jacobi fields* which are commonly used. For more details see [Gr] and for example [Lee].

3.1 Fermi coordinates and fields

3.1 Definition
The **normal bundle** *of P in M is given by*

$$\nu := \{(p, v) \,|\, p \in P, v \in (T_pP)^\perp\}. \tag{3.1}$$

Here $(T_pP)^\perp$ stands for the orthogonal complement of T_pP in T_pM.

3.2 Remark
Let $\pi : \nu \to P$ be defined by letting $\pi(p, v) = p$. Then by lemma 2.28 it's not difficult to show that (ν, P, π) is a $(n-q)$-dimensional vector bundle. We'll identify P with the zero section of ν. Thus P can be regarded as a submanifold of ν as well as a submanifold of M. By this it follows that

$$T_{(p,0)}\nu = T_pP \oplus (T_pP)^\perp.$$

3.3 Definition

(1) A vector $u \in (T_pP)^\perp$ is called a **radial vector**.

(2) A vector $x \in T_pP$ is called a **spherical vector**.

3.4 Definition

(1) The set

$$\mathcal{E}_\nu := \{(p,v) \in \nu |\, v \in \mathcal{E}_p\}$$

is called the **domain for the exponential map of the normal bundle**.

(2) The map $\exp_\nu : \mathcal{E}_\nu \to M$ *defined by*

$$\exp_\nu(p,v) = \exp_p(v),$$

is called the **exponential map of the normal bundle** ν.

3.5 Lemma

Let $p \in P$. *Then* $((\exp_\nu)_*)_{(p,0)}|_{T_0((T_pP)^\perp)}$ *is the canonical identification of* $T_0((T_pP)^\perp)$ *with* $(T_pP)^\perp$ *and* $((\exp_\nu)_*)_{(p,0)}|_{T_0(T_pP)}$ *is the canonical inclusion of* T_pP *in* T_pM.

□

3.6 Lemma

There exists a neighborhood U *of* P *in* ν *such that* $\exp_\nu|_U : U \to \exp_\nu(U)$ *is a diffeomorphism.*

□

3.7 Definition

We denote by $\mathcal{O}_P$ *the largest neighborhood of the zero section of* ν *for which* $\exp_\nu : \mathcal{O}_P \to \exp_\nu(\mathcal{O}_P)$ *is a diffeomorphism. Remark 3.2 and the last lemma guarantees the existence of* $\mathcal{O}_P$.

3.8 Definition

Let $p \in V \subset P$ *where* $y = (y_1, \dots, y_q) : V \to V' \subset \mathbb{R}^q$ *is a chart on* P, *and let* $E_{q+1}, \dots, E_n$ *be a orthonormal local frame for* ν. *The* **Fermi coordinates** $(x_1, \dots, x_n)$ *of* $P \subset M$ *centered at* p *relative to the given coordinate system* $(y_1, \dots, y_q)$ *on* P *and the given orthonormal local frame* $E_{q+1}, \dots, E_n$ *of* ν *are defined by*

$$x_a\left(\exp_\nu\left(\sum_{j=q+1}^{n} t_j E_j(p')\right)\right) = y_a(p') \qquad (a = 1, \dots, q), \tag{3.2}$$

$$x_i\left(\exp_\nu\left(\sum_{j=q+1}^{n} t_j E_j(p')\right)\right) = t_i \qquad (i = q+1, \dots, n), \tag{3.3}$$

for $p' \in V$, *provided that* $\sum_{j=q+1}^{n} t_j E_j(p') \in \mathcal{O}_P$. *We make the notational convention that* $x_a\,(1 \leqslant a \leqslant q)$ *are given by (3.2) and* $x_i\,(q+1 \leqslant i \leqslant n)$ *are given by (3.3).*

The figure above shows how the normal bundle ν and Fermi coordinates could look like when $\dim M = 2$ and $\dim P = 1$. The figure should look a little bit like a tip of a finger. $U \subset \exp_\nu(\mathcal{O}_P)$ is a neighborhood of $p \in P$ in M. $y_1 : V \to y_1(V) \subset \mathbb{R}$ is a chart on P in a neighborhood of p. (E_2) is a local (orthonormal) frame for the normal bundle ν and is indicated by several vectors on the figure. $m \in U \subset M$ is a point which we want to describe. There exist a unique elements $p' \in P$ and $t_2 \in \mathbb{R}$ such that $m = \exp_\nu(t_2 E_2(p'))$ if we assume that $\exp_\nu(tE_2(p')) \in U$ for all $t > 0$ with $t_2/t \geqslant 1$ and $t = 0$. On the figure ξ stands for $\xi_{E_2(p')}$. The Fermi coordinates of m is thus given by $x_1(m) = y_1(p') \in \mathbb{R}$ and $x_2(m) = t_2 \in \mathbb{R}$.

3.9 Lemma
Let $(x_1, \ldots, x_n)$ be a system of Fermi coordinates centered at $p \in P$. Then the coordinate vector fields

$$\frac{\partial}{\partial x_{q+1}}, \ldots, \frac{\partial}{\partial x_n}$$

are orthonormal when restricted to P.

□

The following lemma is very useful when dealing with Fermi coordinates.

3.10 Lemma
Let $(p, u) \in \nu$. So $\xi := \xi_u$ is a geodesic normal to P at p. Then there exist a system of Fermi coordinates $(x_1, \ldots, x_n)$, centered at p such that for all t with $(p, tu) \in \mathcal{O}_P$ we have

$$\left.\frac{\partial}{\partial x_{q+1}}\right|_{\xi(t)} = \xi'(t) \tag{3.4}$$

and

$$\left.\frac{\partial}{\partial x_a}\right|_p \in T_pP, \qquad \left.\frac{\partial}{\partial x_i}\right|_p = (T_pP)^\perp \tag{3.5}$$

for $1 \leqslant a \leqslant q$ *and* $q+1 \leqslant i \leqslant n$. *Moreover,*

$$(x_\alpha \circ \xi)(t) = t\delta_{\alpha,q+1} \tag{3.6}$$

for all α.

□

3.11 Definition
Let $p \in P$. *Assume that* $U \subset \exp_\nu(\mathcal{O}_P) \subset M$ *and we have a system of Fermi coordinates* $(x_1, \ldots, x_n)$ *centered at* p.

(1) *If* $A \in \mathcal{X}(U)$ *can be written as*

$$A = \sum_{a=1}^{q} c_a \frac{\partial}{\partial x_a}$$

where $c_a \in \mathbb{R}$ *for* $1 \leqslant a \leqslant q$ *then* A *is called a* **tangential Fermi field**. *The* q*-dimensional vector space of all tangential Fermi fields centered at* $p \in P$ *is denoted by*

$$\mathcal{X}(P,p)^\top. \tag{3.7}$$

(2) *If* $X \in \mathcal{X}(U)$ *can be written as*

$$X = \sum_{i=q+1}^{n} d_i \frac{\partial}{\partial x_i}$$

where $d_i \in \mathbb{R}$ *for* $q+1 \leqslant i \leqslant n$ *then* X *is called a* **normal Fermi field**. *The* $(n-q)$*-dimensional vector space of all normal Fermi fields centered at* $p \in P$ *is denoted by*

$$\mathcal{X}(P,p)^\perp. \tag{3.8}$$

(3) *We denote by*

$$\mathcal{X}(P,p) = \mathcal{X}(P,p)^\top \oplus \mathcal{X}(P,p)^\perp \tag{3.9}$$

the space of all Fermi fields centered at $p \in P$.

3.12 Definition
Let $(x_1, \ldots, x_n)$ *be a system of Fermi coordinates of* P. *We define*

$$\sigma := \sqrt{\sum_{i=q+1}^{n} x_i^2} \quad \text{and} \quad N := \sum_{i=q+1}^{n} \frac{x_i}{\sigma}\frac{\partial}{\partial x_i} \tag{3.10}$$

3.13 Remark
It's noteworthy to observe that when P is a point, σ and N are the familiar ***radial distance function*** and ***radial vector field*** (See for example [Lee]). N is also known as the ***Gauss map*** when P is a point. It can be shown that the definitions of σ and N are independent of Fermi coordinates.

3.14 Lemma
Let $m \in M$. If there exist a unique unit speed geodesic ξ from P to m which is orthogonal to P then

$$\sigma(m) = \operatorname{dist}(P, m) \textit{ and } N_{\xi(s)} = \xi'(s).$$

Hence we have that σ is defined on $\exp_\nu(\mathcal{O}_P)$ and N is defined on $\exp_\nu(\mathcal{O}_P)\backslash P$.

PROOF: We may assume that $\xi(0) = p$. Let $b \geqslant 0$ be the number which satisfies $\xi(b) = m$. We choose the Fermi coordinates described in lemma 3.10. Therefore by (3.4) and (3.6) we have

$$\sigma(m) = \sqrt{\sum_{i=q+1}^{n} x_i^2(\xi(b))} = \sqrt{x_{q+1}^2(\xi(b))} = b = \operatorname{dist}(m, P)$$

and, if $b \neq 0$

$$N_{\xi(s)} = \sum_{i=q+1}^{n} \frac{x_i(\xi(s))}{\sigma(\xi(s))} \frac{\partial}{\partial x_i}\bigg|_{\xi(s)} = \frac{s}{s} \frac{\partial}{\partial x_{q+i}}\bigg|_{\xi(s)} = \xi'(s)$$

□

In the following lemma some basic properties of σ and N are stated.

3.15 Lemma
Let $X, Y \in \mathcal{X}(P, p)^\perp$ and $A, B \in \mathcal{X}(P, p)^\top$. Then on $\exp_\nu(\mathcal{O}_P)$ we have

$$\nabla_N N = 0 \tag{3.11}$$
$$[X, A] = [N, A] = [X, Y] = [A, B] = 0 \tag{3.12}$$
$$[N, \sigma X] = X(\sigma)N \tag{3.13}$$
$$\nabla_N \nabla_N U + R_{NU}N = 0 \qquad (\textit{for } U = A + \sigma X). \tag{3.14}$$

3.16 Definition
Let $f \in C^\infty(M)$. The vector field $\operatorname{grad} f \in \mathcal{X}(M)$ defined by setting

$$\langle \operatorname{grad} f, X \rangle = Xf \tag{3.15}$$

for all $X \in \mathcal{X}(M)$ is called the **gradient vector field** *of f.*

3.17 Remark
The well known Gauss lemma (see for example [Lee]) and it's corollaries has an extension for submanifolds. I'll let it be sufficient to state these generalizations without proof.

3.18 Theorem (The generalized Gauss lemma)

$$N = \operatorname{grad}\sigma \;\; on \;\; \exp_\nu(\mathcal{O}_P)\backslash P. \tag{3.16}$$

□

3.19 Corollary
N is the unit normal to each of the tubular hypersurfaces $\sigma =$ constant about P of M.

□

3.20 Corollary
$\exp_\nu : \mathcal{E}_\nu \to M$ *preserves the lengths of radial vectors and orthogonality between radial and spherical vectors.*

□

3.2 Definition of a general tube

General agreement

For the rest of this chapter P is allowed to have a boundary.

3.21 Definition

(1) Let $m \in M$. The set of all geodesics $\xi : [a, 0] \to M$ where $a \leqslant 0$ such that $\xi(0) = m$ is denoted by

$$\Xi_m$$

(2) Let $r > 0$. The set

$$T(P, r) := \{m \in M \,|\, \exists\, \xi \in \Xi_m, L(\xi) \leqslant r : \xi \text{ meets } P \text{ orthogonally}\} \tag{3.17}$$

is called a **tube** *of radius r about P.*

3.22 Remark
We will assume that $\exp_\nu : \{(p, v) \in \nu \,|\, \|v\| \leqslant r\} \to T(P, r) \subset \exp_\nu(\mathcal{O}_P)$ is a diffeomorphism. If M is complete and the closure of P is compact then this can always be achieved for sufficiently small $r > 0$. We can then write

$$T(P, r) = \bigcup_{p \in P} \{\exp_p(v) \,|\, v \in (T_p P)^\perp, \|v\| \leqslant r\}. \tag{3.18}$$

3.23 Definition
The hypersurface

$$P_t = \{m \in T(p,r) | \operatorname{dist}(m, P) = t\} \tag{3.19}$$

is called the **tubular hypersurface** *at a distance t from P.*

3.24 Definition
Let ∇ be the Levi-Civita connection on M and R be the curvature transformation on M.

(1) *We denote by $R_N : \mathcal{X}(\exp_\nu(\mathcal{O}_P)\backslash P) \to \mathcal{X}(\exp_\nu(\mathcal{O}_P)\backslash P)$ the operator which is defined by*

$$R_N U = R_{NU} N. \tag{3.20}$$

It's easy to see that R_N is a tensor field.

(2) *The operator $S : \mathcal{X}(\exp_\nu(\mathcal{O}_P)\backslash P) \to \mathcal{X}(\exp_\nu(\mathcal{O}_P)\backslash P)$ defined by*

$$SU = -\nabla_U N \tag{3.21}$$

is called the **shape operator**.

(3) *We define the* **covariant derivative** *of S by*

$$\nabla_U(S)V = \nabla_U SV - S\nabla_U V \tag{3.22}$$

for $U, V \in \mathcal{X}(\exp_\nu(\mathcal{O}_P)\backslash P)$. For more details about covariant derivative of a tensor field, see for example [Lee].

3.25 Lemma

(1) *$S(fU) = f(SU)$ for all $U \in \mathcal{X}(\exp_\nu(\mathcal{O}_P)\backslash P)$ and $f \in C^\infty(\exp_\nu(\mathcal{O}_P)\backslash P)$. Thus S is a tensor field.*

(2) *$SN = 0$;*

(3) *If $U, V \in \mathcal{X}(\exp_\nu(\mathcal{O}_P)\backslash P)$ satisfies $\langle U, N\rangle = \langle V, N\rangle = 0$ then $\langle SU, V\rangle = \langle SV, U\rangle$.*

PROOF:

(1) $S(fU) = -\nabla_{fU} N = -f\nabla_U N = f(SU)$.

(2) $SN = -\nabla_N N = 0$ by (3.11).

(3) We have $\langle U, N\rangle = \langle V, N\rangle = 0$. Hence

$$\begin{aligned}
&\langle SU, V\rangle - \langle SV, U\rangle \\
= \;& -\langle \nabla_U N, V\rangle + \langle \nabla_V N, U\rangle \\
= \;& \langle N, \nabla_U V\rangle - U\langle N, V\rangle - \langle N, \nabla_V U\rangle + V\langle N, U\rangle && \text{(by def. 2.63(4))} \\
= \;& \langle N, [U, V]\rangle && \text{(by def 2.63(3))} \\
= \;& \langle \operatorname{grad}\sigma, [U, V]\rangle && \text{(by the gen. Gauss lemma)} \\
= \;& ([U, V])\sigma && \text{(by definition of grad)} \\
= \;& U(V\sigma) - V(U\sigma) = U\langle N, V\rangle - V\langle N, U\rangle = 0
\end{aligned}$$

and the lemma follows.

□

3.26 Lemma

$$\nabla_N(S) = S^2 + R_N \text{ on } \exp_\nu(\mathcal{O}_P)\backslash P. \tag{3.23}$$

PROOF: For every point $p \in \exp_\nu(\mathcal{O}_P)\backslash P$ it suffices to show that (3.23) is valid for vector fields of the form $U = A + \sigma X$, where $A \in \mathcal{X}(P,p)^\top$ and $X \in \mathcal{X}(P,p)^\perp$ since every vector field on $\exp_\nu(\mathcal{O}_P)\backslash P$ is locally spanned by vector fields of this form. Since by (3.12) and (3.13), $[N, A] = 0$ and $[N, \sigma X] = X(\sigma)N$, we have

$$[N, U] = [N, A + \sigma X] = [N, A] + [N, \sigma X] = X(\sigma)N. \tag{3.24}$$

Now

$$\begin{aligned}
& \nabla_N(S)U \\
= \ & \nabla_N SU - S\nabla_N U && \text{(definition of covariant derivative)} \\
= \ & -\nabla_N\nabla_U N - S([N,U] + \nabla_U N) && \text{(definition of } S \text{ and def. 2.63(3))} \\
= \ & -\nabla_N\nabla_U N + \nabla_U\nabla_N N - S[N,U] + S^2U && \text{(by (3.11) and def. of } S) \\
= \ & R_{NU}N - \nabla_{[N,U]}N - S[N,U] + S^2U && \text{(by (2.38))} \\
= \ & R_N U - \nabla_{X(\sigma)N}N - SX(\sigma)NU + S^2U && \text{(by def. of } R_N \text{ and (3.24))} \\
= \ & R_N U - X(\sigma)\nabla_N N - X(\sigma)SNU + S^2U && \text{(by lemma 3.25(1) and (2.31))} \\
= \ & R_N U + S^2U && \text{(by (3.11) and 3.25(2)),}
\end{aligned}$$

and the proof is complete.

□

3.27 Corollary
For $t > 0$ we define $S(t), R(t)$ and $S'(t)$ to be the restriction to P_t of S, R_N and $\nabla_N(S)$ respectively. Then we have

$$S'(t) = S(t)^2 + R(t). \tag{3.25}$$

A differential equation of this form is called a **Riccati differential equation**.

□

3.2.1 The second fundamental form of submanifolds

3.28 Definition
Let $A, B \in \mathcal{X}(P,p)^\top$ and ∇ be the Levi-Civita connection on M. We define the map $L : \mathcal{X}(P,p)^\top \times \mathcal{X}(P,p)^\top \to \mathcal{X}(P,p)^\perp$, $L(A,B) \mapsto L_A B$ to be the vector field normal to P which satisfies

$$\langle L_A B, X\rangle = \langle \nabla_A B, X\rangle \tag{3.26}$$

for all $X \in \mathcal{X}(P,p)^\perp$.

3.29 Remark
L is clearly linear with respect to smooth functions on P in the first coordinate. Since by (2.33) we have

$$\begin{aligned}\langle L_A(fB), X\rangle &= \langle \nabla_A(fB), X\rangle \\ &= \langle (Af)B, X\rangle + \langle f\nabla_A B, X\rangle \\ &= \langle f L_A B, X\rangle\end{aligned}$$

L is also linear with respect to smooth functions on P in the second coordinate. Thus L is a tensor field on P.

3.30 Definition
(1) The map $T : \mathcal{X}(P,p)^\top \times \mathcal{X}(P,p)^\top \times \mathcal{X}(P,p)^\perp \to C^\infty(P)$ defined by

$$T(A, B, X) = \langle L_A B, X\rangle$$

is called the **second fundamental form** *of P. Since L is a tensor field, T is also a tensor field.*

(2) Let $p \in P$, $A, B \in \mathcal{X}(P,p)^\top$, and $X \in \mathcal{X}(P,p)^\perp$. We denote by $a, b \in T_pP$ and $u \in (T_pP)^\perp$ the vectors of A, B and X at the point p respectively. Then we define

$$T_{abu} := T(A, B, X)(p).$$

Furthermore we define $T_{ab} = L_A B(p)$.

(3) The map $T_u : T_pP \to T_pP$ defined by

$$\langle T_u(a), b\rangle = T_{abu}$$

is called the **Weingarten map**.

3.31 Remark
Let's assume that P is a point. Then N is the familiar Gauss map, as already mentioned. Furthermore we have that S is the ordinary Weingarten map. Since we have

$$\begin{aligned}\langle L_A B, N\rangle =&\langle \nabla_A B, N\rangle \\ =& -\langle B, \nabla_A N\rangle + A\langle B, N\rangle \\ =&\langle -\nabla_A N, B\rangle = \langle SA, B\rangle\end{aligned} \tag{3.27}$$

we see that T_u and S coincide when P is a point so the definitions above are generalizations of well known concepts.

3.32 Remark
The Weingarten map is clearly linear and by using lemma 3.25(3) and (3.27) it's easy to show that it is also symmetric.

The following theorem is taken from spectral theory and is a basic theorem in the field. It's proof can be found in many books about linear algebra.

3.33 Theorem (Spectral theorem)
Let V be a n-dimensional Euclidean inner product space over $\mathbb{R}$ and let $A : V \to V$ be a linear map which is symmetric with respect to the inner product. Then there exists an orthonormal basis for V such that the matrix of A with respect to this basis has a diagonal form. The elements of this basis are eigenvectors for A and the elements on the diagonal are the corresponding eigenvalues. We say that the orthonormal basis **diagonalizes** *A.*

□

3.34 Remark
Let $p \in P$ and $u \in (T_pP)^\perp$. By the spectral theorem and remark 3.32 we can find an orthonormal basis $(f_1, \dots, f_q)$ of T_pP that diagonalizes $T_{\xi'_u(0)} : T_pP \to T_pP$. We extend these vectors to unit vector fields $(F_1(t), \dots, F_q(t))$ along ξ_u such that $F_a(t)$ is an eigenvector for $S(t)$ for all a and all t. Then we have eigenvalues $\kappa_a(t)$ corresponding to these eigenvectors, and we can write

$$S(t)F_a(t) = \kappa_a(t)F_a(t) \tag{3.28}$$

for $a = 1, \dots, q$. We denote the remaining eigenvalues of $S(t)$ by $\kappa_{q+2}(t), \dots, \kappa_n(t)$. Hence there are unit vector fields $F_{q+2}(t), \dots, F_n(t)$ along ξ_u such that

$$S(t)F_i(t) = \kappa_i(t)F_i(t) \tag{3.29}$$

for $i = q+2, \dots, n$. By letting $F_{q+1}(t) = \xi'_u(t)$ we have that $\{F_1(t), \dots, F_n(t)\}$ is a local frame along ξ_u.

Note that

$$\operatorname{trace} S(t) = \sum_{\alpha=1,\dots,q,q+2,\dots,n} \kappa_\alpha(t). \tag{3.30}$$

3.35 Definition

(1) *$\kappa_1, \dots, \kappa_q, \kappa_{q+2}, \dots, \kappa_n$ in the above remark are called the* **principal curvature functions** *of S and $F_1, \dots, F_q, F_{q+2}, \dots, F_n$ are called the* **principal curvature vector fields** *of S.*

(2) *The eigenfunctions of S of any orientable hypersurface are called the* **principal curvatures** *of the hypersurface.*

From the definition we have the following lemma.

3.36 Lemma
Let $t > 0$ and assume that P_t is orientable. Then the restriction of the principal curvature functions to P_t are the principal curvatures of P_t.

□

3.37 Corollary
If F_α $(1 \leqslant \alpha \leqslant n)$ are differentiable at t then

$$\kappa_\alpha'(t) = \kappa_\alpha(t)^2 + R_{\xi'(t)F_\alpha(t)\xi'(t)F_\alpha(t)}.$$

PROOF: By (3.28) and (3.29) we have that κ_α is differentiable for all α. By the generalized Gauss lemma $\xi'(t)$ has the same properties as N. Furthermore since $\nabla_{\xi'}F_\alpha = F'$ and $\langle F_\alpha', F_\alpha\rangle = 0$ we have

$$\begin{aligned}
& R_{\xi' F_\alpha \xi' F_\alpha} \\
= \; & \langle R_{\xi' F_\alpha}\xi', F_\alpha\rangle \\
= \; & \langle R_{\xi'}F_\alpha, F_\alpha\rangle && \text{(by defininiton of } R_N) \\
= \; & \langle \nabla_{\xi'}(S)F_\alpha, F_\alpha\rangle - \langle S^2 F_\alpha, F_\alpha\rangle && \text{(by lemma 3.26)} \\
= \; & \langle \nabla_{\xi'}SF_\alpha, F_\alpha\rangle - \langle S\nabla_{\xi'}F_\alpha, F_\alpha\rangle - \kappa_\alpha^2 && \text{(def. of covariant derivative)} \\
= \; & \langle \nabla_{\xi'}SF_\alpha + SF_\alpha', F_\alpha\rangle - 2S\langle F_\alpha', F_\alpha\rangle - \kappa_\alpha^2 \\
= \; & \langle SF_\alpha, F_\alpha\rangle' - \kappa_\alpha^2 = \kappa_\alpha' - \kappa_\alpha^2
\end{aligned}$$

and the proof is complete.

□

3.3 The infinitesimal change of volume

General agreement

For the rest of the chapter we assume that M and P are oriented unless otherwise stated.

Let ω be the Riemannian volume form on M and ω_ν be the Riemannian volume form on ν. Since $\dim \Omega^n(\nu) = 1$ we have that $\exp_\nu^*(\omega) = f\omega_\nu$ where $f : \nu \to \mathbb{R}$ is a positive real valued function on ν. With this in mind we'll make the following definition:

3.38 Definition

(1) *The function* chvol : $\exp_\nu(\mathcal{O}_P) \to \mathbb{R}$ *is defined such that the following equation is satisfied:*

$$\exp_\nu^*(\omega) = (\mathrm{chvol} \circ \exp_\nu)\omega_\nu. \tag{3.31}$$

We note that chvol *is everywhere non-negative.*

(2) *For a point $p \in P$ and a unit vector $u \in (T_pP)^\perp$ the function*

$$\vartheta_u(t) = (\mathrm{chvol} \circ \exp_\nu)(p, tu) \tag{3.32}$$

defined for all $(p, tu) \in \mathcal{O}_P$ is called the **infinitesimal change of volume function** *of P* **in the direction** *u.*

3.39 Lemma
Let $(x_1,\ldots,x_n)$ be a system of Fermi coordinates for P centered at $p\in P$ which has the same orientation as ω, i.e. $\omega\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)>0$. If $(y_1,\ldots y_q)$ are system of Fermi coordinates on P such that

$$\left\{\left.\frac{\partial}{\partial x_1}\right|_p,\ldots,\left.\frac{\partial}{\partial x_q}\right|_p\right\}$$

forms an orthonormal basis of T_pP then

$$\vartheta_u(t)=\omega\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)(\exp_\nu(p,tu)). \tag{3.33}$$

Proof: By lemma 3.9 we have that

$$\left\{\left.\frac{\partial}{\partial x_{q+1}}\right|_p,\ldots,\left.\frac{\partial}{\partial x_n}\right|_n\right\}$$

is orthonormal basis for $(T_pP)^\perp$. Hence

$$\left.\frac{\partial}{\partial x_1}\right|_p,\ldots,\left.\frac{\partial}{\partial x_n}\right|_n$$

is an orthonormal basis for $T_{(p,0)}\nu=(T_pP)\oplus(T_pP)^\perp$ and since ω_ν is a volume form on ν, its value on this basis is equal to 1. We'll evaluate both sides of equation (3.31) on the basis above at the point (p,tu). The left hand side yields:

$$\begin{aligned}\exp_\nu^*(\omega)\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)(p,tu)&=\omega\left((\exp_\nu)_*\frac{\partial}{\partial x_1},\ldots,(\exp_\nu)_*\frac{\partial}{\partial x_n}\right)\exp_\nu(p,tu)\\&=\omega\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)\exp_\nu(p,tu);\end{aligned} \tag{3.34}$$

by lemma 3.5, and the right hand side yields:

$$(\mathrm{chvol}\circ\exp_\nu)\omega_\nu\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)(p,tu)=(\mathrm{chvol}\circ\exp_\nu)(p,tu)=\vartheta_u(t), \tag{3.35}$$

where we have used the remarks above. Combining (3.34) and (3.35) we get (3.33).

□

The following theorem will be vital for our further study. The proof is a little technical and I choose to omit it.

3.40 Theorem
Let $(p,u)\in\nu$. Then along the unit speed geodesic ξ_u we have

$$\frac{\vartheta_u'(t)}{\vartheta_u(t)}=-\left(\frac{n-q-1}{1}+\mathrm{trace}\,S(t)\right) \tag{3.36}$$

for all $t>0$ if $(p,tu)\in\mathcal{O}_P$.

□

3.41 Definition
We define

$$\begin{aligned} V_P^M(r) &:= \text{ the } n\text{-dimensional volume of } T(P,r), \\ A_P^M(r) &:= \text{ the } (n-1)\text{-dimensional volume of } P_r, \end{aligned}$$

3.42 Remark
We often refer to $\mathrm{A}_P^M(r)$ as the *area* of P_r. This terminology was used in chapter 1.

3.43 Lemma
We assume as before that the exponential map $\exp_\nu : \{(p,v) \in \nu | \|v\| \leqslant r\} \to T(P,r)$ *is a diffeomorphism. Then*

$$A_P^M(r) = r^{n-q-1} \int_P \int_{S^{n-q-1}} \vartheta_u(r) du\, dP. \tag{3.37}$$

PROOF: Let V be a neighborhood of the zero section of ν. Define $s : V \to \mathbb{R}$ by

$$s(p,v) = \|v\|. \tag{3.38}$$

By the generalized Gauss Lemma and lemma 3.14 we have that $s = \sigma \circ \exp_\nu$. We let $*$ denote the Hodge star operator either on M or on ν. By remark 2.86 we have that $ds \wedge * ds$ is the volume form of ν and thus $* ds$ is a volume form for the hypersurface $\{(p,v) \in \nu | \, \|v\| = t, p \in P\}$ on ν. Now $d\sigma \wedge * d\sigma$ is the volume form of M in a neighborhood of P. We thus have (by corollaries of the generalized Gauss lemma) that $* d\sigma$ is the volume form of P_t for each t. We have that the map $\exp_\nu^* : \Omega^n(M) \to \Omega^n(\nu)$ preserves wedge product (by theorem 2.42(2)) and we can thus calculate that

$$\begin{aligned} \exp_\nu^*(d\sigma) \wedge \exp_\nu^*(* d\sigma) &= \exp_\nu^*(d\sigma \wedge * d\sigma) \\ &= \left\{ \omega \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \circ \exp_\nu \right\} ds \wedge * ds. \end{aligned}$$

Since σ is a smooth function we have by theorem 2.42(4) and (2.15) that

$$\exp_\nu^*(d\sigma) = d(\exp_\nu^*(\sigma)) = d(\sigma \circ \exp_\nu) = ds \tag{3.39}$$

and thus

$$\exp_\nu^*(* d\sigma) = \left\{ \omega \left(\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \right) \circ \exp_\nu \right\} * \, ds.$$

P_r has a compact closure, so the integral of $* d\sigma$ over P_r is $\mathrm{A}_P^M(r)$. Hence we get by definition of integration of form on manifolds (see (2.22)) that

$$\begin{aligned} \mathrm{A}_P^M(r) &= \int_{P_r} * d\sigma = \int_{\exp_\nu^{-1}(P_r)} \exp_\nu^*(* d\sigma) \\ &= \int_P \int_{S^{n-q-1}(r)} \left\{ \omega \left(\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \right) \circ \exp_\nu \right\} d\tilde{u}\, dP \qquad (3.40) \\ &= r^{n-q-1} \int_P \int_{S^{n-q-1}} \left\{ \omega \left(\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \right) \circ \exp_\nu \right\} du\, dP \end{aligned}$$

Here dP is the volume form of P, $d\tilde{u}$ the volume form on $S^{n-q-1}(r)$ and du is the volume form of S^{n-q-1}. Now the lemma follows from lemma 3.39.

□

3.44 Lemma
Assuming the same assumptions as in the previous lemma we have

$$\frac{d}{dr} V_P^M(r) = A_P^M(r). \tag{3.41}$$

PROOF:

$$\begin{aligned} & \mathrm{V}_P^M(r) \\ = \ & \int_{T(P,r)} d\sigma \wedge * \, d\sigma \\ = \ & \int_{\exp_\nu^{-1}(T(P,r))} \exp_\nu^*(d\sigma) \wedge \exp_\nu^*(* \, d\sigma) \quad \text{(by (2.22))} \\ = \ & \int_0^r \int_{\exp_\nu^{-1}(P_r)} ds \wedge \exp_\nu(* \, d\sigma) \quad \text{(by (3.39))} \\ = \ & \int_0^r \mathrm{A}_P^M(s) \, ds \quad \text{(by (3.40))} \end{aligned}$$

and the lemma follows from the fundamental theorem of the calculus.

□

The following lemma is easy to prove:

3.45 Lemma
For $M = \mathbb{R}^n$ we have

$$\vartheta_u(t) = \det(\delta_{a,b} - tT_{abu}) \tag{3.42}$$

if the right hand side is non-negative.

□

3.46 Remark
(3.42) is not valid for M in general. We'll see that the previous lemma will be a key concept in deriving Weyl's Tube formula later in this chapter.

3.4 The Bishop-Günther inequalities

In this section the volume of tubes around P will be calculated when P is a point in some special kind of manifold M. In that case the volume of $T(P,r)$ is the volume of a geodesic ball. The following lemma is easily verified.

3.47 Lemma
Let $k : (0, t_0] \to \mathbb{R}$ be a differentiable function which satisfies

$$k' \geqslant k^2 + \lambda, \qquad k(t) \xrightarrow[t \to 0+]{} -\infty, \tag{3.43}$$

where $\lambda \in \mathbb{R}$. Then

$$k(t) \geqslant \frac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})} \tag{3.44}$$

for $0 < t \leqslant t_0$. If the inequality in (3.43) in the assumption is reversed then the inequality in (3.44) is also reversed. Note that $\lambda = 0$ is a removable singularity.

□

3.48 Definition

(1) *Let $m \in M$ and ξ be a geodesic with $\xi(0) = m$. Let $t_0 > 0$ which satisfies*

There exists an admissible curve γ on M with $\gamma(0) = m$ and $\gamma(t) = \xi(t)$ such that $L(\gamma(t)) < L(\xi(t))$;

and assume that for all t such that $0 < t < t_0$, the curve $\xi|_{[0,t]}$ is minimizing, then t_0 is said to be a **cut point** *of m* **along the geodesic** *ξ.*

(2) *The set of all cut points of m is called the* **cut locus** *of m.*

3.49 Theorem

Let M be a complete Riemannian manifold. Assume that r is less than or equal to the distance between m and its cut locus. For $\lambda \in \mathbb{R}$ we have

$$K^M \geqslant \lambda \Rightarrow V_m^M(r) \leqslant \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt;$$

$$K^M \leqslant \lambda \Rightarrow V_m^M(r) \geqslant \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt.$$

PROOF: We shall first prove the latter statement. Then $R_{\xi'(t)F_i(t)\xi'(t)F_i(t)} \leqslant \lambda$. Thus the functions $k_i(t)$ satisfies the Riccati differential inequality

$$k_i' \leqslant k_i^2 + \lambda.$$

by corollary 3.37. By lemma 3.47 we have

$$k_i(t) \leqslant \frac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})},$$

and thus

$$\operatorname{trace} S(t) \leqslant \frac{-(n-1)\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}.$$

By theorem 3.40 and the above observation we obtain the equality

$$\frac{\vartheta_u'(t)}{\vartheta_u(t)} \geqslant \frac{-(n-1)}{t} + \frac{(n-1)\sqrt{\lambda}\cos(t\sqrt{\lambda})}{\sin(t\sqrt{\lambda})}.$$

The integral of $1/\tan(at)$ where $a \in \mathbb{R}$ is $\ln(\sin(at))/a$ for $0 < t < \pi/(2a)$ so we get

$$\ln \vartheta_u(t) \geqslant C - (n-1)\ln(t) + (n-1)\ln(\sin(t\sqrt{\lambda}))$$

where $C \in \mathbb{R}$ or, equivalently

$$\vartheta_u(t) = e^C \left(\frac{\sin(t\sqrt{\lambda})}{t}\right)^{n-1}$$

Since $\vartheta_u(0) = 1$ we see that $e^C = 1/\lambda^{(n-1)/2}$. Making use of lemma 3.43 and lemma 3.44 we have shown the desired inequality. The proof of the former statement is exactly the same, where we reverse all inequalities.

□

3.50 Definition
If $K^M = \lambda \in \mathbb{R}$ then M is called a **space of constant sectional curvature** *and is denoted by*

$$\mathbf{K}^n(\lambda).$$

The following corollary follows immediately from the last theorem.

3.51 Corollary
If $M = \mathbf{K}^n(\lambda)$ is a space of constant sectional curvature, then

$$V_m^M(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt, \tag{3.45}$$

for all $m \in M$.

□

3.52 Remark

(1) It is well known that $\mathbf{K}^n(\lambda)$ is locally isometric to one of three classical spaces depending on the value of λ. Those spaces are

 (a) The Euclidean space, when $\lambda = 0$;

 (b) A sphere, when $\lambda > 0$;

 (c) A hyperbolic space $\lambda < 0$.

(2) For a great survey of hyperbolic spaces, I recommend [Rat]. For more about spaces of constant sectional curvature, see for example [Wo].

3.53 Corollary
It can be shown that $S^n(r)$ is a space of constant sectional curvature λ where $\lambda = \pi/r$. By setting $s = t\sqrt{\lambda}$ in (3.45) and evaluate the volume at $r = \pi$ we get a formula for the volume *of $S^n(r)$, since the geodesic ball exactly covers $S^n(r)$. The result is*

$$V_m^{S^n(r)}(\pi) = \frac{2\pi^{n/2}}{\Gamma(n/2)\lambda^{n/2}} \int_0^\pi (\sin\, s)^{n-1}\, ds. \tag{3.46}$$

□

3.54 Remark
The integral in the last corollary can easily be calculated explicitly and is done so in most calculus books. It is left to the reader to show that the result for the volume of $S^n(r)$ in (3.46) is the same as the area of $S^n(r)$ as a hypersurface given in chapter 1. This should not come as a surprise, since this is exactly the same thing we're doing, though we use different words for it, and different method to find the answer.

3.55 Remark
In this section we have seen how we could get an estimate of the volume of a geodesic ball in terms of the sectional curvature K^M, and that we could get some nice equalities when K^M is constant. In the general theory of tubes often an estimate for the volume is all we can hope to have and many articles in the field today gives some estimates of the volume of a tube for more general manifolds.

3.5 Weyl's tube formula

3.5.1 Double forms

3.56 Definition

(1) *A smooth covariant $(p+q,0)$-tensor field $\alpha : \mathcal{X}(M)^p \times \mathcal{X}(M)^q \to C^\infty(M)$ which is antisymmetric in the first p and the last q variables is called a* **double form** *of type (p,q) on M. We write $\alpha(X_1,\dots,X_p)(Y_1,\dots,Y_q)$ to denote the value on vector fields $X_1,\dots,X_p,Y_1,\dots,Y_q \in \mathcal{X}(M)$.*

(2) *Let $X_1,\dots,X_p \in \mathcal{X}(M)$. Then we define a mapping $\alpha(X_1,\dots,X_p) : \mathcal{X}(M)^q \to C^\infty(M)$, $(Y_1,\dots,Y_q) \mapsto \alpha(X_1,\dots,X_p)(Y_1,\dots,Y_q)$. Then $\alpha(X_1,\dots,X_p)$ is a q-form by definition.*

(3) *Let α be a double form of type (p,q) and β be a double form of type (r,s). We define an* **exterior product** $\wedge$ *of the double forms α and β by*

$$\begin{aligned}\wedge(\alpha,\beta)(X_1,\dots,X_{p+r})(Y_1,\dots,Y_{q+s}) &:= (\alpha\wedge\beta)(X_1,\dots,X_{p+r})(Y_1,\dots,Y_{q+s})\\ &= \frac{1}{p!q!r!s!}\sum_{\rho\in\mathfrak{S}_{p+r}}\sum_{\sigma\in\mathfrak{S}_{q+s}} \varepsilon_\rho\,\varepsilon_\sigma\,\alpha(X_{\rho(1)},\dots,X_{\rho(p)})(X_{\sigma(1)},\dots,X_{\sigma(q)})\\ &\quad\cdot\beta(X_{\rho(p+1)},\dots,X_{\rho(p+r)})(X_{\sigma(q+1)},\dots,X_{\sigma(q+s)}).\end{aligned}$$

(4) *If $p=q$ and $\alpha(X_1,\dots,X_p)(Y_1,\dots,Y_p) = \alpha(Y_1,\dots,Y_p)(X_1,\dots,X_p)$ for all $X_1,Y_1,\dots,X_p,Y_p \in \mathcal{X}(M)$ we say that α is a* **symmetric double form**.

3.57 Remark

(1) The exterior product of double forms is associative and satisfies $\alpha\wedge\beta = (-1)^{pr+qs}\beta\wedge\alpha$, for α and β as in the previous definition. The double forms of type $(p,0)$ along with the exterior product agrees with ordinary differential forms and the ordinary exterior product of differential forms.

(2) The Riemannian metric tensor field $g = \langle\cdot,\cdot\rangle$ is a symmetric double form of type $(1,1)$ and the curvature tensor R is a symmetric double form of type $(2,2)$. Thus $g^c := \underbrace{g \wedge \cdots \wedge g}_{\text{c times}}$ is a double form of type (c,c) and $R^c := \underbrace{R \wedge \cdots \wedge R}_{\text{c times}}$ is a double form of type $(2c,2c)$.

3.58 Definition

Let $(E_1,\ldots,E_n)$ be an orthonormal local frame on M. For a double form α of type (p,q) we let $C^0(\alpha) = \alpha$ and define $C^c(\alpha)$ inductively for $c > 0$ by

$$\begin{aligned} &C^c(\alpha)(X_1,\ldots,X_{p-c})(Y_1,\ldots,Y_{q-c}) \\ =&\sum_{a=1}^{n} C^{c-1}(\alpha)(X_1,\ldots,X_{p-c},E_a)(Y_1,\ldots,Y_{q-c},E_a). \end{aligned}$$

The operators C^c are called **contraction operators**. *It's easy to show that C^c are independent of the choice of the orthonormal local frame.*

3.59 Definition

Let R be the curvature tensor of P. We note that $C^{2c}(R^c)$ is a smooth function on P. We'll assume that $\int_P C^{2c}(R^c)\,dP$ converges. Then the real number

$$k_{2c}(P) = \frac{1}{c!(2c)!}\int_P C^{2c}(R^c)\,dP \tag{3.47}$$

is called the $(2c)^{\text{th}}$ **integrated mean curvature** *of P.*

3.60 Remark

It's easy to calculate k_{2c} for small c. For example we have $k_0(P) = \mathrm{Vol}(P)$ (i.e. the volume of P) and $k_2(P) = \frac{1}{2}\int_P \tau\,dP$. For large numbers of c it seems non-trivial to calculate k_{2c} explicitly.

3.5.2 Invariants

3.61 Definition

Let V be an n-dimensional inner product space.

(1) We let $W_k = \underbrace{V^* \otimes \cdots \otimes V^*}_{\text{k times}}$.

(2) For $g \in O(n,\mathbb{R})$ we define a map $g : W_k \to W_k$ by letting

$$g(\phi)(v_1 \otimes \cdots \otimes v_k) = \phi(g^{-1}v_1 \otimes \cdots \otimes g^{-1}v_k).$$

$O(n,\mathbb{R})$ acts on W_k by this definition.

(3) For $\sigma \in \mathfrak{S}_k$ we define a map $\sigma : W_k \to W_k$ by letting

$$\sigma(\phi)(v_1 \otimes \cdots \otimes v_k) = \phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}).$$

$\mathfrak{S}_k$ acts on W_k by this definition.

(4) *A map $P : W_k \to \mathbb{R}$ which has a symmetric multilinear extension to the tensor product $\underbrace{W_k \otimes \cdots \otimes W_k}_{\text{h times}}$ is called a* **polynomial of degree** *h on W_k. We denote the space of all such polynomials by*

$$P_h(W_k). \tag{3.48}$$

(5) *For $g \in O(n, \mathbb{R})$ we define a map $g : P_h(W_k) \to P_h(W_k)$ by letting*

$$g(P)(\phi) = P(g^{-1}\phi).$$

$O(n, \mathbb{R})$ acts on $P_h(W_k)$ by this definition.

(6) *Let $P \in P_h(W_k)$. If $g(P) = P$ for all $g \in O(n)$ then we say that P is* **invariant**.

(7) *Let $(e_1, \ldots, e_n)$ be an orthonormal basis for V. We set*

$$P_\sigma(\phi) = \sum_{a_1,\ldots,a_m=1}^{n} \sigma(\phi)(e_{a_1}, e_{a_1}, \ldots, e_{a_m}, e_{a_m})$$

for $\phi \in W_{2m}$ and $\sigma \in \mathfrak{S}_{2m}$. It is straightforward to check that the definition is independent of which orthonormal basis we choose.

(8) *We define a map $\otimes^h \phi \in W_{hk}$ by letting*

$$(\otimes^h \phi)(v_1 \otimes \cdots \otimes v_{hk}) = \phi(v_1 \otimes \cdots \otimes v_k) \cdots \phi(v_{k(h-1)+1} \otimes \cdots \otimes v_{hk})$$

for $\phi \in W_k$ and $v_1, \ldots, v_{hk} \in V$.

3.62 Definition
Let $hk = 2m$, where $m \in \mathbb{N}$ and let $\sigma \in \mathfrak{S}_{2m}$. The polynomial $P_\sigma \in P_h(W_k)$ defined by

$$P_\sigma(\phi) = \sum_{a_1,\ldots,a_m=1}^{n} \sigma(\otimes^h \phi)(e_{a_1}, e_{a_1}, \ldots, e_{a_m}, e_{a_m}) \tag{3.49}$$

where $\phi \in W_k$ is called the **elementary invariant** *corresponding to σ.*

The following theorem is a key theorem in the theory of invariants. I'll not go through its proof.

3.63 Theorem
Every invariant polynomial is a sum of products of the elementary invariants.

□

3.64 Remark
Invariant theory has many applications and is used for example in algebraic topology. For some discussion see for example [Spi]. [Gilk] is also interesting.

3.5.3 Some definitions and lemmas

3.65 Definition
Let V be an n-dimensional positive definite inner product space. We denote by S^{n-1} the unit sphere in V, where we use the metric obtained from the inner product on V. We define a function $I_s : W_s \to \mathbb{R}$ by

$$I_s(\phi) = \int_{S^{n-1}} \phi(v, \ldots, v)\, dv. \tag{3.50}$$

3.66 Remark

(1) $\int_{S^{n-1}} dv$ is independent of which inner product space V we use.

(2) I_s is a polynomial of degree 1 and by spherical symmetry it's easy to see that it is invariant.

3.67 Definition
For an even positive number q we define

$$q) = 1 \cdot 3 \cdots (q-1) \tag{3.51}$$

We let $0) = 1$.

3.68 Definition
We define

$$\mathfrak{Q}_s = \{\sigma \in \mathfrak{S}_{2s} |\, \sigma(2t-1) < \sigma(2t+1) \text{ for } t = 1, \ldots, s-1 \text{ and } \sigma(2t-1) < \sigma(2t) \text{ for } t = 1, \ldots, s\}.$$

3.69 Lemma
The cardinality of $\mathfrak{Q}_s$ is

$$\operatorname{card}(\mathfrak{Q}_s) = 2s) = 1 \cdot 3 \cdots (2s-1) = \frac{(2s)!}{2^s s!}. \tag{3.52}$$

PROOF: We'll prove (3.52) by induction. It's easy to check the formula directly for small numbers of s. Assume that the formula is valid for $s = t-1$, where $t > 3$. We'll show that then (3.52) is also valid for $s = t$. Clearly $\sigma(1) = 1$ for all $\sigma \in \mathfrak{Q}_t$. We then have two choices for $\sigma(2) \in \mathfrak{Q}_t$, namely $\sigma(2) = 2$ or $\sigma(2) = 3$. We'll calculate the numbers of elements in $\mathfrak{Q}_t$ separately depending on their value of $\sigma(2)$ and then add the numbers together.

(1) $\sigma(2) = 2$: Elements in $\mathfrak{Q}_t$ of this form clearly has the same cardinality as $\mathfrak{Q}_{t-1}$ which is

$$1 \cdot 3 \cdots (2t-3) \tag{3.53}$$

according to our induction hypothesis.

(2) $\sigma(2) = 3$: We first choose $i, j \in [\![2, 2s]\!]\setminus\{3\}$, $i \neq j$, such that $\sigma(i) = 2$ and $\sigma(j) = 4$. We can do this in $(2t-2)\cdot(2t-3)$ ways. For every such choice we have $\operatorname{card}(\mathfrak{Q}_{t-2})$ possibilities for the rest of the elements in $[\![1, 2t]\!]$ which is $1\cdot 3\cdots(2t-5)$ by our induction hypothesis. Multiplying these numbers we get the whole cardinality of elements in $\mathfrak{Q}_t$ of this form, namely

$$(2t-2)\cdot(2t-3)\cdot 1\cdot 3\cdots(2t-5). \tag{3.54}$$

By adding (3.53) and (3.54) we get that $\operatorname{card}(\mathfrak{Q}_t) = 2t)$ which proves (3.52).

□

3.70 Remark

For $\phi \in W_{2s}$ and $\sigma \in \mathfrak{S}_{2s}$ we have

$$\begin{aligned}&\sum_{\sigma\in\mathfrak{S}_{2s}}\sum_{a_1,\ldots,a_s=1}^{n}\sigma(\phi)(e_{a_1}, e_{a_1}, \ldots, e_{a_s}, e_{a_s})\\ =&2^s s!\sum_{\sigma\in\mathfrak{Q}_s}\sum_{a_1,\ldots,a_s=1}^{n}\sigma(\phi)(e_{a_1}, e_{a_1}, \ldots, e_{a_s}, e_{a_s}),\end{aligned}$$

since the inner sum is independent of orthonormal basis.

3.71 Definition

Let $F \in W_s$ be an integrable function. Then we define the **average** *of F over the unit sphere to be the real number*

$$\langle F\rangle = \frac{\int_{S^{n-1}} F\,du}{\int_{S^{n-1}} du}. \tag{3.55}$$

3.72 Theorem

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V and denote by $(u_1, \ldots, u_n)$ its dual basis. For $c = \sum_{k=1}^n i_k$, we have

$$\langle u_{j_1}^{i_1}\cdots u_{j_n}^{i_n}\rangle = \frac{i_1)\cdots i_n)}{n(n+2)\cdots(n+c-2)}, \tag{3.56}$$

where $i_1, \ldots, i_n \in \mathbb{N}$.

□

3.73 Remark

The proof of the above theorem when $V = \mathbb{R}^n$ can be found in [Gr] but I see nothing wrong with applying the proof to general positive definite inner product spaces.

3.74 Remark

Let $\rho \in \mathfrak{S}_s$, and $f : \mathfrak{S}_s \to \mathbb{R}$ be a function. Then

$$\sum_{\sigma\in\mathfrak{S}_s} f(\rho\sigma) = \sum_{\sigma\in\mathfrak{S}_s} f(\sigma\rho) = \sum_{\sigma\in\mathfrak{S}_s} f(\sigma). \tag{3.57}$$

3.75 Lemma

(1) If s is odd, then $I_s = 0$;

(2) Let $\phi \in W_{2s}$. Then I_{2s} is given by

$$I_{2s}(\phi) = \frac{2\pi^{n/2}}{n(n+2)\cdots(n+2s-2)\Gamma(n/2)} \sum_{\sigma\in\mathfrak{Q}_s} \sum_{i_1,\ldots,i_s=1}^{n} \sigma(\phi)(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}) \tag{3.58}$$

where $(e_1, \ldots, e_n)$ is any orthonormal basis of V.

PROOF: The former statement follows trivially from theorem 3.63. By the same theorem there are constants b_σ such that

$$I_{2s}(\phi) = \sum_{\sigma\in\mathfrak{S}_{2s}} b_\sigma \left(\sum_{i_1,\ldots,i_s=1}^{n} \sigma(\phi)(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}) \right). \tag{3.59}$$

We denote by $(u_1, \ldots, u_n)$ the dual basis of $(e_1, \ldots, e_n)$ as before. Let $\phi = u_{j_1}^2 \cdots u_{j_s}^2$. By (1.2) and theorem 3.72 we have that

$$I_{2s}(u_{j_1}^2 \cdots u_{j_s}^2) = \frac{2\pi^{n/2}}{n(n+2)\cdots(n+2s-2)\Gamma(n/2)}, \tag{3.60}$$

independent of $j_1, \ldots, j_s$ as long as $j_l \neq j_k$ for $l \neq k$. Now let $\rho \in \mathfrak{S}_{2s}$. Then we have

$$\begin{aligned}
&I_{2s}(u_{j_1}^2 \cdots u_{j_s}^2) \\
=&I_{2s}(\rho^{-1}(u_{j_1}^2 \cdots u_{j_s}^2)) \qquad \text{(since } I_{2s} \text{ is an invariant polynomial)} \\
=&\sum_{\sigma\in\mathfrak{S}_{2s}} b_\sigma \left(\sum_{i_1,\ldots,i_s=1}^{n} \sigma(\rho^{-1}(u_{j_1}^2 \cdots u_{j_s}^2))(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}) \right) \qquad \text{(by (3.59))} \\
=&\sum_{\sigma\in\mathfrak{S}_{2s}} b_{\sigma\rho} \left(\sum_{i_1,\ldots,i_s=1}^{n} \sigma(u_{j_1}^2 \cdots u_{j_s}^2)(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}) \right) \qquad \text{(by (3.57))} \\
=&2^s s! \sum_{\sigma\in\mathfrak{Q}_s} b_{\sigma\rho} \left(\sum_{i_1,\ldots,i_s=1}^{n} \sigma(u_{j_1}^2 \cdots u_{j_s}^2)(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}) \right). \qquad \text{(by remark 3.70)}
\end{aligned}$$

The inner sum is always zero unless $\sigma = \mathrm{id}_{\mathfrak{S}_s}$ but then it is equal to one. Thus the last line is equal to $2^s s! b_\rho$. We deduce that b_σ are the same for all σ, namely

$$b_\sigma = \frac{2\pi^{n/2}}{n(n+2)\cdots(n+2s-2)\Gamma(n/2)2^s s!}. \tag{3.61}$$

Since $\mathrm{card}(\mathfrak{S}_{2s})/\mathrm{card}(\mathfrak{Q}_s) = 2^s s!$ the lemma now follows from lemma 3.69.

□

3.76 Corollary

Assume that $\phi \in W_{2s}$ is symmetric. Then (3.58) simplifies to

$$I_{2s}(\phi) = \frac{1 \cdot 3 \cdots (2s-1) \cdot 2\pi^{n/2}}{n(n+2)\cdots(n+2s-2)\Gamma(n/2)} \sum_{i_1,\dots,i_s=1}^{n} \phi(e_{i_1}, e_{i_1}, \dots, e_{i_s}, e_{i_s}).$$

PROOF: This follows immediately from the previous lemma and lemma 3.69.

□

3.77 Theorem

Let $M = \mathbb{R}^n$, and assume that P has a second fundamental form T and curvature tensor R^P. Let S_p^{n-q-1} denote the unit sphere in $(T_pP)^\perp$. Then

$$\begin{aligned}
&\int_{S_p^{n-q-1}} \det(\delta_{a,b} - tT_{abu})\, du \\
=&\frac{2\pi^{(n-q)/2}}{\Gamma((n-q)/2)} \sum_{c=0}^{[\frac{q}{2}]} \frac{C^{2c}((R^P)^c)t^{2c}}{c!(2c)!(n-q)(n-q+2)\cdots(n-q+2c-2)} \\
=&2\pi^{(n-q)/2)} \sum_{c=0}^{[\frac{q}{2}]} \frac{C^{2c}((R^P)^c)t^{2c}}{c!(2c)!\Gamma((n-q)/2+c)2^c}.
\end{aligned}$$

Note that we define $(n-q)(n-q+2)\cdots(n-q+2c-2) = 1$ if $c = 0$.

PROOF : We define the mapping ψ_c be setting

$$\psi_c(u_1,\dots,u_c) = \frac{1}{(c!)^2} \sum_{a_1,\dots,a_c=1}^{q} \sum_{\sigma,\pi \in \mathfrak{S}_c} \varepsilon_\sigma \varepsilon_\pi T_{a_{\sigma(1)}a_{\pi(1)}u_1} \cdots T_{a_{\sigma(c)}a_{\pi(c)}u_c} \tag{3.62}$$

for $u_1, \dots, u_c \in (T_pP)^\perp$. By remark 3.57 it's straightforward to show that ψ_c is symmetric and by methods from linear algebra it can be shown that we can expand $\det(\delta_{a,b} - tT_{abu})$ by minors which gives us:

$$\det(\delta_{a,b} - tT_{abu}) = \sum_{c=0}^{q} \psi_c(u,\dots,u)(-t)^c, \tag{3.63}$$

for $u \in (T_pP)^\perp$. We want to integrate ψ_c over S^{n-q-1}. We note that due to symmetry the integral of ψ_{2c+1} is equal to 0. Let $(e_{q+1},\dots,e_n)$ be an orthonormal basis for $(T_pP)^\perp$. By corollary 3.76 we have that

$$\begin{aligned}
I_{2c}(\psi_{2c}) &= \frac{1 \cdot 3 \cdots (2c-1) \cdot 2\pi^{(n-q)/2} \sum_{i_1,\dots,i_c=q+1}^{n} \psi_{2c}(e_{i_1}, e_{i_1}, \dots, e_{i_c}, e_{i_c})}{(n-q)(n-q+2)\cdots(n-q+2c-2)\Gamma((n-q)/2)} \\
&= \frac{(2c)!2\pi^{(n-q)/2}}{c!4^c\Gamma((n-q)/2+c)} \sum_{i_1,\dots,i_c=q+1}^{n} \psi_{2c}(e_{i_1}, e_{i_1}, \dots, e_{i_c}, e_{i_c}).
\end{aligned} \tag{3.64}$$

We will have use for the following lemma which is a special case of an equality known as the ***Gauss equation***. It's proof can be found for example in [On].

3.78 Lemma
For a submanifold $P \subset \mathbb{R}^n$ the following equation is valid:

$$R^P_{abcd} = \sum_{i=q+1}^{n} (T_{ace_i}T_{bde_i} - T_{ade_i}T_{bce_i}) = \langle T_{ac}, T_{bd}\rangle - \langle T_{ad}, T_{bc}\rangle.$$

□

From this we obtain

$$\begin{aligned}&\sum_{i_1,\dots,i_c=q+1}^{n} \psi_{2c}(e_{i_1}, e_{i_1}, \dots, e_{i_c}, e_{i_c})\\ &= \frac{1}{((2c)!)^2} \sum_{i_1,\dots,i_c=q+1}^{n} \sum_{a_1,\dots,a_c=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma\varepsilon_\pi \Big\{ T_{a_{\sigma(1)}a_{\pi(1)}e_{i_1}} T_{a_{\sigma(2)}a_{\pi(2)}e_{i_1}} \\ &\cdots T_{a_{\sigma(2c-1)}a_{\pi(2c-1)}e_{i_c}} T_{a_{\sigma(2c)}a_{\pi(2c)}e_{i_c}} \Big\}.\end{aligned} \tag{3.65}$$

We have that

$$\begin{aligned}\langle T_{a_{\sigma(s)}a_{\pi(t)}}, T_{a_{\sigma(u)}a_{\pi(v)}}\rangle &= \langle T_{a_{\sigma(s)}a_{\pi(t)}}, \sum_{i=q+1}^{n} \Big(\langle T_{a_{\sigma(u)}a_{\pi(v)}}, e_i\rangle e_i\Big)\rangle \\ = \sum_{i=q+1}^{n} \langle T_{a_{\sigma(s)}a_{\pi(t)}}, \langle T_{a_{\sigma(u)}a_{\pi(v)}}, e_i\rangle e_i\rangle &= \sum_{i=q+1}^{n} \langle T_{a_{\sigma(s)}a_{\pi(t)}}, e_i\rangle \langle T_{a_{\sigma(u)}a_{\pi(v)}}, e_i\rangle \\ &= \sum_{i=q+1}^{n} T_{a_{\sigma(s)}a_{\pi(t)}e_i} T_{a_{\sigma(u)}a_{\pi(v)}e_i}\end{aligned}$$

since $(e_{q+1}, \dots, e_n)$ is an orthonormal basis for $(T_pP)^\perp$. Hence we can simplify the right hand side of (3.65) to

$$\frac{1}{((2c)!)^2} \sum_{a_1,\dots,a_c=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma\varepsilon_\pi \Big\{ \langle T_{a_{\sigma(1)}a_{\pi(1)}}, T_{a_{\sigma(2)}a_{\pi(2)}}\rangle \cdots \langle T_{a_{\sigma(2c-1)}a_{\pi(2c-1)}}, T_{a_{\sigma(2c)}a_{\pi(2c)}}\rangle \Big\} \tag{3.66}$$

3.79 Lemma
The expression (3.66) above is the same as

$$\begin{aligned}&\frac{1}{2^c((2c)!)^2} \sum_{a_1,\dots,a_c=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma\varepsilon_\pi \Big\{ \Big(\langle T_{a_{\sigma(1)}a_{\pi(1)}}, T_{a_{\sigma(2)}a_{\pi(2)}}\rangle - \langle T_{a_{\sigma(1)}a_{\pi(2)}}, T_{a_{\sigma(2)}a_{\pi(1)}}\rangle\Big) \\ &\cdots \Big(\langle T_{a_{\sigma(2c-1)}a_{\pi(2c-1)}}, T_{a_{\sigma(2c)}a_{\pi(2c)}}\rangle - \langle T_{a_{\sigma(2c-1)}a_{\pi(2c)}}, T_{a_{\sigma(2c)}a_{\pi(2c-1)}}\rangle\Big) \Big\}\end{aligned} \tag{3.67}$$

PROOF: Let $j_1, \ldots, j_c \in \{0,1\}$. We define $\rho_{j_1,\ldots,j_c} \in \mathfrak{S}_{2c}$ to be $(1,2)^{j_1} \cdots (2c-1, 2c)^{j_c}$. We note that

$$\varepsilon_{\rho_{j_1,\ldots,j_c}} (-1)^{\sum_{i=1}^c j_i} = 1. \tag{3.68}$$

Now (3.67) can be written as

$$\begin{aligned}
&\frac{1}{2^c((2c)!)^2} \sum_{a_1,\ldots,a_c=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma \varepsilon_\pi \sum_{j_1,\ldots,j_c=0}^{1} (-1)^{j_1} \langle T_{a_{\sigma(1)}a_{\pi(1+j_1)}}, T_{a_{\sigma(2)}a_{\pi(2-j_1)}} \rangle \\
&\cdots (-1)^{j_c} \langle T_{a_{\sigma(2c-1)}a_{\pi(2c-1+j_c)}}, T_{a_{\sigma(2c)}a_{\pi(2c-j_c)}} \rangle \\
&= \frac{1}{2^c((2c)!)^2} \sum_{a_1,\ldots,a_c=1}^{q} \sum_{j_1,\ldots,j_c=0}^{1} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma \varepsilon_{\pi\rho_{j_1,\ldots,j_c}} \\
&(-1)^{j_1} \langle T_{a_{\sigma(1)}a_{\pi\rho_{j_1,\ldots,j_c}(1+j_1)}}, T_{a_{\sigma(2)}a_{\pi\rho_{j_1,\ldots,j_c}(2-j_1)}} \rangle \\
&\cdots (-1)^{j_c} \langle T_{a_{\sigma(2c-1)}a_{\pi\rho_{j_1,\ldots,j_c}(2c-1+j_c)}}, T_{a_{\sigma(2c)}a_{\pi\rho_{j_1,\ldots,j_c}(2c-j_c)}} \rangle
\end{aligned}$$

which gives us (3.66) since the last expression is independent of $j_1, \ldots, j_c$ by (3.68).

□

We now use the Gauss equation and obtain

$$\begin{aligned}
&\sum_{i_1,\ldots,i_c=q+1}^{n} \psi_{2c}(e_{i_1}, e_{i_1}, \ldots, e_{i_c}, e_{i_c}) \\
&= \frac{1}{2^c((2c)!)^2} \sum_{a_1,\ldots,a_c=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma \varepsilon_\pi \left\{ R^P_{a_{\sigma(1)}a_{\sigma(2)}a_{\pi(1)}a_{\pi(2)}} \cdots R^P_{a_{\sigma(2c-1)}a_{\sigma(2c)}a_{\pi(2c-1)}a_{\pi(2c)}} \right\} \\
&= \frac{2^c}{((2c)!)^2} \sum_{a_1,\ldots,a_c=1}^{q} (R^P)^c(a_1, \ldots, a_{2c})(a_1, \ldots, a_{2c}) \\
&= \frac{2^c}{((2c)!)^2} C^{2c}((R^P)^c).
\end{aligned}$$

By this and (3.64) we get

$$I_{2s}(\psi_{2c}) = \frac{2\pi^{(n-q)/2} C^{2c}((R^P)^c)}{c!(2c)!2^c\Gamma((n-q)/2+c)}.$$

Hence by (3.63) we obtain

$$\begin{aligned}
\int_{S^{n-q-1}} \det(\delta_{a,b} - tT_{abu})\, du &= \sum_{c=0}^{q} (-t)^c \int_{S^{n-q-1}} \psi_c(u, \ldots, u)\, du \\
&= \sum_{c=0}^{[\frac{q}{2}]} I_{2s}(\psi_{2c}) t^{2c} = 2\pi^{(n-q)/2} \sum_{c=0}^{[\frac{q}{2}]} \frac{C^{2c}((R^P)^c) t^{2c}}{c!(2c)!2^c\Gamma((n-q)/2+c)}
\end{aligned}$$

which was what was wanted.

□

Now we're ready to prove Weyl's Tube formula. In order to prevent misunderstanding I'll write the theorem exactly as it is written in [Gr].

3.80 Theorem (Weyl's Tube Formula)
Let P be a q-dimensional topologically embedded submanifold in Euclidean space $\mathbb{R}^n$. Assume that P has compact closure, and that every point in the tube $T(P,r)$ has a unique shortest geodesic connecting it with P. Then the volume $V_P^{\mathbb{R}^n}(r)$ of $T(P,r)$ is given by

$$V_P^{\mathbb{R}^n}(r) = \frac{(\pi r^2)^{(n-q)/2}}{\Gamma((n-q)/2+1)} \sum_{c=0}^{[\frac{q}{2}]} \frac{k_{2c}(P) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)}. \tag{3.69}$$

PROOF: By lemma 3.43 we have that

$$\mathrm{A}_P^{\mathbb{R}^n}(t) = t^{n-q-1} \int_P \int_{S^{n-q-1}} \vartheta_u(r) du\, dP, \tag{3.70}$$

where $\vartheta_u(r) = \det(\delta_{a,b} - tT_{abu})$ by lemma 3.45. By theorem 3.77 and definition 3.59 we therefore have

$$\begin{aligned} \mathrm{A}_P^{\mathbb{R}^n}(r) &= \frac{2\pi^{(n-q)/2}}{\Gamma((n-q)/2)} \sum_{c=0}^{[\frac{q}{2}]} \frac{k_{2c}(P) t^{n-q-1+2c}}{(n-q)(n-q+2)(n-q+4)\cdots(n-q+2c-2)} \\ &= \frac{\pi^{(n-q)/2}}{\Gamma((n-q)/2+1)} \sum_{c=0}^{[\frac{q}{2}]} \frac{k_{2c}(P) t^{n-q-1+2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c-2)}. \end{aligned}$$

Now, since $\mathrm{V}_P^{\mathbb{R}^n}(r) = \int_0^r \mathrm{A}_P^{\mathbb{R}^n}(t)\, dt$ by lemma 3.44, Weyl's Tube Formula follows.

□

3.81 Corollary
The coefficients in (3.69) depends only on the curvature of P and thus the tube volume $V_P^{\mathbb{R}^n}(r)$ depends only on P and r and not on the particular way that P is embedded in $\mathbb{R}^n$.

□

3.5.4 Calculating the volume of $B_p^n(r)$

I'll now calculate the volume of $B_p^n(r)$ in two ways. We have already seen one way how to do this in chapter 1. Now we'll use other methods. We observe that $B_p^n(r)$ is just a geodesic ball with center p and radius r.

(1) **Using corollary 3.51**: By remark 3.52 we have that the sectional curvature of $\mathbb{R}^n$ is 0. Thus we have by corollary 3.51 that

$$\begin{aligned} B_p^n(r) &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \lim_{\lambda\to 0} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r t^{n-1} \lim_{\lambda\to 0} \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1} dt \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r t^{n-1} dt = \frac{2\pi^{n/2} r^n}{n\Gamma(n/2)}, \end{aligned}$$

which agrees with (1.3).

(2) **Using Weyl's tube formula**: As mentioned in remark 3.60, $k_0(P) = \mathrm{Vol}(P)$. So when P is a point we have $\mathrm{Vol}(P) = 1$. From Weyl's tube formula we have

$$\begin{aligned} B_p^n(r) &= \frac{(\pi r^2)^{(n-0)/2}}{\Gamma((n-0)/2+1)} \sum_{c=0}^{[\frac{0}{2}]} \frac{k_{2c}(P) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)} \\ &= \frac{(\pi r^2)^{n/2}}{\Gamma(n/2+1)} k_0(P) r^0 \\ &= \frac{2\pi^{n/2} r^n}{n\Gamma(n/2)}, \end{aligned}$$

which is the same result as before.

3.6 Kähler manifolds

3.82 Remark
In this section I'll introduce some concept from the theory of complex manifolds. Some of the lemmas and theorems in this section will be stated without proof. For more details see [Gr]. For some introduction to complex manifolds, see for example [1].

3.83 Definition
Let N be a smooth manifold. A (1,1)-tensor field $J : \mathcal{X}(N) \to \mathcal{X}(N)$ which satisfies $J^2 = -I$ is called an **almost complex structure** *on N. N together with this structure is called an* **almost complex manifold**.

3.84 Remark
If J and N are as in the above definition, then J is a field of linear transformations which sends every $m \in N$ to a linear map $J_m : T_mN \to T_mN$ such that $J_m^2 = -I_m$, where I_m is the identity map on T_mN. If $\dim N = n$, then we have

$$(-1)^n = \det(-I_m) = \det(J_m^2) = (\det(J_m))^2,$$

for all $m \in N$, so N is even dimensional.

3.85 Definition

Let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric on M, and assume that M has an almost complex structure J.

(1) $\langle \cdot, \cdot \rangle$ is said to be an **almost Hermitian metric** *and M to be an* **almost Hermitian manifold** *if $\langle JX, JY \rangle = \langle X, Y \rangle$ for all $X, Y \in \mathcal{X}(M)$.*

(2) Let M be an almost Hermitian manifold. The 2-form $F : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}$ defined by setting $F(X, Y) = \langle JX, Y \rangle$ for all $X, Y \in \mathcal{X}$ is called a **Kähler form** *of M* **with respect to** *J.*

3.86 Lemma

An almost complex n-dimensional manifold N is orientable and can be made into an almost Hermitian manifold.

Proof: N has a Riemannian metric $(\cdot, \cdot)$ by remark 2.59. We define another Riemannian metric $\langle \cdot, \cdot \rangle : \mathcal{X}(N) \times \mathcal{X}(N) \to \mathbb{R}$ by

$$\langle X, Y \rangle = (X, Y) + (JX, JY).$$

Now

$$\langle JX, JY \rangle = (JX, JY) + (-X, -Y) = (JX, JY) + (-1)^2 (X, Y) = \langle X, Y \rangle,$$

so $\langle \cdot, \cdot \rangle$ is an almost Hermitian metric on M. By remark 3.84 $\dim N$ is even, so $F^{n/2}$ is well defined. It is straightforward to show that $F^{n/2}$ is a non-vanishing n-form and thus N is orientable by remark 2.23.

□

3.87 Definition

Let M be an almost Hermitian manifold, and let $m \in M$.

(1) A real orthonormal basis of the form

$$(1, 1^*, \ldots, n, n^*) := (e_1, Je_1, \ldots, e_n, Je_n)$$

is called a **holomorphic orthonormal frame**. *Note that $Ji = i^*$, $Ji^* = -i$ for all $i = 1, \ldots, n$.*

(2) Let $X \in \mathcal{X}(M)$. Then we define the covariant derivative of J as

$$\nabla_X(J) = \nabla_X J - J\nabla_X.$$

(See also definition 3.22). If $\nabla_X(J)Y = 0$ for all $X, Y \in \mathcal{X}$ then J is said to be **parallel**.

(3) If J is parallel then M is called a **Kähler manifold**.

The following lemma shows the naturality of J. I'll go through the proof since I find it instructive.

3.88 Lemma
Let M be a Kähler manifold, and let $X, Y \in \mathcal{X}(M)$. Then

$$\begin{aligned} R_{JXJY} &= R_{XY} && (\textit{Kähler identidy}) \\ \rho(JX, JY) &= \rho(X, Y) \end{aligned}$$

PROOF : We have for all $X, Y, Z, W \in \mathcal{X}(M)$ that

$$\begin{aligned} & \langle R_{JXJY} Z, W\rangle \\ = & \langle R_{ZW} JX, JY\rangle && \text{(by (2.41))} \\ = & \langle (\nabla_{[Z,W]} - [\nabla_Z, \nabla_W]) JX, JY\rangle && \text{(by (2.38))} \\ = & \langle (\nabla_{[Z,W]} - \nabla_Z\nabla_W + \nabla_W\nabla_Z) JX, JY\rangle && \text{(def. of Lie brackets)} \\ = & \langle J(\nabla_{[Z,W]} - \nabla_Z\nabla_W + \nabla_W\nabla_Z) X, JY\rangle && \text{(since } J \text{ is parallel)} \\ = & \langle JR_{ZW} X, JY\rangle && \text{(by (2.38))} \\ = & \langle R_{ZW} X, Y\rangle && (M \text{ is almost Hermitian}) \\ = & \langle R_{XY} Z, W\rangle && \text{(by (2.41))} \end{aligned}$$

so this proves the first part.
Let $(E_1, \ldots, E_{2n})$ be a real local orthonormal frame for TM. Then $(JE_1, \ldots, JE_{2n})$ is also a local orthonormal frame for TM. By using the same definitions and theorems as before we have

$$\begin{aligned} \rho(JX, JY) &= \sum_{a=1}^{2n} R_{JXE_aJYE_a} = \sum_{a=1}^{2n} R_{JXJE_aJYJE_a} = \sum_{a=1}^{2n} \langle R_{JXJE_a} JY, JE_a\rangle \\ &= \sum_{a=1}^{2n} \langle R_{XE_a} JY, JE_a\rangle = \sum_{a=1}^{2n} \langle R_{JYJE_a} X, E_a\rangle = \sum_{a=1}^{2n} \langle R_{YE_a} X, E_a\rangle \\ &= \rho(Y, X) = \rho(X, Y). \end{aligned}$$

□

3.89 Remark
For a holomorphic orthonormal frame we have by (2.39)–(2.42)

$$\begin{aligned} \rho(X, Y) &= \sum_{i=1}^{n} (R_{xiyi} + R_{xi^*yi^*}) = \sum_{i=1}^{n} (R_{xiy^*i^*} - R_{xi^*y^*i}) \\ &= \sum_{i=1}^{n} (-R_{ixy^*i^*} - R_{y^*ixi^*}) = \sum_{i=1}^{n} R_{xy^*ii^*}. \end{aligned}$$

3.90 Definition

(1) *Let N be an almost complex manifold. A two dimensional subspace $T_m\Pi$ of T_mN which can be written on the form* span$\{x, Jx\}$ *where $x \in T_mN$ is called a* **holomorphic section**.

(2) *Let M be a Kähler manifold. The sectional curvature (see definition 2.45) restricted to holomorphic sections of tangent spaces is called the* **holomorphic sectional curvature** *and is denoted by*

$$K_{\text{hol}}.$$

3.91 Remark

(1) If M is Hermitian then $\langle JX, X\rangle = 0$ for all $X \in \mathcal{X}(M)$.

(2) K_{hol} can be regarded as a function that assigns each unit tangent vector x to M a real number $K_{\text{hol}}(x)$. We extend the definition to all tangent vectors x to M by

$$K_{\text{hol}}(x) = R_{xJxxJx}.$$

3.92 Definition

Let M be an almost Hermitian manifold with an almost Hermitian structure J.

(1) If there exist a constant λ such that

$$K_{\text{hol}}(x) = 4\lambda\|x\|^4 \tag{3.71}$$

for all x tangent to M, then M is said to have a **constant sectional curvature** 4λ.

(2) If N is a Kähler manifold of dimension $2n$ which satisfies (3.71) we say that N is a **space of constant holomorphic sectional curvature** 4λ *and denote it by*

$$\mathbb{K}^n_{\text{hol}}(\lambda).$$

Note that if we regard N as a complex manifold, its complex dimension is n.

The following theorem is well known in the theory of complex manifolds.

3.93 Theorem

Let $\mathbb{K}^n_{\text{hol}}(\lambda)$ be a Kähler manifold of constant holomorphic sectional curvature 4λ. Then it's sectional curvature of linearly independent x and y is given by

$$K_{xy} = \lambda\left(1 + \frac{3\langle Jx, y\rangle^2}{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}\right).$$

Thus

$$\lambda < K < 4\lambda.$$

□

3.94 Definition

Let R denote the curvature tensor of M.

(1) As before we define the covariant derivate of R as a tensor field by

$$\begin{aligned}\nabla_V(R)_{WXYZ} = V\left(R_{WXYZ}\right) - R_{\nabla_V WXYZ} \\ - R_{W\nabla_V XYZ} - R_{WX\nabla_V YZ} - R_{WXY\nabla_V Z}\end{aligned}$$

for $V, X, Y, Z, W \in \mathcal{X}(M)$. (For more details see for example page 53 in [Lee]).

(2) M is said to be **locally symmetric** *if $\nabla R = 0$.*

3.95 Definition
Let P and ν be as before.

(1) Let $(p, u) \in \nu$. Then we define a map $R_u : T_pM \to T_pM$ by

$$R_u x = R_{ux} u$$

for $x \in T_pM$. We note the similarities between R_u and R_N. By (2.41) it follows that R_u is symmetric, i.e. $\langle R_u x, y\rangle = \langle x, R_u y\rangle$ for all $x, y \in T_pM$.

(2) Let ξ be a unit speed geodesic on M. Then we define a map $R(t) : T_{\xi(t)}M \to T_{\xi(t)}M$ by

$$R(t)x = R_{\xi'(t)x}\xi'(t).$$

This generalizes $R(t)$ defined in corollary 3.27.

(3) Let $(p, u) \in \nu$, $T_u : T_pP \to T_pP$ be the Weingarten map and $I : (T_pP)^\perp \to (T_pP)^\perp$ be the identity map.. Since we have the ordinary identification $T_pM = T_pP \oplus (T_pP)^\perp$ we have a natural map $T_u \oplus I : T_pM \to T_pM$, defined in an obvious way.

(4) If for all $(p, u) \in \nu$, where u is a unit vector we have

$$T_u \oplus I \text{ is diagonalizable } \Leftrightarrow R_u \text{ is diagonalizable}$$

then P is said to be **compatible with** *M.*

3.6.1 A vital Lemma without proof

3.96 Lemma
(1) $\mathbf{K}^n(\lambda)$ is locally symmetric.

(2) $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ is locally symmetric.

(3) Any submanifold of $\mathbf{K}^n(\lambda)$ is compatible with $\mathbf{K}^n(\lambda)$.

(4) Any complex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ is compatible with $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$.

(5) If M is locally symmetric, and $P \subset M$ is a submanifold of M and compatible with M then the eigenvectors of the shape operator of the tubular hypersurface P_t can be chosen parallel along geodesics normal to P.

□

3.6.2 Volume of a geodesic ball in $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$

3.97 Lemma

The volume of a geodesic ball in $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ is given by

$$V_m^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r) = \frac{1}{n!}\left(\frac{\pi}{\lambda}\right)^n (\sin(r\sqrt{\lambda}))^{2n}.$$

PROOF: Let $u \in T_m\mathbb{K}^n_{\mathrm{hol}}(\lambda)$, with $\|u\| = 1$. Let $e_1 = u$ and extend e_1 to a holomorphic orthonormal frame $(e_1, Je_1, \ldots, e_n, Je_n)$ for $T_m\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ which diagonalizes R_u. Let $(E_1(t), \ldots, JE_n(t))$ be a holomorphic frame along ξ which coincides with $(e_1, Je_1, \ldots, e_n, Je_n)$ at m. Clearly m is compatible with M and thus we have by theorem 3.96 that $E_1(t), \ldots, JE_n(t)$ are eigenvectors both for $R(t)$ and $S(t)$. Here $S(t)$ is the shape operator at $\xi(t)$ of the geodesic ball

$$\{m' \in \mathbb{K}^n_{\mathrm{hol}}(\lambda) \,|\, \mathrm{dist}(m, m') = t\}. \tag{3.72}$$

Thus by theorem 3.93 we have the following Riccati differential equations for the principal curvature functions on (3.72):

$$\begin{aligned} \kappa_2' &= \kappa_2^2 + 4\lambda \\ \kappa_i' &= \kappa_i^2 + \lambda \qquad (3 \leqslant i \leqslant 2n) \end{aligned}$$

where

$$k_i(t) \xrightarrow[t\to 0+]{} -\infty,$$

for $i = 2, \ldots, 2n$. By theorem 3.47 we have

$$\kappa_2(t) = \frac{-2\sqrt{\lambda}}{\tan(2t\sqrt{\lambda})}, \qquad \kappa_i(t) = \frac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}, \quad (3 \leqslant i \leqslant 2n) \tag{3.73}$$

By theorem 3.40 we have

$$\frac{\vartheta_u'(t)}{\vartheta_u(t)} = -\frac{2n-1}{t} + \frac{2\sqrt{\lambda}}{\tan(2t\sqrt{\lambda})} + \frac{(2n-2)\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}.$$

We solve this in a similar fashion as in the proof of theorem 3.49 and get

$$\ln\vartheta_u(t) = -(2n-1)\ln(t) + \ln(\sin(2t\sqrt{\lambda})) + (2n-2)\ln(\sin(t\sqrt{\lambda})) + C$$

where $C \in \mathbb{R}^n$, or equivalently

$$t^{2n-1}\vartheta_u(t) = e^C \sin(2t\sqrt{\lambda})(\sin(t\sqrt{\lambda}))^{2n-2}.$$

Due to our initial condition $\vartheta_u(0) = 1$, we can easily calculate that $e^C = \frac{1}{2}\lambda^{-n+1/2}$. Since $\frac{1}{2}\sin(2t\sqrt{\lambda}) = \sin(t\sqrt{\lambda})\cos(t\sqrt{\lambda})$ we have that

$$t^{2n-1}\vartheta_u(t) = (\sin(t\sqrt{\lambda}))^{2n-1}\cos(t\sqrt{\lambda})\lambda^{-n+1/2}. \tag{3.74}$$

Now we'll use (1.2), lemmas (3.43) and (3.44) along with the above to obtain the desired volume, namely

$$
\begin{aligned}
V_m^{\mathbb{K}^n_{\text{hol}}(\lambda)}(r) &= \int_0^r \int_{S^{2n-1}} (\sin(t\sqrt{\lambda}))^{2n-1} \cos(t\sqrt{\lambda})\lambda^{-n+1/2}\, du\, dt \\
&= \frac{2\pi^n}{\Gamma(n)} \int_0^r (\sin(t\sqrt{\lambda}))^{2n-1} \cos(t\sqrt{\lambda})\lambda^{-n+1/2}\, dt \\
&= \frac{1}{n!}\left(\frac{\pi}{\lambda}\right)^n (\sin(r\sqrt{\lambda}))^{2n}.
\end{aligned}
$$

□

3.98 Definition
If $\vartheta_u(t)$ is independent of which unit tangent vector u on M we choose, then M is said to be a **harmonic space***.*

3.6.3 Complex projective space $\mathbb{C}P^n(\lambda)$

$\mathbb{C}P^n(\lambda)$ is a complex manifold, defined to be the space of complex lines in $\mathbb{C}^{n+1}$. As the notation indicates, then $\mathbb{C}P^n(\lambda)$ is a harmonic space. If we define the following equivalence relation on S^{2n+1}

$$(z_0, \ldots, z_n) \sim (w_0, \ldots, w_n) \Leftrightarrow \exists\, a \in \mathbb{C},\ |a| = 1 : \forall i : z_i = a w_i,$$

where $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$ then it is can be show that $\mathbb{C}P^n(1)$ is the set of equivalence classes of $\sim$. So $\mathbb{C}P^n(1)$ is a manifold and the projection $\pi : S^{2n+1} \to \mathbb{C}P^n(1)$ is clearly differentiable and surjective.

More important is that $\mathbb{C}P^n(1)$ can be made into a Kählerian manifold, by using the so-called *Fubini-Study* metric. For details of its construction and references see [Gr]. Thus it follows that $\mathbb{C}P^n(\lambda)$ is $\mathbb{C}P^n(1)$ with the Fubini-Study metric divided by λ^2.

3.99 Theorem
The volume of $\mathbb{C}P^n(\lambda)$ is given by

$$\frac{1}{n!}\left(\frac{\pi}{\lambda}\right)^n$$

PROOF: We have that

$$\mathrm{A}_m^{\mathbb{C}P^n(\lambda)}(t) = \frac{2\pi^n}{(n-1)!\lambda^{n-1/2}} (\sin(t\sqrt{\lambda}))^{2n-1} \cos(t\sqrt{\lambda}) \tag{3.75}$$

by lemmas 3.97 and 3.43. Since $\vartheta_u(t)$ is independent of u so is the distance from m to the first zero of $\vartheta_u(t)$. So for the first $t > 0$ such that $\mathrm{A}_m^{\mathbb{C}P^n(\lambda)}(t) = 0$ is when the geodesic ball

$$\{m' \in \mathbb{C}P^n(\lambda) |\, \mathrm{dist}(m, m') < t\}$$

exactly covers $\mathbb{C}P^n$. This apparently happens the first time when $\cos(t\sqrt{\lambda}) = 0$, i.e. when $t = \pi/(2\sqrt{\lambda})$. By lemma 3.44 we obtain the volume of $\mathbb{C}P^n$, namely

$$\begin{aligned} \mathrm{V}_m^{\mathbb{C}P^n}(\pi/(2\sqrt{\lambda})) &= \int_0^{\pi/(2\sqrt{\lambda})} \mathrm{A}_m^{\mathbb{C}P^n(\lambda)}(t)\,dt \\ &= \frac{2\pi^n}{(n-1)!\lambda^{n-1/2}} \int_0^{\pi/(2\sqrt{\lambda})} (\sin(t\sqrt{\lambda}))^{2n-1}\cos(t\sqrt{\lambda})\,dt \\ &= \frac{2\pi^n}{(n-1)!\lambda^{n-1/2}} \left[\frac{1}{2n\sqrt{\lambda}} \sin(t\sqrt{\lambda})^{2n}\right]_0^{\pi/(2\sqrt{\lambda})} = \frac{1}{n!}\left(\frac{\pi}{\lambda}\right)^n, \end{aligned}$$

which gives the desired volume.

□

Chapter 4

Invariant measure on $Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R})$

The main aim of this chapter is to calculate an *invariant measure* of the quotient space

$$Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R}) = \{Sl(n,\mathbb{Z}) \cdot x |\, x \in Sl(n,\mathbb{R})\}.$$

I'll also make a few speculations about fibre bundles and their relation to volume.

4.1 Topics from linear algebra

4.1 Definition
An element in $M(n,\mathbb{R})$ which has 1 in the (i,j)-th entry and 0 elsewhere is denoted by $E_{i,j}$. The elements $E_{i,j} \in M(n,\mathbb{R})$ where $i \neq j$ and $E_{i,i} - E_{i+1,i+1}$ where $i \in [\![1, n-1]\!]$ form a basis for $\mathfrak{sl}(n,\mathbb{R})$ [Hum]. This basis is called the **zero trace basis**.

4.2 Definition
Let X be a matrix. We will denote the i-th row by L_i. The following procedures

(1) $L_i \to \alpha L_i$, meaning multiply row i by a non-zero scalar α;

(2) $L_i \leftrightarrow L_j$, meaning interchange rows i and j;

(3) $L_j \to \alpha L_i + L_j$, meaning linear combination $\alpha L_i + L_j$ replaces row j;

are called **row operations** *and we write $r(X)$ for a row operator r which is applied to X. For two row operators r, s we define $sr(X) := s(r(X))$. Two $(n \times m)$-matrices X, X' are said to be* **row equivalent** *if there exist a finite sequence of row operations $(r_1, \dots, r_t)$ such that $r_t r_{t-1} \cdots r_1(X) = X'$.*

4.3 Remark
It's well known that every non-singular $(n \times n)$-matrix is row-equivalent to the identity matrix I_n. Furthermore if $(r_1, \dots, r_t)$ is a sequence of row operations such that $r_t r_{t-1} \cdots r_1(X) = I_n$ then

$$r_t r_{t-1} \cdots r_1(I_n) = X^{-1}.$$

4.4 Definition

(1) *We define* $S_{i,j} := I_n - E_{i,i} - E_{j,j} + E_{i,j} - E_{j,i} \in Sl(n,\mathbb{Z})$ *for all* i, j *such that* $i \neq j$.

(2) *Let* $x \in \mathbb{R}$. *Then we define the functions*

  (a) FLOOR(x) *which rounds* x *to the nearest integer towards minus infinity;*

  (b) FIX(x) *which rounds* x *to the nearest integer towards zero.*

(3) *Let* A *be a* $(n \times n)$*-matrix. For* $k \in [\![1, n]\!]$ *we define*

$$Q_k(A) := \{a_{k,k}, a_{k+1,k}, \ldots a_{n,k}\}.$$

If the number of non-zero elements in $Q_k(A)$ *equals* 1 *we let* $q(Q_k(A))$ *be the line number* i *of the non-zero element in* $Q_k(A)$.

(4) *Let* `N,N'`$\in \mathbb{N}$, *where* `N`$\leqslant$`N'` *and denote by* `P(i)` *an expression (or a finite sequence of expressions) which depends on* `i`. *The following*

```
for i=N:N'
   P(i)
end
```

means that we apply `P(i)` *for all integer values* `i` *between* `N` *and* `N'` *in an increasing order including* `N` *and* `N'`.

4.5 Theorem

Let $x \in M_m(n,\mathbb{Z})$ *and let* $Sl(n,\mathbb{Z}) \cdot x$ *be it's orbit. Then there exist a unique element*

$$y = \begin{bmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & \ddots & \vdots \\ y_{n,1} & \cdots & b_{n,n} \end{bmatrix} \in Sl(n,\mathbb{Z}) \cdot x$$

which satisfies the following conditions:

(1) $y_{j,j} > 0$ *for all* j;

(2) $y_{i,j} = 0$ *for all* i, j *where* $i > j$;

(3) $0 \leqslant y_{i,j} < y_{j,j}$ *for all* i, j *where* $i < j$.

PROOF: *Existence:*
Let

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \in Sl(n,\mathbb{Z}) \cdot x.$$

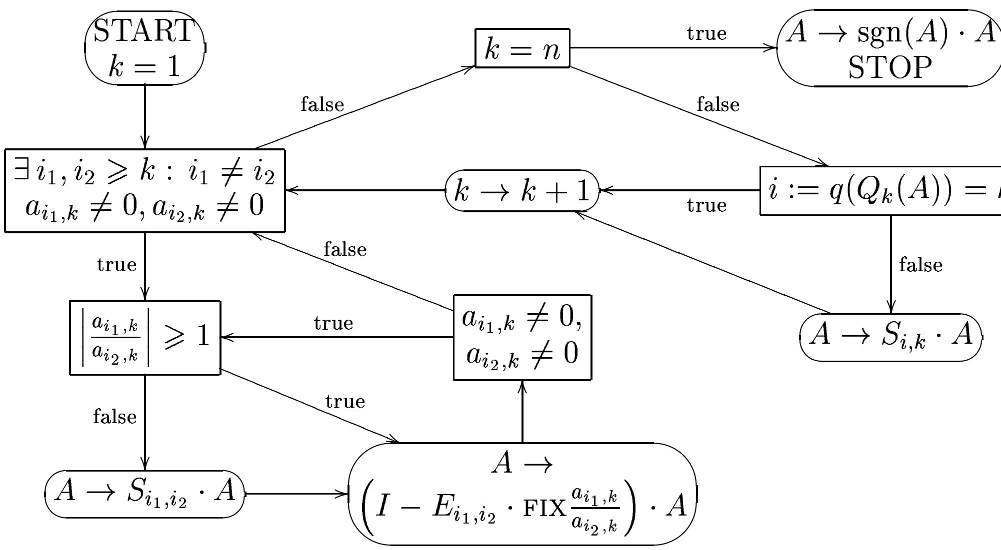
For $a \in \mathbb{R}$ we define

$$\operatorname{sgn}(a) := \begin{cases} 1 & \text{if } a > 0; \\ 0 & \text{if } a = 0; \\ -1 & \text{if } a < 0. \end{cases}$$

We define

$$\operatorname{sgn}(A) = \begin{bmatrix} \operatorname{sgn}(a_{1,1}) & \cdots & \operatorname{sgn}(a_{1,n}) \\ \vdots & \ddots & \vdots \\ \operatorname{sgn}(a_{n,1}) & \cdots & \operatorname{sgn}(a_{n,n}) \end{bmatrix}$$

In the following diagram we show how we can multiply A from the left side with elements in $Sl(n, \mathbb{Z})$ to obtain a matrix which satisfies conditions (1) and (2):



A rounded box means that we perform what is written in the box. If the box is squared then we check if the statement in the box is true or false and then follow a suitable arrow. Since every cycle through the lowest three or four boxes brings one number in $Q_k(A)$ closer to zero (though maybe reversing the order of terms), this diagram eventually comes to an end. In the last rounded box we have that $\operatorname{sgn}(A) \in Sl(n, \mathbb{Z})$ since $\det M_m(n, \mathbb{Z}) \geqslant 1$. After this procedure we apply the following to A.

```
for j=2:n
   for i=1:l-1
      A=(I-E_{i,j}FLOOR(a_{i,j}/a_{j,j}))*A
   end
end
```

This gives the existence.

Uniqueness: Let's assume that $A, A' \in Sl(n, \mathbb{Z}) \cdot x$ which satisfies the given conditions. Then there exist $C \in Sl(n, \mathbb{Z})$ such that $A' = CA$ or, equivalently $C = A'A^{-1}$. We will first prove the following lemma

4.6 Lemma
If we write

$$B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ & \ddots & \vdots \\ 0 & & b_{n,n} \end{bmatrix} := \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{bmatrix}^{-1}$$

then $b_{i,i} = a_{i,i}^{-1}$ *for all* i *and*

$$b_{l,k} = -\sum_{j=0}^{k-l-1} \frac{a_{l,k-j}}{a_{l,l}} \cdot b_{k-j,k}, \tag{4.1}$$

for all l, k *such that* $l < k$.

PROOF: We define the following row operations

(1) $r_{i,i} : L_i \to a_{i,i}^{-1} L_i$ for all i;

(2) $r_{l,k} : L_l \to -a_{l,k} a_{l,l}^{-1} L_k + L_l$ for all l, k such that $l < k$.

We also define the following combinations of row operators

(1) $R = r_{1,1} r_{2,2} \cdots r_{n,n}$

(2) $R_k = r_{1,k} r_{2,k} \cdots r_{k-1,k}$, (for all $k \geqslant 2$).

Then $R_2 R_3 \cdots R_n R(A) = I_n$ and thus $B = R_2 R_3 \cdots R_n R(I_n)$ by remark 4.3. This is just a Gauss elimination process. We clearly have that $b_{i,i} = a_{i,i}^{-1}$ for all i. We want to calculate $b_{l,k}$ for $l < k$. We shall look at how $R_2 \cdots R_k$ reacts on $R_{k+1} \cdots R_n R(I_n)$, namely:

$$B = R_2 \cdots R_k \left[\begin{array}{ccc|ccc} b_{1,1} & & 0 & \cdot & \cdot & \cdot \\ & \ddots & & \cdot & \cdot & \cdot \\ 0 & & b_{k,k} & b_{k,k+1} & \cdots & b_{k,n} \\ \hline & & & b_{k+1,k+1} & \cdots & b_{k+1,n} \\ & 0 & & & \ddots & \vdots \\ & & & & & b_{n,n} \end{array} \right].$$

We note that the (l,k)-th entry in $R_{k+1} \cdots R_n R(I_n)$ is 0 for $l < k$. Furthermore the (j,k)-th entry in $R_{j+1} \cdots R_n R(I_n)$ is $b_{j,k}$. Now $R_j = r_{1,j} r_{2,j} \cdots r_{l,j} \cdots r_{j-1,j}$ for all j with $l < j \leqslant k$. The only terms in R_j which alters the (l,k)-th entry when R_j is applied to $R_{j+1} \cdots R_n R(I_n)$ is $r_{l,j}$, namely $r_{l,j} : L_l \to -a_{l,j} a_{l,l}^{-1} L_j + L_l$. Thus the term

$$-\frac{a_{l,j}}{a_{l,l}} \cdot b_{j,k}$$

is added to the (l,k)-th entry when we apply R_j to $R_{j+1} \cdots R_n R(I_n)$. Evidently we obtain (4.1) by summing over all j with $l < j \leqslant k$.

□

We'll continue with the proof of the theorem. Now

$$C = \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ & \ddots & \vdots \\ \mathbf{0} & & c_{n,n} \end{bmatrix} = \begin{bmatrix} a'_{1,1} & \cdots & a'_{1,n} \\ & \ddots & \vdots \\ \mathbf{0} & & a'_{n,n} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ & \ddots & \vdots \\ \mathbf{0} & & b_{n,n} \end{bmatrix}.$$

and thus $c_{l,k}$ can be written as

$$c_{l,k} = \sum_{j=0}^{k-l} a'_{l,k-j} \cdot b_{k-j,k}. \tag{4.2}$$

By previous lemma we have that $c_{i,i} = a_{i,i}^{-1} \cdot a'_{i,i}$. Since $a_{i,i}, a'_{i,i}, c_{i,i} \in \mathbb{N}$ for all i we deduce that $a_{i,i} | a'_{i,i}$ for all i. But since

$$\prod_i a_{i,i} = \det A = m = \det A' = \prod_i a'_{i,i}$$

we conclude that $a_{i,i} = a'_{i,i}$ for all i.
We'll now show that $a_{l,k} = a'_{l,k}$ for $l < k$. To do that we'll use induction. Let's first assume that $a_{l,k} = a'_{l,k}$ for all $l > r$. We'll then show that $a_{r,r+1} = a'_{r,r+1}$. Now we have by (4.2) and (4.1) that

$$\begin{aligned} c_{r,k} &= \sum_{j=0}^{k-r} a'_{r,k-j} \cdot b_{k-j,k} = \sum_{j=0}^{k-r-1} a'_{r,k-j} \cdot b_{k-j,k} + a_{r,r} \cdot b_{r,k} \\ &= \sum_{j=0}^{k-r-1} \left(a'_{r,k-j} \cdot b_{k-j,k} - a_{r,k-j} \cdot b_{k-j,k} \right) \end{aligned}$$

For $k = r + 1$ this reduces to

$$c_{r,r+1} = \frac{a'_{r,r+1} - a_{r,r+1}}{a_{r+1,r+1}}$$

Since $0 \leqslant a_{r,r+1}, a'_{r,r+1} < a_{r+1,r+1}$ and $c_{r,r+1} \in \mathbb{N}_0$ we deduce that $a_{r,r+1} = a'_{r,r+1}$.
Now assume that $a_{r,k} = a'_{r,k}$ for all $l > r$ and all $k < s$, where $l < s \leqslant n$. We'll show that $a_{r,s} = a'_{r,s}$. We have

$$\begin{aligned} c_{r,s} &= \sum_{j=0}^{s-r-1} \left(a'_{r,s-j} \cdot b_{s-j,s} - a_{r,s-j} \cdot b_{s-j,s} \right) \\ &= \frac{a'_{r,s} - a_{r,s}}{a_{s,s}} + \sum_{j=1}^{s-r-1} \left(a'_{r,s-j} \cdot b_{s-j,s} - a_{r,s-j} \cdot b_{s-j,s} \right) = \frac{a'_{r,s} - a_{r,s}}{a_{s,s}}, \end{aligned}$$

where the last equality signs follows from our induction assumption. As before we can conclude that $a_{r,s} = a'_{r,s}$. This observations shows that $a_{l,k} = a'_{l,k}$ for all l, k hence $A = A'$ proving the uniqueness.
The proof of the theorem is complete.

□

4.2 Topics from number theory

4.2.1 Dirichlet theorem

4.7 Definition

(1) *Let D be a subset in $\mathbb{R}^n\backslash\{0\}$. Let C be the set of all half-lines from the origin which intersects D. The set of all points in C is called a* **cone** *with* **section** *D.*

(2) *Let $(e_1,\ldots,e_n)$ be a basis for $\mathbb{R}^n$. Then the set*

$$\mathbb{Z}e_1+\cdots+\mathbb{Z}e_n$$

is called a **lattice** *for $\mathbb{R}^n$.*

(3) *Let $M=\mathbb{Z}e_1+\cdots+\mathbb{Z}e_n$ be a lattice for $\mathbb{R}^n$. The set*

$$[0,1)e_1+\cdots+[0,1)e_n$$

is called the **fundamental parallelepiped** *of the lattice M. $\mathbb{R}^n$ spanned by $e_1,\ldots,e_n$.*

(4) *A* **lattice** *of a Lie group G is a discrete subgroup $\Gamma\subset G$ such that G/Γ has a finite volume relative to the (induced) G-invariant measure.*

(5) *Let X be a smooth manifold, and Γ be a discrete group. A* **fundamental domain** *of Γ is a subset $D\subset X$ such that the subsets γD, where $\gamma\in\Gamma$ have no common interior points and form a locally finite covering of X. For more details on fundamental domains, see for example [Rat].*

The proof for the following theorem can be found in [Bo]. The proof is simple and I choose to omit it.

4.8 Theorem (Dirichlet theorem)
Let X be a cone in $\mathbb{R}^n\backslash\{0\}$ with vertex at the origin. Let F be a positive function defined over X which satisfies

(1) *$F(tx)=t^nF(x)$, $\forall x\in X, t>0$;*

(2) *$T:=\{x\in X|\ F(x)<1\}$ is bounded and has a non-zero n-dimensional volume $v(T)$.*

Then for any lattice M in $\mathbb{R}^n$ the series

$$f(s)=\sum_{x\in M\cap X}F(x)^{-s} \tag{4.3}$$

is convergent for $s>1$, and

$$\lim_{s\to 1+}(s-1)f(s)=\frac{v(T)}{\Delta} \tag{4.4}$$

where Δ is the volume of the fundamental parallelepiped of the lattice M.

□

4.9 Remark
Let $n \in \mathbb{N}$ be greater than 1 and let X be a cone in $M(n,\mathbb{R})$ with a section D. We define a function $F : M(n,\mathbb{R}) \to \mathbb{R}$ by

$$F(x) = (\det x)^n.$$

By Dirichlet theorem we have

$$v(T) = \lim_{s \to 1+} (s-1) f(s), \qquad f(s) = \sum_{x \in M \cap X} (\det x)^{-ns}, \tag{4.5}$$

where $M := M(n,\mathbb{Z})$.

By theorem 4.5 the number of orbits for the left action of the group $Sl(n,\mathbb{Z})$ on $M_m(n,\mathbb{Z})$ is the same as the number of matrices which satisfies conditions (1)-(3) in theorem 4.5. Let's count those matrices.
First we see in how many ways we can choose elements $y_{1,1},\ldots,y_{n,n}$ on the diagonal. Clearly that is just the number of ways the number m can be written as a product of n ordered natural numbers $d_1,\ldots,d_n$. Now let's look at the possible matrix which has $y_{1,1},\ldots,y_{n,n}$ on the diagonal. Let $\tau_{l,k}$ be the number of possibilities we can fill the (l,k)-th entry for $l < k$. Then the number of matrices which has $y_{1,1},\ldots,y_{n,n}$ on the diagonal is $\prod_{l<k} \tau_{l,k}$. But by condition (3) in theorem 4.5 we have that $\tau_{l,k} = y_{k,k}$ for $l < k$ so we have

$$\prod_{l<k} b_{l,k} = y_{2,2}^1 \cdot y_{3,3}^2 \cdots y_{n,n}^{n-1}.$$

By letting $d_k = y_{n-k+1,n-k+1}$ for $k = 1,\ldots,n$ we have proven the following lemma:

4.10 Lemma
Let a_m be the number of orbits for the left action of the group $Sl(n,\mathbb{Z})$ on $M_m(n,\mathbb{Z})$. Then

$$a_m = \sum d_1^{n-1} d_2^{n-2} \cdots d_{n-1} \tag{4.6}$$

where the sum is taken over all the representations of the number $m = d_1 \cdot d_2 \cdots d_n$, where $d_i \in \mathbb{N}$.

□

4.11 Lemma
The function $f(s)$ in (4.5) can be written as

$$f(s) = \zeta(ns)\zeta(ns-1)\cdots\zeta(ns-n+1), \tag{4.7}$$

where ζ is the **Riemann zeta function** *defined by*

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \qquad (\text{for all } z \in \{w \in \mathbb{C} | \operatorname{Re}(w) > 1\}). \tag{4.8}$$

PROOF: We shall expand the terms on the right side in (4.7). We have

$$\begin{aligned}
&\zeta(ns)\zeta(ns-1)\cdots\zeta(ns-n+1)\\
=&\sum_{i_1=1}^{\infty} i_1^{-ns}\sum_{i_2=1}^{\infty} i_2^{-ns+1}\cdots\sum_{i_n=1}^{\infty} i_n^{-ns+n-1}\\
=&\sum_{k_1,k_2,\ldots,k_n=1}^{\infty} i_{1_{k_1}}^{-ns} i_{2_{k_2}}^{-ns}\cdots i_{n_{k_n}}^{-ns}\cdot i_{2_{k_2}} i_{3_{k_2}}^{2}\cdots i_{n_{k_n}}^{n-1}
\end{aligned}$$

which is evidently the same as $f(s)$.

□

It's easy to show that (see for example [Bo])

$$(s-1)\zeta(ns-n+1)\longrightarrow\frac{1}{n}\qquad\text{as } s\to 1+ \tag{4.9}$$

and thus we have by Dirichlet theorem the following corollary:

4.12 Corollary
The function $v(T)$ in (4.5) is given by

$$v(T)=\frac{1}{n}\zeta(2)\cdots\zeta(n) \tag{4.10}$$

□

4.2.2 Calculating $\omega_\infty(Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R}))$

4.13 Lemma
Two left invariant n-forms on an n-dimensional Lie group G are proportional to each other.

PROOF: Let ω and ω' be two n-forms on G. We have that $\omega = f\omega'$ where f is a real function. Then for every $\sigma\in G$ we have

$$f\omega' = l_\sigma^*(f\omega') = (f\circ l_\sigma)l_\sigma^*(\omega') = (f\circ l_\sigma)\omega'$$

so f must be a constant.

□

4.14 Definition
Let ω be a left invariant n-forms on an n-dimensional Lie group G. Then ω is said to be a **left invariant density** *on G.*

4.15 Remark
(1) By lemma 4.13 all left invariant densities on a Lie-group G is identical up to a constant. We can thus choose a specific left invariant density by giving up its value at one point.

(2) If G is compact a left invariant density ω on G induces a left invariant Haar measure on G by requiring that

$$\int_G \omega = \omega_\infty(G).$$

(3) If Γ is a discrete group in G such that the closure of G/Γ is compact a left invariant density ω on G induces a left invariant Haar measure on ω_∞ on G by requiring that for a measurable fundamental domain D for Γ in G we have

$$\int_D \omega = \omega_\infty(D).$$

I will not go through the proof of this but these results are taken from measure theory. For some discussion on Haar measure see for example [Fo].

We denote by x_{ij} the standard coordinates in $M(n,\mathbb{R})$. $dx_{11} \wedge \cdots \wedge dx_{nn}$ is the form which defines the Lebesgue measure in $M(n,\mathbb{R})$. $Sl(n,\mathbb{R})$ is a hypersurface in $M(n,\mathbb{R})$. We choose a connected measurable fundamental domain D for $Sl(n,\mathbb{Z})$ in $Sl(n,\mathbb{R})$. It can be shown that this is indeed possible, but I'll not show this here. We let ω be a left invariant density on $Sl(n,\mathbb{R})$ and denote by ω_∞ its deduced Haar measure. We define the measure of $Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R})$ to be $\omega_\infty(D)$ and we write $\omega_\infty(Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R}))$ to indicate the measure of $Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R})$. Since ω_∞ is a left invariant measure on $Sl(n,\mathbb{R})$ our definition is independent of the choice if D. For more details and references on this subject see for example [Rat].
The mapping $\phi : I \times D \to T$, defined by

$$\phi(t,y) = ty, \tag{4.11}$$

where $I = (0,1]$, is clearly a homeomorphism. We extend the mapping ϕ to a homomorphism of groups $\phi : \mathbb{R}^* \times Sl(n,\mathbb{R}) \to Gl(n,\mathbb{R})$, $(t,y) \mapsto ty$.

4.16 Lemma
The form

$$\beta := (\det(\cdot))^{-n} dx_{11} \wedge \cdots \wedge dx_{nn}$$

is a left invariant density on $Gl(n,\mathbb{R})$.

PROOF: Let $A \in Gl(n,\mathbb{R})$. For $x = (x_{1,1},\ldots,x_{n,n})$ we define

$$C = \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{bmatrix} := \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} \cdot \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix}.$$

and $c_{i,j}$ is given by

$$c_{i,j} = \sum_{k=1}^{n} A_{i,k} \cdot x_{k,j}. \tag{4.12}$$

We then have by (2.16) that

$$
\begin{aligned}
&(l_A^*\beta)_x \\
=&(\det l_A x)^{-n}\det(J_{i,j}(l_A))_x\, dx_{11}\wedge\cdots\wedge dx_{nn} \\
=&(\det l_A x)^{-n}\begin{vmatrix}
\frac{\partial C_{1,1}}{\partial x_{1,1}} & \cdots & \frac{\partial C_{1,n}}{\partial x_{1,1}} & \cdots & \cdots & \frac{\partial C_{n,1}}{\partial x_{1,1}} & \cdots & \frac{\partial C_{n,n}}{\partial x_{1,1}} \\
\vdots & \ddots & \vdots & & & \vdots & \ddots & \vdots \\
\frac{\partial C_{1,1}}{\partial x_{1,n}} & \cdots & \frac{\partial C_{1,n}}{\partial x_{1,n}} & \cdots & \cdots & \frac{\partial C_{n,1}}{\partial x_{1,n}} & \cdots & \frac{\partial C_{n,n}}{\partial x_{1,n}} \\
\vdots & & \vdots & \ddots & & \vdots & & \vdots \\
\vdots & & \vdots & & \ddots & \vdots & & \vdots \\
\frac{\partial C_{1,1}}{\partial x_{n,1}} & \cdots & \frac{\partial C_{1,n}}{\partial x_{n,1}} & \cdots & \cdots & \frac{\partial C_{n,1}}{\partial x_{n,1}} & \cdots & \frac{\partial C_{n,n}}{\partial x_{n,1}} \\
\vdots & \ddots & \vdots & & & \vdots & \ddots & \vdots \\
\frac{\partial C_{1,1}}{\partial x_{n,n}} & \cdots & \frac{\partial C_{1,n}}{\partial x_{n,n}} & \cdots & \cdots & \frac{\partial C_{n,1}}{\partial x_{n,n}} & \cdots & \frac{\partial C_{n,n}}{\partial x_{n,n}}
\end{vmatrix} dx_{11}\wedge\cdots\wedge dx_{nn} \\
=&(\det A)^{-n}(\det x)^{-n}\begin{vmatrix}
A & & & \mathbf{0} \\
& A & & \\
& & \ddots & \\
\mathbf{0} & & & A
\end{vmatrix} dx_{11}\wedge\cdots\wedge dx_{nn} \\
=&(\det A)^{-n}(\det x)^{-n}(\det A)^n dx_{11}\wedge\cdots\wedge dx_{nn} = (\det x)^{-n}dx_{11}\wedge\cdots\wedge dx_{nn} = \beta_x.
\end{aligned}
$$

□

If we let $n = 1$ in the above lemma we get the following corollary.

4.17 Corollary
$(\cdot)^{-1}dt$ is left invariant form on $\mathbb{R}^*$.

□

4.18 Lemma
If β on $\Omega(H)$ is a left invariant form and $\phi : G \to H$ is a homomorphism of Lie groups then $\phi^*(\beta)$ is a left invariant form on $\Omega(G)$.

PROOF: Since ϕ is a homomorphism it's easy to show that $\phi\circ l_g = l_{\phi(g)}\circ\phi$. Therefore for all $g\in G$ we have

$$
\begin{aligned}
l_g^*\phi^*(\beta) &= (\phi\circ l_g)^*\beta = (l_{\phi(g)}\circ\phi)^*\beta \\
&= \phi^* l_{\phi(g)}^*\beta = \phi^*(\beta),
\end{aligned}
$$

which proves the lemma.

□

4.19 Corollary
(1) $\phi^*((\det(\cdot))^{-n}dx_{11}\wedge\cdots\wedge dx_{nn}) = \alpha(\cdot)^{-1}dt\wedge\omega$, where $\alpha\in\mathbb{R}^*$.

(2) $\phi^*(dx_{11} \wedge \cdots \wedge dx_{nn}) = \alpha(\cdot)^{n^2-1} dt \wedge \omega$.

where $\alpha \in \mathbb{R}$.

PROOF:

(1) In previous lemmas we have shown that $\phi^*((\det(\cdot))^{-n} dx_{11} \wedge \cdots \wedge dx_{nn})$ and $(\cdot)^{-1} dt \wedge \omega$ are both left invariant forms on $R^* \times Sl(n, \mathbb{R})$, so the result follows from lemma 4.13.

(2) We have

$$
\begin{aligned}
&\phi^*((\det(\cdot))^{-n} dx_{1,1} \wedge \cdots \wedge dx_{n,n})(t, y)\\
=&\phi^*((\det(\cdot))^{-n})(t, y)\phi^*(dx_{1,1} \wedge \cdots \wedge dx_{n,n})\\
=&((\det(\cdot))^{-n} \circ \phi)(t, y)\phi^*(dx_{1,1} \wedge \cdots \wedge dx_{n,n})\\
=&(\det ty)^{-n}\phi^*(dx_{1,1} \wedge \cdots \wedge dx_{n,n})\\
=&(\cdot)^{-n^2}\phi^*(dx_{1,1} \wedge \cdots \wedge dx_{n,n}),
\end{aligned}
$$

and the results follows.

□

4.20 Remark

$T_E Gl(n, \mathbb{R})$ is naturally isomorphic to $\mathbb{R}^{n^2}$. Since $T_1\mathbb{R}^*$ is naturally isomorphic to $\mathbb{R}$ we have that $T_{(1,E)}(\mathbb{R}^* \times Sl(n, \mathbb{R}))$ is isomorphic to $\mathbb{R} \times \mathfrak{sl}(n, \mathbb{R})$ by remark 2.55. Thus via this isomorphism a vector in $T_{(1,E)}(\mathbb{R}^* \times Sl(n, \mathbb{R}))$ can be written on the form (s, A) where $s \in \mathbb{R}$ and $A \in \mathfrak{sl}(n, \mathbb{R})$.

4.21 Lemma

Let $s \in \mathbb{R}$ and $A \in \mathfrak{sl}(n, \mathbb{R})$. Then

(1) $\phi_*(s, \mathbf{0}) = sE$;

(2) $\phi_*(0, A) = A$.

PROOF:

(1) Let $\tau :]a, b[\to \mathbb{R}^* \times Sl(n, \mathbb{R})$ be defined by

$$\tau(t) = (1, E) + (st, \mathbf{0})$$

τ is thus a representation curve for the tangent vector $(s, \mathbf{0})$. So we have

$$
\begin{aligned}
\phi_*(s, \mathbf{0}) = \left.\frac{d}{dt}\right|_{t=0} \phi \circ \tau(t) &= \left.\frac{d}{dt}\right|_{t=0} ((1 + st)E)\\
&= sE \in T_{\phi(1,E)} Gl(n, \mathbb{R}).
\end{aligned}
$$

(2) Let $\tau :]a,b[\to \mathbb{R}^* \times Sl(n,\mathbb{R})$ be defined by

$$\tau(t) = (1,E) + (0,tA).$$

τ is thus a representation curve for the tangent vector $(0,A)$. Thus we get

$$\begin{aligned}\phi_*(0,A) &= \frac{d}{dt}\bigg|_{t=0} \phi \circ \tau(t) = \frac{d}{dt}\bigg|_{t=0} (E + tA) \\ &= A \in T_{\phi(1,E)} Gl(n,\mathbb{R}).\end{aligned}$$

□

In the next two lemmas we let A_n be the $(n \times n)$-matrix defined by

$$\mathrm{A}_n := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

4.22 Lemma
$|A_n| = (-1)^{n+1} n$

PROOF: I'll use induction. The postulate is obvious for $n = 1, 2$. Let's assume it's right for $n < p$. We'll expand A_p on the first column, i.e.

$$|A_p| = 1 \begin{vmatrix} -1 & & & \mathbf{0} \\ 1 & -1 & & \\ & \ddots & \ddots & \\ \mathbf{0} & & & -1 \end{vmatrix} - |A_{p-1}|.$$

If p is even the first term on the right hand side equals -1 and by our induction hypothesis we have that $|A_{p-1}| = p - 1$. Hence $|A_p| = -p$. On the other hand if p is odd the first term on the right hand side equals 1 and by our induction hypothesis we have that $|A_{p-1}| = -(p-1)$. Thus $|A_p| = p$. We have thus proved the lemma.

□

4.23 Lemma
$(dx_{11} \wedge \cdots \wedge dx_{nn})_E(E, E_{11} - E_{22}, \ldots) = \pm n$

PROOF: By making the suitable change of the permutation of $dx_{11} \wedge \cdots \wedge dx_{nn}$ we have that

$$(dx_{11} \wedge \cdots \wedge dx_{nn})_E(E, E_{11} - E_{22}, \ldots) = \pm \begin{vmatrix} \mathrm{A}_n & 0 \\ 0 & \mathrm{I}, \end{vmatrix}$$

and the results follows from the previous lemma.

□

4.24 Theorem

$$\phi^*(dx_{11}\wedge\cdots\wedge dx_{nn})_{(1,E)}(1, E_{11}-E_{22},\dots) = \pm n.$$

PROOF:

$$\begin{aligned} & \phi^*(dx_{11}\wedge\cdots\wedge dx_{nn})_{(1,E)}((1,\mathbf{0}),(0,E_{11}-E_{22}),\cdots) \\ = \; & (dx_{11}\wedge\cdots\wedge dx_{nn})_{\phi(1,E)}(\phi_*(1,\mathbf{0}),\phi_*(0,E_{11}-E_{22}),\cdots) \\ = \; & (dx_{11}\wedge\cdots\wedge dx_{nn})_E(E,E_{11}-E_{22},\cdots) && \text{(by lemma 4.21)} \\ = \; & \pm n && \text{(by lemma 4.23).} \end{aligned}$$

□

4.25 Remark
A natural selection of our invariant density ω is to require that at the identity its value on the zero trace basis is equal to ± 1. By previous theorem we deduce that $|\alpha| = n$ in corollary 4.19.

Thus

$$\begin{aligned} v(T) &= \int_T dx_{11}\wedge\cdots\wedge dx_{nn} = \int_D\int_0^1 \phi^*(dx_{11}\wedge\cdots\wedge dx_{nn}) \\ &= \int_D\int_0^1 nt^{n^2-1}dt\wedge\omega = \frac{1}{n}\omega_\infty(D). \end{aligned} \tag{4.13}$$

By combining (4.10) and the above equation we have

4.26 Theorem

$$\omega_\infty(Sl(n,\mathbb{Z})\backslash Sl(n,\mathbb{R})) = \zeta(2)\cdots\zeta(n) \tag{4.14}$$

□

4.3 Volume of $SO(n,\mathbb{R})$ and further study

$SO(n,\mathbb{R})$ is a Lie group of dimension $n(n-1)/2$. In [Fl, He] it is shown how a basis of linearly independent left invariant 1-forms $\omega_{i,j}$ where $1 \leqslant i < l \leqslant n$ can be constructed. The resulting basis is known as the ***Maurer-Cartan basis***. We define an invariant density on $SO(n,\mathbb{R})$ by

$$\alpha_n = \bigwedge_{i>j}\omega_{i,j}, \tag{4.15}$$

where we require that the induced measure on $SO(n,\mathbb{R})$,

$$\alpha_n(SO(n,\mathbb{R})) := \int_{SO(n,\mathbb{R})} \alpha_n,$$

is equal to the volume of $SO(n,\mathbb{R})$ with respect to the standard density. We look at $SO(n-1,\mathbb{R})$ as a subgroup of $SO(n,\mathbb{R})$ by

$$SO(n-1,\mathbb{R}) = \{A \in SO(n,\mathbb{R}) |\, a_{1,1} = 1,\ a_{i,1} = a_{1,i} = 0 \text{ for all } i = 2,\dots,n\}. \tag{4.16}$$

Let's write $\alpha_n = u_{n-1} \wedge \tilde{\alpha}_{n-1}$ where $u_{n-1} = \bigwedge_{i>1} \omega_{i,1}$ and $\tilde{\alpha}_{n-1} = \bigwedge_{i>j>1} \omega_{i,j}$. Then it can be shown that

$$\alpha_n(SO(n,\mathbb{R})) = \alpha_{n-1}(SO(n-1,\mathbb{R})) \cdot u(S^{n-1}) \tag{4.17}$$

where u is the volume form on S^{n-1}. From this we can calculate $\alpha_n(SO(n,\mathbb{R}))$ for all n, and I'll do that in the following theorem.

4.27 Theorem
For $n \in \mathbb{N}$ we have

$$\alpha_n(SO(n,\mathbb{R})) = 2^{n-1}\pi^{n(n+1)/4} \prod_{a=1}^{n} \Gamma\left(\frac{a}{2}\right)^{-1}. \tag{4.18}$$

PROOF: We'll use induction. From chapter 1 we have that

$$u(S^m) = \frac{2\pi^{(m+1)/2}}{\Gamma\left(\frac{m+1}{2}\right)}.$$

(4.18) is obvious for $n = 1$. Let's assume it's true for $n = p-1$, where $p \geqslant 2$. By (4.17) and our induction hypothesis we have

$$\begin{aligned}
\alpha_p(SO(p,\mathbb{R})) &= \alpha_{p-1}(SO(p-1,\mathbb{R})) \cdot u_{p-1}(S^{p-1}) \\
&= 2^{p-2}\pi^{(p-1)p/4} \prod_{a=1}^{p-1} \Gamma\left(\frac{a}{2}\right)^{-1} \cdot \frac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)} \\
&= 2^{p-1}\pi^{(p^2-p)/4+2p/4} \prod_{a=1}^{p} \Gamma\left(\frac{a}{2}\right)^{-1} \\
&= 2^{p-1}\pi^{p(p+1)/4} \prod_{a=1}^{p} \Gamma\left(\frac{a}{2}\right)^{-1}
\end{aligned}$$

so (4.18) is also true for $n = p$ and thus (4.18) is true for all n so the proof is complete.

□

In [Gilm] the volume of many classical Lie groups and quotients of Lie groups is calculated. We have seen that for the *fibre bundle*

$$SO(n-1) \to SO(n) \to S^{n-1}$$

we got a very nice relation between the volume of these spaces. One might ask if this was true in general. Let's look at the following question:

Question:

Let's assume that F and B are compact Riemannian manifolds with invariant densities μ_F and μ_B respectively. If E is a manifold such that $F \to E \to B$ is a fibre bundle, can we find a Riemannian metric on E such that

$$\mu_E(E) = \mu_F(F) \cdot \mu_B(B)$$

where $\mu_E(E)$ is the volume of E with respect to some invariant density μ_E ?

Part of answer:

We saw that this was true, when we calculated the volume of $SO(n, \mathbb{R})$. According to [Mac] the answer to the above question is "yes" if F, E and B are compact Lie groups. MacDonald's proof is above my understanding and so is a simpler proof given by [Ha]. Those who are familiar with characteristic classes and algebraic topology could however find the proofs interesting. For some more discussion of this see: `http://www.lehigh.edu/~dmd1/p2000.html`. Many classical fibre bundles are given in [Br]. [Hus] is also a great book about fibre bundles. I think that studying this further could be very interesting, but it's beyond the scope of this thesis.

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