Toric morphisms between p-compact groups

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Abstract. It is well-known that any morphism between two p-compact groups will lift, non-uniquely, to an admissible morphism between the maximal tori. We identify here a class of p-compact group morphisms, the p-toric morphisms, which can be perceived as generalized rational isomorphisms, enjoying the stronger property of lifting uniquely to a morphism between the maximal torus normalizers. We investigate the class of p-toric morphisms and apply our observations to determine the mapping spaces $\mathrm{map}(\mathrm{BSU}(3), \mathrm{BF}_4)$, $\mathrm{map}(\mathrm{BG}_2, \mathrm{BF}_4)$, and $\mathrm{map}(\mathrm{BSU}(3), \mathrm{BG}_2)$ where the classifying spaces have been completed at the prime $p = 3$.

1. Introduction

The classification up to homotopy of maps between classifying spaces of compact Lie groups is a traditional project of algebraic topology [18, 26]. One line of development started with the investigations 25 years ago by Hubbuck [15, 16] and Adams-Mahmud [1]. They noted the close relationship between maps between classifying spaces and admissible homomorphisms between maximal tori. The regular admissible homomorphisms, in particular, turned out to have especially nice properties. It is the purpose of this paper to study regular admissible morphisms, here called toric admissible morphisms, in light of the more recent theory by Dwyer-Wilkerson [9] of p-compact groups. As case studies, we classify homotopy homomorphisms $\mathrm{SU}(3) \to \mathrm{F}_4$, $\mathrm{G}_2 \to \mathrm{F}_4$, and $\mathrm{SU}(3) \to \mathrm{G}_2$ at the prime $p = 3$.

In order to describe the content in more detail, let X_1 and X_2 be p-compact groups, for the sake of this introduction assumed to be connected, with maximal tori $T(X_1) \to X_1$ and $T(X_2) \to X_2$, respectively. For any morphism $f\colon X_1 \to X_2$ there is a lift $T(f)\colon T(X_1) \to T(X_2)$, unique up the action of the Weyl group of

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X_2, such that the diagram

$$\begin{tikzcd} T(X_1) \arrow[r, "T(f)"] \arrow[d, "i_1"'] & T(X_2) \arrow[d, "i_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

commutes up to conjugacy. As a consequence of uniqueness, the morphism $T(f)$ is *admissible* in the sense that for any element w_1 of the Weyl group of X_1 there exists and element w_2 of the Weyl group of X_2 such that $T(f)w_1 = w_2T(f)$. In general, w_2 is not uniquely determined by w_1, but if it is, we say that f is p-toric (2.1). (As we shall see (2.4), f is p-toric, if and only if the centralizer $C_{X_2}(fi_1T(X_1))$ of the maximal torus of X_1 in X_2 is a maximal torus of X_2. This explains the name.) In that case, the correspondence $w_1 \to w_2$ is a homomorphism of Weyl groups and, by Theorem 3.5, there is a *unique* lift $N(f)\colon N(X_1) \to N(X_2)$ to a map between the maximal torus normalizers such that the diagram

$$\begin{tikzcd} N(X_1) \arrow[r, "N(f)"] \arrow[d] & N(X_2) \arrow[d] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

commutes up to conjugacy, i.e. a p-toric morphism lifts uniquely to a morphism between the maximal torus normalizers.

In many concrete cases the generic morphism is p-toric. As a first example, we consider the case where the domain $X_1 = \mathrm{SU}(3)$, the codomain $X_2 = \mathrm{F}_4$, and the prime $p = 3$. The compact Lie group F_4 contains a unique copy of $\mathrm{SU}(3,3) = \mathrm{SU}(3) \times_{Z(\mathrm{SU}(3))} \mathrm{SU}(3)$ as a subgroup of maximal rank (4.10). Any morphism $\mathrm{SU}(3) \to \mathrm{SU}(3,3)$ is of the form

$$\psi^{(u,v)}\colon \mathrm{SU}(3) \xrightarrow{\Delta} \mathrm{SU}(3) \times \mathrm{SU}(3) \xrightarrow{\psi^u \times \psi^v} \mathrm{SU}(3) \times \mathrm{SU}(3) \to \mathrm{SU}(3,3)$$

where u and v are 3-adic units or zero (2.17). Composing with the inclusion $e\colon \mathrm{SU}(3,3) \to \mathrm{F}_4$ we obtain the morphism $e\psi^{(u,v)}\colon \mathrm{SU}(3) \to \mathrm{F}_4$. Observe that $e\psi^{(u,v)} = e\psi^{(-u,-v)}$ since the inclusion e is invariant under the action of the Weyl group $W_{\mathrm{F}_4}(\mathrm{SU}(3,3))$ [11, 4.3] [24, 8.4] which is of order two generated by the self-map $\psi^{-1} \times_{Z(\mathrm{SU}(3))} \psi^{-1}$ of $\mathrm{SU}(3,3)$ (4.15). These maps $e\psi^{(u,v)}$, $u, v \in \mathbf{Z}_3^* \cup \{0\}$, with the relation $e\psi^{(u,v)} = e\psi^{(-u,-v)}$, turn out to describe $\mathrm{Rep}(\mathrm{SU}(3), \mathrm{F}_4) = [B\mathrm{SU}(3)^\wedge_3, (B\mathrm{F}_4)^\wedge_3]$ completely.

Theorem 1.1. *The map*

$$e \circ -\colon W_{\mathrm{F}_4}(\mathrm{SU}(3,3)) \backslash \mathrm{Rep}(\mathrm{SU}(3), \mathrm{SU}(3,3)) \to \mathrm{Rep}(\mathrm{SU}(3), \mathrm{F}_4)$$

is a bijection when $p = 3$.

See (4.16, 5.7, 6.7) for information about the centralizers [9, 3.5] of these maps. The proof of Theorem 1.1 is divided into three cases: Monomorphisms $\mathrm{SU}(3) \to \mathrm{F}_4$ (4.13), p-toric monomorphisms $\mathrm{PU}(3) \to \mathrm{F}_4$ (5.4), and, the technically most demanding case, non-p-toric monomorphisms $\mathrm{PU}(3) \to \mathrm{F}_4$ (6.1).

As a second example, we consider the case where $X_1 = \mathrm{G}_2$ and $X_2 = \mathrm{F}_4$ and $p = 3$ and reprove a result from Jackowski-McClure-Oliver [19]. To state the theorem, we recall that the compact Lie group G_2 contains a unique copy of $\mathrm{SU}(3)$ as a subgroup of maximal rank (8.5). Thus we may restrict morphisms defined on G_2 to this subgroup $\mathrm{SU}(3) \subset \mathrm{G}_2$.

Theorem 1.2. [19, 3.4] *The restriction map*

$$\mathrm{Rep}(\mathrm{G}_2, \mathrm{F}_4) \to \mathrm{Rep}(\mathrm{SU}(3), \mathrm{F}_4)$$

is a bijection when $p = 3$.

See (7.2) for information about the centralizers of the homotopy morphisms from G_2 to F_4 at the prime $p = 3$.

When working with this paper, I made use of a MAGMA program written by K. Andersen for computing admissible homomorphisms. I also wish to thank C. McGibbon for a clarifying remark.

2. Toric morphisms

In this section I introduce the concept of a p-toric morphism, relate it to other, more familiar, types of morphisms between p-compact groups, and provide examples of morphisms that are p-toric and others that are not.

Let X_1 and X_2 be p-compact groups (or extended p-compact tori [10, 3.12]) with maximal tori $T_1 = T(X_1) \to X_1$, $T_2 = T(X_2) \to X_2$ and Weyl groups $W_1 = W(X_1)$ and $W_2 = W(X_2)$ [9], respectively. Write $\mathrm{Rep}(X_1, X_2)$ for the set $[BX_1, BX_2]$ of conjugacy classes of loop space morphisms [9, §3].

Definition 2.1.
1. *A loop space morphism* $T_1 \to X_2$ *is* p-toric *(or* regular [1, 2.22], [19, 1.3]*) if its centralizer* $C_{X_2}(T_1)$ *is a p-compact toral group* [9, 6.3].
2. *A loop space morphism* $X_1 \to X_2$ *is p-toric if its composition with* $T_1 \to X_1$ *is p-toric.*

Note that the centralizer $C_{X_2}(T_1)$ in (2.1.1) is known to be a p-compact group [9, §6] [10, 2.5].

We shall now consider some alternative criteria for a morphism to be p-toric. For any loop space morphism $f\colon X_1 \to X_2$ between p-compact groups or extended p-compact tori there exists [9, 8.11] [10, 2.14] a loop space morphism

$T(f)\colon T_1 \to T_2$ between the maximal tori such that

$$\begin{tikzcd} T_1 \arrow[r, "T(f)"] \arrow[d, "i_1"'] & T_2 \arrow[d, "i_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd} \tag{2.2}$$

commutes. Moreover, the conjugacy class of $T(f)$ in $\mathrm{Rep}(T_1, T_2)$ is unique up to the action by the Weyl group W_2 of the target [22, 3.5]. The Adams-Mahmud map

$$\mathrm{Rep}(X_1, X_2) \to W_2 \backslash \mathrm{Rep}(T_1, T_2),$$

taking $f \in \mathrm{Rep}(X_1, X_2)$ to the W_2-orbit of $T(f) \in \mathrm{Rep}(T_1, T_2)$, is instrumental in the proofs of 1.1 and 1.2. Note that, by uniqueness of $T(f)$, the image of the Adams-Mahmud map is contained in $W_2 \backslash \mathrm{Adm}(T_1, T_2)$ where

$$\mathrm{Adm}(T_1, T_2) = \{\varphi \in \mathrm{Rep}(T_1, T_2) \mid \varphi W_1 \subseteq W_2 \varphi\} \tag{2.3}$$

is the set of *admissible* homomorphisms. For each element w_1 of the Weyl group W_1 of the domain there are in general several solutions for $w_2 \in W_2$ in the equation $T(f)w_1 = w_2 T(f)$. As we shall shortly see (2.4), the p-toric morphisms are characterized (for connected X_2) as the ones for which w_2 is uniquely determined by w_1.

Let

$$W_2^{T(f)} = \{w_2 \in W_2 \mid w_2 \cdot T(f) = T(f)\}$$

denote the stabilizer subgroup at $T(f)$ for the action of W_2 on $\mathrm{Rep}(T_1, T_2)$. The conjugacy class of this subgroup does not depend of the choice of $T(f)$ but only on f.

In case X_1 and X_2 are extended p-compact tori, there is a short exact sequence of loop spaces

$$T_2 \to C_{X_2}(T(f)T_1) \to W_2^{T(f)}$$

from which we see that

$$f\colon X_1 \to X_2 \text{ is } p\text{-toric} \Leftrightarrow W_2^{T(f)} = \pi_0(C_{X_2}(T_1)) \text{ is a finite } p\text{-group.}$$

In case X_1 and X_2 are p-compact groups, $C_{X_2}(T_1) \to X_2$ is a monomorphism of maximal rank [10, §4], so

$$f\colon X_1 \to X_2 \text{ is } p\text{-toric} \Leftrightarrow C_{X_2}(T_1)_0 \to X_2 \text{ is a maximal torus for } X_2$$

where subscript 0 indicates identity component. If X_2 is assumed to be *connected*, a stronger statement is possible.

Proposition 2.4. *Assume that X_2 is a connected p-compact group. The following are equivalent*

1. *f is p-toric.*
2. *$C_{X_2}(T_1) \to X_2$ is a maximal torus for X_2.*

3. $W_2^{T(f)}$ *is trivial.*

for any p-compact group morphism $f\colon X_1 \to X_2$.

Proof. For general reasons, the centralizer $C_{X_2}(T_1)$ is a connected [21, 3.11] [10, 7.8] p-compact group [10, 2.5] and the evaluation morphism $C_{X_2}(T_1) \to X_2$ a monomorphism of maximal rank [10, 4.3]. Also, any p-compact group with trivial Weyl group is [9, 9.7] [21, 3.7, 3.8] a p-compact torus. These general facts, in combination with [9, 8.11] [21, 3.6], easily imply the proposition. $\square$

Consequently, for any p-toric morphism $f\colon X_1 \to X_2$ with connected target, there is for each element w of the Weyl group of the domain a *unique* element $\chi(f)(w)$ of the Weyl group of the target so that $T(f)w = \chi(f)(w)T(f) \in \mathrm{Rep}(T_1, T_2)$, and $\chi(f)\colon W_1 \to W_2$ is a group homomorphism.

In general, for a possible non-connected target X_2, we consider an enlarged version of diagram (2.2) in the form of the diagram

$$\begin{tikzcd} T_1 \arrow[r, "T(f)"] \arrow[dd, "i_1"'] & T_2 \arrow[d, "i_2'"] \\ & N_2 \arrow[d, "j_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd} \tag{2.5}$$

where $j_2\colon N_2 \to X_2$ is the normalizer [9, 9.8] of the maximal torus. Using that $C_{N_2}(T_1) \to C_{X_2}(T_1)$ is a maximal torus normalizer [22, 3.4.3], we get

$$f \text{ is } p\text{-toric} \Leftrightarrow T_1 \xrightarrow{i_1} X_1 \xrightarrow{f} X_2 \text{ is } p\text{-toric} \tag{2.6}$$
$$\Leftrightarrow C_{N_2}(T_1) \to C_{X_2}(T_1) \text{ is an isomorphism} \tag{2.7}$$
$$\Rightarrow T_1 \xrightarrow{T(f)} T_2 \xrightarrow{i_2'} N_2 \text{ is } p\text{-toric} \tag{2.8}$$
$$\Leftrightarrow W_2^{T(f)} \text{ is a finite } p\text{-group.} \tag{2.9}$$

When $p > 2$, also the converse of the third implication holds because, for odd p, a p-compact group is a p-compact toral group if and only if its Weyl group is a finite p-group [23, 7.9].

In some cases, see e.g. [22, 5.1] or (3.5) below, it is possible to lift f to a loop space morphism $N(f)$ between the maximal torus normalizers such that

$$\begin{tikzcd} N_1 \arrow[r, "N(f)"] \arrow[d, "j_1"'] & N_2 \arrow[d, "j_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd} \tag{2.10}$$

commutes up to conjugacy. In this situation

$$f \text{ is } p\text{-toric} \Rightarrow N(f) \text{ is } p\text{-toric} \tag{2.11}$$

and for $p > 2$ also the converse holds. (Use (2.7, 2.8) to see this.)

In the following examples and elsewhere

- $\mathrm{TRep}(X_1, X_2) \subset \mathrm{Rep}(X_1, X_2)$ denotes the set of conjugacy classes of p-toric morphisms
- $\mathrm{Mono}(X_1, X_2) \subset \mathrm{Rep}(X_1, X_2)$ denotes the set of conjugacy classes of monomorphisms
- $\mathrm{TMono}(X_1, X_2) = \mathrm{Mono}(X_1, X_2) \cap \mathrm{TRep}(X_1, X_2)$
- $\varepsilon_{\mathbf{Q}}(X_1, X_2) \subset \mathrm{Rep}(X_1, X_2)$ is the set of rational isomorphisms [22, 2.1]
- $\varepsilon_{\mathbf{Q}}(X_1) = \varepsilon_{\mathbf{Q}}(X_1, X_1)$ is the monoid of rational automorphisms of X_1
- $\mathrm{Out}(X_1)$ is the group of conjugacy classes of automorphisms of X_1 (the invertible elements of the monoid $\mathrm{Rep}(X_1, X_1)$).

Above, a loop space morphism between extended p-compact tori is a monomorphism if its discrete approximation [10, 3.12] is a group monomorphism.

Example 2.12. *If X_1 and X_2 have the same rank* [9, 5.11]*,*

$$\mathrm{Mono}(X_1, X_2) \subset \mathrm{TRep}(X_1, X_2) \supset \varepsilon_{\mathbf{Q}}(X_1, X_2)$$

because any monomorphism [9, 3.2] *(rational isomorphism* [22, 2.1]*) restricts to an isomorphism (epimorphism) between maximal tori* [21, 3.6] [22, 3.6].

If X_1 and X_2 are locally isomorphic, simple p-compact groups [22, 2.7, 5.4]

$$\mathrm{TRep}(X_1, X_2) = \mathrm{Rep}(X_1, X_2) - \{0\} = \varepsilon_{\mathbf{Q}}(X_1, X_2)$$

because f is p-toric or a rational isomorphism if and only if $T(f)$ is non-trivial if and only if f is non-trivial [22, 6.7].

Example 2.13. *For any p-compact group X and any integer $m > 0$, $\mathrm{TRep}(X, X^m) = (\mathrm{TRep}(X, X))^m$. If X is simple,*

$$\mathrm{TRep}(X, X^m) = (\mathrm{Rep}(X, X) - \{0\})^m = \varepsilon_{\mathbf{Q}}(X)^m \overset{p \mid |W|}{=} \mathrm{Out}(X)^m,$$

where the last identity holds under the assumption that p divides the order of the Weyl group [22, 5.5, 5.6].

Proposition 2.14. *Assume that X_1 is connected and that $z\colon Z_1 \to X_1$ is a central monomorphism* [9, 3.5]*. Then there are bijections*

- $\mathrm{Rep}(X_1/Z_1, X_2) \to \{f \in \mathrm{Rep}(X_1, X_2) \mid f \circ z \textit{ is trivial}\}$
- $\mathrm{TRep}(X_1/Z_1, X_2) \to \{f \in \mathrm{TRep}(X_1, X_2) \mid f \circ z \textit{ is trivial}\}$

induced by the epimorphism $X_1 \to X_1/Z_1$ [9, 3.2, 8.3]*. In fact,* $\mathrm{map}(B(X_1/Z_1), BX_2)$ *is homotopy equivalent to a union of connected components of* $\mathrm{map}(BX_1, BX_2)$.

Proof. The epimorphism of X_1 to X_1/Z_1 induces a homotopy equivalence between $\mathrm{map}(B(X_1/Z_1), BX_2)$ and a collection of components of $\mathrm{map}(BX_1, BX_2)$ [22, 2.10]. This shows the injection of sets of representations, and, when applied with X_1 replaced by T_1, it shows that a morphism $X_1 \to X_2$ is p-toric if and only if its composition with the epimorphism $X_1 \to X_1/Z_1$ is p-toric. $\square$

Proposition 2.15. *Assume that X_1 is simply connected, X_2 is connected, and that $z\colon Z_2 \to X_2$ is a central monomorphism. Then there are bijections*

- $\mathrm{Rep}(X_1, X_2) \to \mathrm{Rep}(X_1, X_2/Z_2)$
- $\mathrm{TRep}(X_1, X_2) \to \mathrm{TRep}(X_1, X_2/Z_2)$

induced by the epimorphism $X_2 \to X_2/Z_2$.

Proof. Obstruction theory (remember that BX_1 is 3-connected [6]) shows that $\mathrm{Rep}(X_1, X_2) = \mathrm{Rep}(X_1, X_2/Z_2)$ and the existence of a short exact sequence of p-compact groups [9, 3.2]

$$K \to C_{X_2}(X_1) \to C_{X_2/Z_2}(X_1)$$

where BK is one component of the homotopy fixed point set $BZ_2^{hX_1}$; in particular K is a p-compact toral group. It follows that $C_{X_2}(X_1)$ is a p-compact toral group if and only if $C_{X_2/Z_2}(X_1)$ is. $\square$

Example 2.16. *For any simply connected, simple p-compact group X and any central monomorphism $Z \to X^m$,*

$$\mathrm{TRep}(X, X^m/Z) = \mathrm{TRep}(X, X^m) = \varepsilon_{\mathbf{Q}}(X)^m \overset{p||W|}{=} \mathrm{Out}(X)^m$$

where the last identity holds if p divides the order of the Weyl group [22, 5.5, 5.6].

Example 2.17. *Let p be an odd prime and let $\mathrm{SU}(p,p)$ denote the quotient of $\mathrm{SU}(p) \times \mathrm{SU}(p)$ be the central subgroup generated by $(\zeta E, \zeta^{-1}E)$ where $\zeta \neq 1$ is a pth root of unity. Then (*2.15*)*

$$\mathrm{Rep}(\mathrm{SU}(p), \mathrm{SU}(p,p)) = \mathrm{Rep}(\mathrm{SU}(p), \mathrm{SU}(p)) \times \mathrm{Rep}(\mathrm{SU}(p), \mathrm{SU}(p))$$
$$\mathrm{TRep}(\mathrm{SU}(p), \mathrm{SU}(p,p)) = \mathrm{Out}(\mathrm{SU}(p)) \times \mathrm{Out}(\mathrm{SU}(p))$$

where [20, 2.5, 3.5] [24, 4.8] $\mathrm{Rep}(\mathrm{SU}(p), \mathrm{SU}(p)) - \{0\} = \mathrm{Out}(\mathrm{SU}(p)) = \mathbf{Z}_p^*$*, the group of units in the ring of p-adic integers. Relative to this identification*

$$\mathrm{Mono}(\mathrm{SU}(p), \mathrm{SU}(p,p)) = \{(u,v) \in (\mathbf{Z}_p^* \cup \{0\})^2 \mid u+v \in \mathbf{Z}_p^*\} \tag{2.18}$$

for [24, 5.2] *the morphism $\psi^{(u,v)}$ defined as the composition*

$$\mathrm{SU}(p) \xrightarrow{\Delta} \mathrm{SU}(p) \times \mathrm{SU}(p) \xrightarrow{\psi^u \times \psi^v} \mathrm{SU}(p) \times \mathrm{SU}(p) \to \mathrm{SU}(p,p)$$

is a monomorphism if and only if $u+v \in \mathbf{Z}_p^*$. The monoid $\mathrm{Rep}(\mathrm{SU}(p,p), \mathrm{SU}(p,p))$ *is (*2.14, 2.15*) isomorphic to a submonoid of* $\mathrm{Rep}(\mathrm{SU}(p) \times \mathrm{SU}(p), \mathrm{SU}(p) \times \mathrm{SU}(p))$ *and, in particular,*

$$\mathrm{Out}(\mathrm{SU}(p,p)) = \{(u,v) \in \mathbf{Z}_p^* \times \mathbf{Z}_p^* \mid u \equiv v \bmod p\} \rtimes \langle \tau \rangle$$

where τ is the automorphism that swaps the two $\mathrm{SU}(p)$*-factors.*

*The set of monomorphisms (*2.18*) consists of two orbits, represented by* $\psi^{(1,1)}$ *and* $\psi^{(1,0)}$*, under the action of the automorphism group* $\mathrm{Out}(\mathrm{SU}(p,p))$*. It follows that the centralizers of the monomorphisms* $\psi^{(u,v)}$ *are*

$$C_{\mathrm{SU}(p,p)}(\psi^{(u,v)}\mathrm{SU}(p)) \cong \begin{cases} Z(\mathrm{SU}(p)) & \text{if } u \neq 0 \text{ and } v \neq 0 \\ \mathrm{SU}(p) & \text{if } u = 0 \text{ or } v = 0 \end{cases} \tag{2.19}$$

i.e. that $\psi^{(u,v)}$ *is centric* [7] *precisely when it is p-toric. (To prove that* $\psi^{(1,1)}$ *is centric one uses the fact that* $Z(\mathrm{SU}(p)) \xrightarrow{\Delta} Z(\mathrm{SU}(p) \times \mathrm{SU}(p)) \to Z(\mathrm{SU}(p,p))$ *is an isomorphism of centers.) In the non-toric case, observe that the projection morphism* $\mathrm{SU}(p) \times \mathrm{SU}(p) \to \mathrm{SU}(p,p)$ *restricts to* $\psi^{(1,0)}$ *on the first factor and to* $\psi^{(0,1)}$ *on the second factor. This gives a factorization*

$$\mathrm{SU}(p) \to C_{\mathrm{SU}(p,p)}(\psi^{(1,0)}\mathrm{SU}(p)) \to \mathrm{SU}(p,p)$$

of $\psi^{(0,1)}$ *through the centralizer of* $\psi^{(1,0)}$ *where the first map is an isomorphism. We conclude that if* $f\colon \mathrm{SU}(p) \to \mathrm{SU}(p,p)$ *is a non-toric monomorphism, so is the evaluation monomorphism* $\mathrm{SU}(p) = C_{\mathrm{SU}(p,p)}(f\mathrm{SU}(p)) \to \mathrm{SU}(p,p)$*. The Weyl group,* $W_{\mathrm{SU}(p,p)}(\psi^{(u,v)}\mathrm{SU}(p))$*, of any monomorphism* $\psi^{(u,v)}$ *is trivial* [24, 8.5]*.*

*Finally, we note that by (*2.14*),*

$$\mathrm{Rep}(\mathrm{PU}(p), \mathrm{SU}(p,p)) = \{(u,v) \in (\mathbf{Z}_p^* \cup \{0\})^2 \mid u+v \in p\mathbf{Z}_p\}$$
$$\mathrm{TRep}(\mathrm{PU}(p), \mathrm{SU}(p,p)) = \{(u,v) \in (\mathbf{Z}_p^*)^2 \mid u+v \in p\mathbf{Z}_p\}$$

so that $\mathrm{Rep}(\mathrm{PU}(p), \mathrm{SU}(p,p)) = \{0\} \cup \mathrm{Mono}(\mathrm{PU}(p), \mathrm{SU}(p,p))$ *and* $\mathrm{Mono}(\mathrm{PU}(p), \mathrm{SU}(p,p)) = \mathrm{TRep}(\mathrm{PU}(p), \mathrm{SU}(p,p))$*.*

Lemma 2.20. *Let* $f\colon X \to Y_1$ *be any morphism and* $g\colon Y_1 \to Y_2$ *a monomorphism between p-compact groups. Then*

$$g \circ f\colon X \to Y_2 \text{ is } p\text{-toric} \Rightarrow f\colon X \to Y_1 \text{ is } p\text{-toric.}$$

Proof. Let T be a maximal torus of X_1. Since composition with Bg, $C_{Y_1}(fiT) \to C_{Y_2}(gfiT)$, is a monomorphism, $C_{Y_2}(gfiT)$ is a p-compact toral group if $C_{Y_1}(fiT)$ is a p-compact toral group [21, 3.5.(1)]. □

The converse of (2.20) is not true in general; take for instance Y_1 to be the maximal torus of Y_2.

3. Lifting p-toric morphisms

In this section I show that all p-toric morphisms between two p-compact groups lift uniquely to p-toric morphisms between the maximal torus normalizers.

Recall that X_1 and X_2 are p-compact groups or extended p-compact tori and that $j_1\colon N_1 \to X_1$ and $j_2\colon N_2 \to X_2$ are normalizers of the respective maximal tori, $i_1\colon T_1 \to X_1$ and $i_2\colon T_2 \to X_2$.

By the very definition of a p-toric morphism, the maps j_1 and j_2 induce maps

$$\mathrm{TRep}(X_1, X_2) \to \mathrm{TRep}(N_1, X_2) \leftarrow \mathrm{TRep}(N_1, N_2) \tag{3.1}$$

of sets of p-toric representations. Our first objective is to prove that the arrow to the right is a bijection. This will enable us to define a map from $\mathrm{TRep}(X_1, X_2)$ to $\mathrm{TRep}(N_1, N_2)$. Note the favorable input provided by the information [22, 3.2] that

$$\mathrm{TRep}(T_1, X_2) \leftarrow \mathrm{TRep}(T_1, N_2) \tag{3.2}$$

is a bijection and

$$C_{X_2}(T_1) \leftarrow C_{N_2}(T_1) \tag{3.3}$$

an isomorphism for any p-toric morphism $T_1 \to N_2$.

For any set $S \subset \mathrm{Rep}(X_1, X_2)$, write $\mathrm{map}(BX_1, BX_2)_S$ for the space of all maps $BX_1 \to BX_2$ homotopic to a member of S.

Lemma 3.4. *The map, induced by j_2,*

$$\mathrm{map}(BN_1, BX_2)_{\mathrm{TRep}(N_1,X_2)} \leftarrow \mathrm{map}(BN_1, BN_2)_{\mathrm{TRep}(N_1,N_2)}$$

is a homotopy equivalence.

Proof. The map of the lemma is the map on homotopy fixed point spaces

$$\mathrm{map}(BN_1, BY_2)_{\mathrm{TRep}(N_1,Y_2)} = \left(\mathrm{map}(BT_1, BY_2)_{\mathrm{TRep}(T_1,Y_2)}\right)^{hW_1}, \quad Y_2 = N_2, X_2,$$

induced by the map

$$\mathrm{map}(BT_1, BX_2)_{\mathrm{TRep}(T_1,X_2)} \leftarrow \mathrm{map}(BT_1, BN_2)_{\mathrm{TRep}(T_1,N_2)}$$

which is known to be a homotopy equivalence (3.2, 3.3). □

This lemma immediately leads to the main result of this section.

Theorem 3.5. *(Cf.* [1, 2.22]*) Let X_1 and X_2 be p-compact groups and $f\colon X_1 \to X_2$ a p-toric morphism. Then there exists a morphism $N(f)\colon N_1 \to N_2$ between extended p-compact tori such that*

$$\begin{tikzcd} N_1 \arrow[r, "N(f)"] \arrow[d, "j_1"'] & N_2 \arrow[d, "j_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

commutes up to conjugacy. Moreover,

- *$N(f)$ is unique up to conjugacy*
- *$N(f)$ is p-toric*
- *$C_{X_2}(fj_1N_1) \leftarrow C_{N_2}(N(f)N_1)$ is an isomorphism of loop spaces*

Proof. The map

$$N\colon \operatorname{TRep}(X_1,X_2) \to \operatorname{TRep}(N_1,N_2) \tag{3.6}$$

is defined as the composition of the map $\operatorname{TRep}(X_1,X_2) \to \operatorname{TRep}(X_1,N_2)$ with the inverse of the bijection $\operatorname{TRep}(N_1,X_2) \leftarrow \operatorname{TRep}(N_1,N_2)$ from (3.1). That $N(f)$ is p-toric is (2.11) and the isomorphism of centralizers is (3.4). □

Example 3.7. *If X is simple and $N \to X$ the normalizer of the maximal torus, the map $\operatorname{TRep}(X,X^m) \to \operatorname{TRep}(N,N^m)$ is injective if $\varepsilon_{\mathbf{Q}}(X) \to \operatorname{Rep}(N,N)$ is injective (*2.13*); e.g. if $X = \operatorname{PU}(p)$, $X = G_2$ and $p=3$, or $X = \operatorname{DI}_2$ and $p=3$* [24].

The above theorem is intended as a tool to facilitate the computation of $\operatorname{TRep}(X_1,X_2)$ in concrete cases. We now address injectivity of (3.6).

Remark 3.8. *According to the homology decomposition theorem of Jackowski-McClure* [17] *and Dwyer-Wilkerson* [8]*, the exists an $\mathbf{F}_p$-equivalence*

$$\operatorname{hocolim}_{\mathbf{A}^{\mathrm{op}}} BC_{X_1}(\nu) \to BX_1$$

where the homotopy colimit is taken over some full subcategory $\mathbf{A}$ of the Quillen category $\mathbf{A}(X_1)$. Let us assume that

- *Any object $\nu\colon V \to X_1$ of $\mathbf{A}$ admits a factorization $\mu\colon V \to T_1$ through the maximal torus and*
- *$N\colon \operatorname{TRep}(C_{X_1}(\nu),X_2) \to \operatorname{TRep}(C_{N_1}(\mu),N_2)$ is injective for all objects $\nu\colon V \to X_1$ of $\mathbf{A}$*

and let now f and f' be two p-toric morphisms with $N(f) = \varphi = N(f')$ for some $\varphi \in \operatorname{TRep}(C_{N_1}(\mu),N_2)$. Then the two possible compositions

$$\begin{tikzcd} C_{X_1}(\nu) \arrow[r, "e(\nu)"] & X_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & X_2 \end{tikzcd}$$

are again p-toric morphisms for $C_{X_2}(fe(\nu)C_{T_1}(\mu)) = C_{X_2}(fi_1T_1)$ is a p-compact torus and similarly for the other morphism f'. Since also,

$$N(f\circ e(\nu)) = \varphi\circ e(\mu) = N(f'\circ e(\nu))$$

we have $f\circ e(\nu) \simeq f'\circ e(\nu)$ for all objects ν of $\mathbf{A}$ by hypothesis. (Here, $e(\nu)\colon C_X(\nu) \to X$ stands for the evaluation monomorphism.) The obstructions to constructing a homotopy between Bf and Bf' lie in

$$\lim\nolimits^i_{\mathbf{A}} \pi_i(\operatorname{map}(BC_{X_1}(\nu),BX_2)_{B(f\circ e(\nu))}), \quad i \geq 1$$

which is an abelian group for $i>1$ but just a set if $i=1$ and the fundamental groups are non-abelian.

It is possible that (3.8) can be generalized to a more general situation using the preferred lifts of [25].

While (3.8) applies to the case where X_1 is center-free, the following lemma can be helpful if X_1 has a non-trivial center [10] [21].

Consider the following situation

$$\begin{tikzcd} Z \arrow[d, "z"'] \arrow[dr, "z_1"'] \arrow[drr, "z_2"] & & \\ X & Y_1 \arrow[r, "g"'] & Y_2 \end{tikzcd}$$

of p-compact groups and loop space morphisms. Let $\mathrm{Rep}(X, Y_1)_{z \to z_1} = \{f \in \mathrm{Rep}(X, Y_1) \mid f \circ z = z_1\}$ denote the set of conjugacy classes of morphisms under Z and $\mathrm{map}(BX, BY)_{z \to z_1}$ the corresponding mapping space.

Lemma 3.9. *(Cf.* [9, 8.4]*.) Assume that $z\colon Z \to X$ is a central monomorphism into the connected p-compact group X and that composition with Bg is an isomorphism $\underline{g}\colon C_{Y_1}(z_1Z) \to C_{Y_2}(z_2Z)$ of centralizers. Then composition with Bg,*

$$Bg \circ -\colon\ \mathrm{map}(BX, BY_1)_{z \to z_1} \to \mathrm{map}(BX, BY_2)_{z \to z_2}$$

is a homotopy equivalence.

Proof. The fibration [9, 8.3] [21, 4.1] $BZ \to BX \to B(X/Z)$ allows us to view $BX = BZ_{h(X/Z)}$ as a homotopy orbit space [9, 9.10] and

$$\mathrm{map}(BX, BY_i) = \mathrm{map}(BZ_{h(X/Z)}, BY_1) = \mathrm{map}(BZ, BY_i)^{h(X/Z)}, \quad i = 1, 2,$$

as homotopy fixed point spaces. Composition with $Bg\colon BY_1 \to BY_2$,

$$\begin{aligned} \mathrm{map}(BX, BY_1)_{z \to z_1} = \mathrm{map}(BZ, BY_1)^{h(X/Z)}_{Bz_1} &\to \mathrm{map}(BZ, BY_2)^{h(X/Z)}_{Bz_2} \\ &= \mathrm{map}(BX, BY_2)_{z \to z_2} \end{aligned}$$

is a homotopy equivalence because [9, 10.2] it is induced by the map

$$\mathrm{map}(BZ, BY_1)_{Bz_1} = BC_{Y_1}(z_1) \to BC_{Y_2}(z_2) = \mathrm{map}(BZ, BY_2)_{Bz_2}$$

which by assumption is a homotopy equivalence. □

Here is a typical application of (3.9). In the diagram

$$\begin{tikzcd} V \arrow[d, "z_1"'] \arrow[dr, "\overline{z}_2"'] \arrow[drr, "z_2"] & & \\ X_1 & C_{X_2}(V) \arrow[r, "e(V)"'] & X_2 \end{tikzcd}$$

V is an elementary abelian p-group, z_1 a central monomorphism, z_2 a monomorphism , and $\overline{z}_2$ the canonical factorization of z_2 through its centralizer [9, 8.2]. Since the evaluation monomorphism $e(V)\colon C_{X_2}(V) \to X_2$ clearly [9, 8.2] satisfies the hypothesis of (3.9) we see that

$$\mathrm{map}(BX_1, BC_{X_2}(V))_{z_1 \to \overline{z}_2} \to \mathrm{map}(BX_1, BX_2)_{z_1 \to z_2} \tag{3.10}$$

is a homotopy equivalence.

Definition 3.11. *Let R be a subset of* $\mathrm{Rep}(X_1, X_2)$. *We say that R is T-determined if the implication*

$$f|T(X_1) = g|T(X_1) \Rightarrow f = g$$

holds for all $f \in R$ and all $g \in \mathrm{Rep}(X_1, X_2)$.

Example 3.12. *If the order of $W(X_1)$ is prime to p, then*

$$\mathrm{Rep}(X_1, X_2) = W(X_2)\backslash \mathrm{Adm}(T(X_1), T(X_2)) \tag{3.13}$$

where $\mathrm{Adm}(T(X_1), T(X_2))$ *consists of the admissible homomorphisms (*2.3*). Thus* $\mathrm{Rep}(X_1, X_2)$ *is T-determined in this case. The bijection (*3.13*) follows by exploiting the $H^*\mathbf{F}_p$-equivalence $BN(X_1) \to BX_1$* [23, 3.12].

Remark 3.14. *Let $S_1 \to G_1 \to \pi_0(G_1)$ and $S_2 \to G_2 \to \pi_0(G_2)$ be two extensions of finite groups, $\pi_0(G_1)$ and $\pi_0(G_2)$, by p-compact tori, S_1 and S_2. Let* $\mathrm{Hom}(G_1, G_2) = [BG_1, *; BG_2]$ *denote the set of* based *and* $\mathrm{Rep}(G_1, G_2) = [BG_1, BG_2] = \pi_0(G_2)\backslash \mathrm{Hom}(G_1, G_2)$ *the set of* free *homotopy classes of maps of BG_1 into BG_2.*

The two functors π_1 and π_2 define a map

$$\mathrm{Hom}(G_1, G_2) \to \mathrm{Hom}_{(\pi_0(G_1),\pi_0(G_2))}(S_1, S_2) \tag{3.15}$$

into the set $\mathrm{Hom}_{(\pi_0(G_1),\pi_0(G_2))}(S_1, S_2)$ *of pairs* $(\chi, \phi) \in \mathrm{Hom}(\pi_0(G_1), \pi_0(G_2)) \times \mathrm{Hom}(S_1, S_2)$ *such that ϕ is χ-equivariant. The fibre over (χ, ϕ) is either empty or in bijection with the set*

$$\pi_0(\mathrm{map}(BS_1, BS_2)^{\pi_0(G_1)}_{B\phi}) = H^2(\pi_0(G_1); \pi_2(BS_2)) = H^1_\chi(\pi_0(G_1); \check{S}_2) \tag{3.16}$$

where $\pi_0(G_1)$ acts on $\check{S}_2$, the discrete approximation to S_2, through χ.

If we put $w_2 \cdot (\chi, \phi) = (w_2\chi w_2^{-1}, w_2\phi)$ for all $w_2 \in \pi_0(G_2)$ and all $(\chi, \phi) \in \mathrm{Hom}_{(\pi_0(G_1),\pi_0(G_2))}(S_1, S_2)$ *then (*3.15*) becomes $\pi_0(G_2)$-equivariant, so it descends to a map*

$$\mathrm{Rep}(G_1, G_2) \to \pi_0(G_2)\backslash \mathrm{Hom}_{(\pi_0(G_1),\pi_0(G_2))}(S_1, S_2) \tag{3.17}$$

of $\pi_0(G_2)$-orbit sets. The fibre over the orbit $\pi_0(G_2)(\chi, \phi)$ is either empty or in bijection with the orbit set

$$\pi_0(G_2)^{(\chi,\phi)}\backslash H^1_\chi(\pi_0(G_1), \check{S}_2)$$

*for the action of the stabilizer group $\pi_0(G_2)^{(\chi,\phi)}$, consisting of all $w_2 \in \pi_0(G_2)$ such that $w_2\chi = \chi w_2$ and $w_2\phi = \phi$, on the fibre (*3.16*).*

Proposition 3.18. *Let (χ, ϕ) be an element of $Hom_{(\pi_0(G_1),\pi_0(G_2))}(S_1, S_2)$ and suppose that the stabilizer subgroup $\pi_0(G_2)^{(\chi,\phi)}$ acts transitively on the cohomology group $H^1_\chi(\pi_0(G_1), \check{S}_2)$. Then at most one element of* $\mathrm{Rep}(G_1, G_2)$ *is mapped to the orbit $\pi_0(G_2)(\chi, \phi)$ under the map (*3.17*).*

For later reference, I record here a non-realizability result.

Lemma 3.19. *(Cf.* [19, 1.8]*) Let $f\colon X_1 \to X_2$ be a p-compact group morphism where p is odd and X_1 is connected. Assume that*

- *$\pi_1(T(f))$ is injective, and*
- *p divides the order of the Weyl group W_1.*

Then p does not divide $\pi_1(T(f))$ in $\mathrm{Hom}(\pi_1(T_1), \pi_1(T_2))$.

Proof. By fixed point theory [10, 2.10, 2.14], f lifts to a morphism $N_p(f)\colon \mathrm{Syl}_p(N_1) \to \mathrm{Syl}_p(N_2)$ of the p-normalizers. The assumption that $\pi_1(T(f))$ be injective implies, since W_1 is faithfully represented in $\pi_1(T_1)$ [9, 9.7], that $\pi_0(N_p(f))$ embeds the Sylow p-subgroup of W_1 into W_2.

Choose a monomorphism $\mu\colon \mathbf{Z}/p \to \mathrm{Syl}_p(N_1)$ such that also $\pi_0(\mu)\colon \mathbf{Z}/p \to \mathrm{Syl}_p(W_1)$ is injective. This is possible since the epimorphism $\mathrm{Syl}_p(N_1) \to \mathrm{Syl}_p(W_1)$ admits a section when p is odd [2]. Note that the composition $N_p(f)\mu$ is a monomorphism since it induces a monomorphism on component groups. Consider now the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Syl}_p(N_1) \arrow[r, "N_p(f)"] \arrow[d, "j_p"] & \mathrm{Syl}_p(N_2) \arrow[d] \\
\mathbf{Z}/p \arrow[ur, "\mu"] \arrow[r, "j_p\mu"] \arrow[dr, "\mu'"'] & X_1 \arrow[r, "f"] & X_2 \\
 & T_1 \arrow[u] \arrow[r, "T(f)"'] & T_2 \arrow[u, "i_2"']
\end{tikzcd}$$

where μ' is a lift of $j_p\mu$ [9, 4.7, 5.6]. Since $N_p(f)\mu$ is monomorphic, so is $i_2T(f)\mu'$ by commutativity of the diagram. However, this map would be trivial were $\pi_1(T(f))$ divisible by p. □

The rest of the paper consists of an analysis of the special case where $X_1 = \mathrm{SU}(3)$ or G_2, $X_2 = F_4$, and the prime $p = 3$.

4. Embeddings of SU(3) in F_4

In this section we apply the concepts of the previous sections to investigate monomorphisms from SU(3) to F_4 at the prime $p = 3$. First, a few facts about the Quillen category $\mathbf{A}(\mathrm{F}_4)$ of F_4. (See [28] for more details.)

Lemma 4.1. [14, 7.4][28, 8.2.2] *Let E^1 be an elementary abelian group of order 3^1. The set $\mathrm{Mono}(E^1, \mathrm{F}_4)$ of conjugacy classes of monomorphisms of E^1 into F_4 has three elements e^1_1, e^1_2, e^1_3. The centralizers of these three elements are connected 3-compact groups with Weyl groups of order 36, 48, and 48, respectively. The centralizer $C_{\mathrm{F}_4}(e^1_1)$ of e^1_1 is isomorphic to* $\mathrm{SU}(3,3)$. *The automorphism group* $\mathrm{Aut}(E^1)$ *acts trivially on* $\mathrm{Mono}(E^1, \mathrm{F}_4)$.

Lemma 4.2. [14, 7.4][28, 8.2.4], [27, 7.5] *Let E^2 be an elementary abelian group of order 3^2. The set* $\mathrm{Mono}(E^2, \mathrm{F}_4)/\mathrm{Aut}(E^2)$ *of isomorphism classes of conjugacy classes of monomorphisms of E^2 into F_4 has 5 elements, $e^2_1, e^2_2, e^2_3, e^2_4, e^2_5$, with Quillen automorphism groups of order* 8, 4, 12, 12, 48, *and with centralizer Weyl groups of order* 4, 6, 6, 8, 3, *respectively. The centralizer, $C_{\mathrm{F}_4}(e^2_5)$, of e^2_5 is a 3-compact toral group of maximal rank with component group $\pi_0(C_{\mathrm{F}_4}(e^2_5))$ of order* 3. *There are no maps in the Quillen category from e^1_2 or e^1_3 to e^2_5.*

*Proofs of (*4.1*) and (*4.2*).* With computer assistance it is easy to determine, using [24, 2.6] and [22, 3.2], that $\mathrm{Mono}(E^1, \mathrm{F}_4)$ is a trivial $\mathrm{Aut}(E^1)$-set containing three elements whose centralizers are connected 3-compact groups with Weyl groups of order 36, 48, 48, respectively. See [19, 3.3] for the precise structure of $C_{\mathrm{F}_4}(a)$. Since each centralizer of E^1 is connected, any monomorphism $E^2 \to \mathrm{F}_4$ will factor through the maximal torus. □

The Quillen automorphism group referred to in (4.2) consists of all automorphism of E^2 that leaves $e^2_i \in \mathrm{Mono}(E^2, \mathrm{F}_4)$ invariant.

We now show that for any monomorphism of SU(3) or SU(3,3) to F_4 the triangles

$$\begin{tikzcd} & E^1 \arrow[dl, "z"'] \arrow[dr, "e^1_1"] & \\ \mathrm{SU}(3) \arrow[rr, tail] & & \mathrm{F}_4 \end{tikzcd} \qquad \begin{tikzcd} & E^1 \arrow[dl, "z"'] \arrow[dr, "e^1_1"] & \\ \mathrm{SU}(3,3) \arrow[rr, tail] & & \mathrm{F}_4 \end{tikzcd} \tag{4.3}$$

where $z\colon E^1 \to \mathrm{SU}(3)$ and $z\colon E^1 \to \mathrm{SU}(3,3)$ are centers, will commute up to conjugacy. This observation is the key to the classification of monomorphisms of $\mathrm{SU}(3) \rightarrowtail \mathrm{F}_4$.

Lemma 4.4. 1. $\mathrm{Mono}(\mathrm{SU}(3), \mathrm{F}_4)_{z \to e^1_1} = \mathrm{Mono}(\mathrm{SU}(3), \mathrm{F}_4)$.
2. $\mathrm{Mono}(\mathrm{SU}(3,3), \mathrm{F}_4)_{z \to e^1_1} = \mathrm{Mono}(\mathrm{SU}(3,3), \mathrm{F}_4)$.

The proof of this lemma uses *admissible homomorphisms* (2.3) which we now discuss.

Let $\mathbf{Z}_3$ denote the ring of 3-adic integers. The Weyl group $W_1 = W(\mathrm{SU}(3))$ of SU(3) is [24, 3.8, 3.13] $\langle \sigma, \tau \rangle \subseteq \mathrm{Aut}(\Sigma_0(\mathbf{Z}_3^3))$ where $\Sigma_0(\mathbf{Z}_3^3)$ is the free $\mathbf{Z}_3$-module with basis $(1, -1, 0), (0, 1, -1) \in \mathbf{Z}_3^3$ and σ and τ have matrices

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

with respect to this basis. The Weyl group $W(\mathrm{F}_4) = W(\mathrm{F}_4) < \mathrm{GL}(4, \mathbf{Z}_3)$ of F_4 is [3] [24, 3.13] the group (of order $1152 = 384 \cdot 3$)

$$W(\mathrm{F}_4) = W(B_4)E \cup W(B_4)H_1 \cup W(B_4)H_2 \tag{4.5}$$

where $W(B_4)$ is the reflection group (of order $384 = 2^4 \cdot 4!$) of all signed permutation matrices, and H_1 and H_2 are the matrices

$$H_1 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad H_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

satisfying $H_1^2 = E = H_2^2, H_2H_1 = -H_2, H_1H_2 = \mathrm{diag}(-1,1,1,1)H_1$.

We say that a linear map $A\colon \Sigma_0(\mathbf{Z}_3^3) \to \mathbf{Z}_3^4$ is *admissible* if $AW(\mathrm{SU}(3)) \subseteq W(\mathrm{F}_4)A$. The linear map $A(u,v)\colon \Sigma_0(\mathbf{Z}_3^3) \to \mathbf{Z}_3^4$, $u,v \in \mathbf{Z}_3$, for instance, with matrix

$$\boxed{A(u,v) = \begin{pmatrix} -u & v \\ u & v-u \\ 0 & v+u \\ -2v & v \end{pmatrix} = u\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ -2 & 1 \end{pmatrix}} \tag{4.6}$$

with respect to the chosen basis for $\Sigma_0(\mathbf{Z}_3^3)$ and the canonical basis for $\mathbf{Z}_3^4$, is admissible. Indeed, $A(u,v)$ is χ-equivariant where $\chi\colon W(\mathrm{SU}(3)) \to W(\mathrm{F}_4)$ is the group homomorphism given by

$$\chi(\sigma) = \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \chi(\tau) = \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{4.7}$$

The next lemma classifies the admissible homomorphisms. Note that $A(u,v)$ and $-A(u,v)$ lie in the same orbit under the action of $W(\mathrm{F}_4)$ as $-E \in W(\mathrm{F}_4)$.

Lemma 4.8. 1. *Let $A\colon \Sigma_0(\mathbf{Z}_3^3) \to \mathbf{Z}_3^4$ be a linear map. Then A is admissible with respect to $W(\mathrm{SU}(3))$ and $W(\mathrm{F}_4)$ if and only if $A \in W(\mathrm{F}_4)A(u,v)$ for some 3-adic integers $u,v \in \mathbf{Z}_3$.*
2. *$A(u,v)$ is split injective if and only if $u+v$ is a 3-adic unit.*
3. *The map*

$$\begin{aligned} \langle(-1,-1)\rangle \backslash (\mathbf{Z}_3)^2 &\to W(\mathrm{F}_4)\backslash \mathrm{Hom}_{\mathbf{Z}_3}(\Sigma_0(\mathbf{Z}_3^3), \mathbf{Z}_3^4) \\ \pm(u,v) &\to W(\mathrm{F}_4)A(u,v) \end{aligned}$$

is injective.

Proof. 1. Using a computer, it is possible to show that up to inner automorphisms, any admissible homomorphism $\Sigma_0(\mathbf{Z}_3^3) \to \mathbf{Z}_3^4$ must be χ-equivariant. Given this, one simply solves the system of linear equations $Aw = \chi(w)A$ for A where w runs through a generating set for $W(\mathrm{SU}(3))$.

2. The matrix $A(u,v)$ is equivalent to the matrix

$$\begin{pmatrix} u-2v & 0 \\ 0 & 2v-u \\ 3u & 0 \\ -u & v \end{pmatrix}$$

which is split injective if and only if $u-2v$ or, equivalently, $(u-2v)+3v=u+v$ is a 3-adic unit.

3. The claim is that for any w in $W(\mathrm{F}_4)$ the set of solutions to the homogeneous system of linear equations

$$wA(u_1,v_1)-A(u_2,v_2)=0$$

in the four unknowns (u_1,v_1,u_2,v_2) is contained in the diagonal $(u_1,v_1)=(u_2,v_2)$ or in the anti-diagonal $(u_1,v_1)=-(u_2,v_2)$. This is easily verified on a computer. □

Our interest in the admissible homomorphisms lies in the fact that the induced homomorphism $\pi_1(T(f))$ is admissible for any lift $T(f)\colon T(\mathrm{SU}(3))\to T(\mathrm{F}_4)$ to the maximal tori of any morphism $f\colon \mathrm{SU}(3)\to\mathrm{F}_4$. Thus we must have $\pi_1(T(f))\in W(\mathrm{F}_4)A(u,v)$ for some 3-adic integers u and v. However, as we shall shortly see, not all the homomorphisms $A(u,v)$ are induced in this way from morphisms $\mathrm{SU}(3)\to\mathrm{F}_4$.

The proof of (4.4) follows immediately from (4.8.1).

Proof of Lemma 4.4. 1. Let $f\colon \mathrm{SU}(3)\to\mathrm{F}_4$ be any monomorphism. Then $\pi_1(T(f))$ is admissible, so we may assume that $\pi_1(T(f))=A(u,v)$ for some 3-adic integers $u,v\in\mathbf{Z}_3$. The restriction $fz\colon E^1\to\mathrm{F}_4$ of f to the center $z\colon E^1\to\mathrm{SU}(3)$ of $\mathrm{SU}(3)$ is given by

$$A(u,v)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} u+v \\ u+v \\ u+v \\ 0 \end{pmatrix} \tag{4.9}$$

where we have reduced modulo 3. Since fz is a monomorphism, $u+v\not\equiv 0 \bmod 3$ and then the stabilizer in $W(\mathrm{F}_4)$ of $(u+v,u+v,u+v,0)\in(\mathbf{Z}/3)^4$ has order 36. Thus $fz\simeq e_1^1\in\mathrm{Mono}(E^1,\mathrm{F}_4)$.

2. Let $f\colon \mathrm{SU}(3,3)\to\mathrm{F}_4$ be any monomorphism and choose some monomorphism $g\colon \mathrm{SU}(3)\to\mathrm{SU}(3,3)$ such that $gz=z$, e.g. $g=\psi^{(1,0)}$. Then $fz=fgz=e_1^1$. □

Let $e\colon \mathrm{SU}(3,3)=C_{\mathrm{F}_4}(e_1^1)\to\mathrm{F}_4$ denote the inclusion of the centralizer of e_1^1 into F_4; this map is described in detail in [19, 3.3].

Corollary 4.10. *The maps*

$$\mathrm{Mono}(\mathrm{SU}(3),\mathrm{SU}(3,3))_{z\to z}\xrightarrow{e\circ -}\mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4)$$

$$\mathrm{Out}(\mathrm{SU}(3,3))_{z\to z}\xrightarrow{e\circ -}\mathrm{Mono}(\mathrm{SU}(3,3),\mathrm{F}_4)$$

are bijections.

Proof. By (3.9) and (4.4),

$$\mathrm{Mono}(\mathrm{SU}(3),\mathrm{SU}(3,3)_{z\to z} = \mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4)_{z\to e_1^1} = \mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4)$$

and similarly for morphisms from $\mathrm{SU}(3,3)$. □

Lemma 4.11. *Let* $\psi^{(u,v)}\colon \mathrm{SU}(3)\to\mathrm{SU}(3,3)$ *be the morphism* (2.17) *indexed by* $u,v\in\mathbf{Z}_3^*\cup\{0\}$. *Then* $W(\mathrm{F}_4)\pi_1(T(e\psi^{(u,v)})) = W(\mathrm{F}_4)A(u,v)$.

Proof. The monomorphism $e\colon \mathrm{SU}(3,3)\to\mathrm{F}_4$ is [19, 3.3] realizable on the level of compact Lie groups as an inclusion $\mathrm{SU}(3,3)\hookrightarrow\mathrm{F}_4$ such that the restriction $\Sigma_0(\mathbf{Z}^3)\times\Sigma_0(\mathbf{Z}^3)\to\Sigma_2(\mathbf{Z}^4)$ to the integral lattices of the composite morphism $\mathrm{SU}(3)\times\mathrm{SU}(3)\twoheadrightarrow\mathrm{SU}(3,3)\hookrightarrow\mathrm{F}_4$ takes $(x_1,x_2,x_3;y_1,y_2,y_3)$ to $(x_1+y_3,x_2+y_3,x_3+y_3,y_1-y_2)$. Thus

$$\begin{pmatrix} 1 & 0 & 0 & -1\\ -1 & 1 & 0 & -1\\ 0 & -1 & 0 & -1\\ 0 & 0 & 2 & -1\end{pmatrix}\begin{pmatrix} u & 0\\ 0 & u\\ v & 0\\ 0 & v\end{pmatrix} = \begin{pmatrix} u & -v\\ -u & u-v\\ 0 & -u-v\\ 2v & -v\end{pmatrix} = -A(u,v)$$

represents $\pi_1(T(e\psi^{(u,v)}))$. □

Lemma 4.12. *Let* u *and* v *be* 3*-adic integers and* $A(u,v)$ *the corresponding admissible homomorphism.*

1. *There exists a morphism* $f\colon\mathrm{SU}(3)\to\mathrm{F}_4$ *such that* $W(\mathrm{F}_4)\pi_1(T(f)) = W(\mathrm{F}_4)A(u,v)$ *if and only if both* u *and* v *are in the set* $\mathbf{Z}_3^*\cup\{0\}$.
2. *There exists a monomorphism* $f\colon\mathrm{SU}(3)\to\mathrm{F}_4$ *such that* $W(\mathrm{F}_4)\pi_1(T(f)) = W(\mathrm{F}_4)A(u,v)$ *if and only if* $u,v\in\mathbf{Z}_3^*\cup\{0\}$ *and* $u+v\in\mathbf{Z}_3^*$.

Proof. We have already seen (4.11) that $A(u,v)$ is realizable for all $u,v\in\mathbf{Z}_3^*\cup\{0\}$.

Suppose, conversely, that $\pi_1(T(f)) = A(u,v)$ for some 3-adic integers, u and v, and some morphism $f\colon\mathrm{SU}(3)\to\mathrm{F}_4$. If f is a monomorphism, then $f = e\psi^{(u,v)}$ for some $u,v\in\mathbf{Z}_3^*\cup\{0\}$ with $u+v\in\mathbf{Z}_3^*$ by (4.10). If f is not a monomorphism, $A(u,v)$ is not split injective [24, 5.2] [21, 3.6.1], so $u+v$ is not a 3-adic unit (4.8.2). □

Theorem 4.13. 1. $\mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4)$ *is* T*-determined.*

2. *The map*

$$\begin{aligned}\langle(-1,-1)\rangle\backslash\{(u,v)\in(\mathbf{Z}_3^*\cup\{0\})^2 | u+v\in\mathbf{Z}_3^*\} &\to \mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4)\\ \pm(u,v) &\to e\psi^{(u,v)}\end{aligned}$$

is a bijection.

Proof. 1. The restriction map $\mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4)\to\mathrm{Mono}(T(\mathrm{SU}(3)),\mathrm{F}_4)$ can be identified to the map

$$\begin{aligned}\{(u,v)\in(\mathbf{Z}_3^*\cup\{0\})^2 \mid u+v\equiv 1 \bmod 3\} &\to W(\mathrm{F}_4)\backslash\mathrm{Hom}(\Sigma_0(\mathbf{Z}_3^3),\mathbf{Z}_3^4)\\ (u,v) &\to W(\mathrm{F}_4)A(u,v)\end{aligned}$$

which is injective by (4.8.3).
2. This is immediate from (2.18) and (4.10). □

Here is an alternative formulation of (4.10): Consider the commutative diagrams

$$\begin{tikzcd} \mathrm{Mono}(\mathrm{SU}(3),\mathrm{SU}(3,3))_{z\to z} \arrow[r] \arrow[dr, "\cong", "e\circ-"'] & \left\langle \psi^{-1}\times\psi^{-1}\right\rangle \backslash \mathrm{Mono}(\mathrm{SU}(3),\mathrm{SU}(3,3)) \arrow[d, "e\circ-"] \\ & \mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Out}(\mathrm{SU}(3,3))_{z\to z} \arrow[r] \arrow[dr, "\cong", "e\circ-"'] & \left\langle \psi^{-1}\times\psi^{-1}\right\rangle \backslash \mathrm{Out}(\mathrm{SU}(3,3) \arrow[d, "e\circ-"] \\ & \mathrm{Mono}(\mathrm{SU}(3,3),\mathrm{F}_4) \end{tikzcd}$$

where the slanted arrows are bijections. The vertical arrows exist because $e(\psi^{-1}\times\psi^{-1}) = e$ by [19, 3.3]. Noting (2.17) that

$$\mathrm{Mono}(\mathrm{SU}(3),\mathrm{SU}(3,3))_{z\to z} = \{(u,v)\in(\mathbf{Z}_3^*\cup\{0\})^2 \mid u+v\equiv 1 \bmod 3\}$$
$$\mathrm{Out}(\mathrm{SU}(3,3))_{z\to z} = \{(u,v)\in(\mathbf{Z}_3^*)^2 \mid u\equiv 1\equiv v \bmod 3\}\rtimes\langle\tau\rangle$$

we see that the vertical arrow in each of the diagrams is a bijection, too, and hence that the vertical arrow of the upper (lower) diagram is a bijection of right Out(SU(3))- (Out(SU(3, 3))-) sets. Thus the action

$$\mathrm{Mono}(\mathrm{SU}(3,3),\mathrm{F}_4)\times\mathrm{Out}(\mathrm{SU}(3,3))\to\mathrm{Mono}(\mathrm{SU}(3,3),\mathrm{F}_4) \tag{4.14}$$

is transitive and the stabilizer subgroup at the centric monomorphism e, i.e. the Weyl group [11, 4.3] [24, 8.4]

$$W_{\mathrm{F}_4}(e\mathrm{SU}(3,3)) = \left\langle\psi^{-1}\times\psi^{-1}\right\rangle \tag{4.15}$$

is cyclic of order two.

The next lemma lists the centralizers of all monomorphisms $\mathrm{SU}(3)\rightarrowtail\mathrm{F}_4$. We let ψ^{-1} denote the automorphism $\psi^{-1}\times_{Z(\mathrm{SU}(3))}\psi^{-1}$ of $T(\mathrm{SU}(3)\times_{Z(\mathrm{SU}(3))}\mathrm{SU}(3)$ [22, 4.3].

Lemma 4.16. *Let* $(u,v)\in(\mathbf{Z}_3^*\cup\{0\})^2$ *and* $u+v\in\mathbf{Z}_3^*$. *If* $uv\neq 0$, *then*

$$C_{\mathrm{F}_4}(e\psi^{(u,v)}\mathrm{SU}(3)) = Z(\mathrm{SU}(3))$$
$$C_{\mathrm{F}_4}(e\psi^{(u,v)}T(\mathrm{SU}(3))) = T(\mathrm{F}_4)$$

If $uv=0$, *then*

$$C_{\mathrm{F}_4}(e\psi^{(u,v)}\mathrm{SU}(3)) = Z(\mathrm{SU}(3))\times_{Z(\mathrm{SU}(3))}\mathrm{SU}(3)$$
$$C_{\mathrm{F}_4}(e\psi^{(u,v)}T(\mathrm{SU}(3))) = T(\mathrm{SU}(3))\times_{Z(\mathrm{SU}(3))}\mathrm{SU}(3)$$

In all cases, $C_{\mathrm{F}_4}(\psi^{-1}) = \psi^{-1}$*.*

Proof. It only remains to determine the map $C_{\mathrm{F}_4}(\psi^{-1})$ induced by ψ^{-1} since the centralizers themselves are given by (2.19, 3.9). Let us, for example, consider the case where $(u, v) = (0, 1)$. Consider the morphism $\mu\colon (\mathrm{SU}(3) \times T(\mathrm{SU}(3))) \times T(\mathrm{SU}(3)) \to \mathrm{SU}(3) \times T(\mathrm{SU}(3)) \to \mathrm{SU}(3,3)$ constructed from the multiplication on the maximal torus and the projection map. Since

$$e\mu((1 \times 1) \times \psi^{-1}) = e(\psi^{-1} \times \psi^{-1})\mu((1 \times 1) \times \psi^{-1}) = e\mu((\psi^{-1} \times \psi^{-1}) \times 1)$$

it follows from (4.17) that $C_{\mathrm{F}_4}(\psi^{-1}) = \psi^{-1}$ on $C_{\mathrm{F}_4}(e\psi^{(0,1)}T(\mathrm{SU}(3)))$. The other cases are similar. $\square$

Lemma 4.17. *If the diagram of p-compact groups*

$$\begin{tikzcd}
X_1 \times X_2 \arrow[d, "\mu"'] & X_1 \times X_2' \arrow[l, "1\times f_2"'] \arrow[r, "f_1\times 1"] & X_1' \times X_2' \arrow[d, "\mu'"] \\
Y \arrow[rr, "h"'] & & Y'
\end{tikzcd}$$

commutes up to conjugacy, so does the induced diagram

$$\begin{tikzcd}
X_1 \arrow[r, "\mathrm{ad}(\mu)"] \arrow[d, "f_1"'] & C_Y(X_2) \arrow[d, "C_h(f_2)"] \\
X_1' \arrow[r, "\mathrm{ad}(\mu')"] & C_{Y'}(X_2')
\end{tikzcd}$$

where the horizontal arrows are adjoints of μ *and* μ'*.*

Corollary 4.18. *Let* N *be a (topological) group with subgroups* $g_1\colon G_1 \to N$ *and* $g_2\colon G_2 \to N$*. Suppose that* $n \in N$ *is an element such that conjugation with* n*,* $c(n)(m) = nmn^{-1}$*,* $m \in N$*, takes* G_1 *into* G_2*. Then conjugation with* n^{-1} *takes the centralizer* $C_N(G_2)$ *into* $C_N(G_1)$ *and the diagram*

$$\begin{tikzcd}
BC_N(G_1) \arrow[r] & \mathrm{map}(BG_1, BN)_{Bg_1} \\
BC_N(G_2) \arrow[r] \arrow[u, "Bc(n^{-1})"] & \mathrm{map}(BG_2, BN)_{Bg_2} \arrow[u, "\overline{Bc(n)}"']
\end{tikzcd}$$

commutes up to homotopy.

Proof. We have $\mu(c(n) \times 1) = c(n)\mu(1 \times c(n^{-1}))$ where μ is group multiplication and where the induced map $Bc(n)\colon BN \to BN$ is homotopic to the identity. $\square$

5. Toric representations of PU(3) in F_4

In this section I classify the p-toric morphisms from PU(3) to F_4 viewed as 3-compact groups. The first step is the determination of the admissible homomorphisms.

Let X be a connected p-compact group with maximal torus $i\colon T \to X$. We want to describe the integral lattice of the central quotients of X. Suppose that Z is a subgroup of the discrete approximation $\check{T} = (\pi_1(T) \otimes \mathbf{Q})/\pi_1(T)$ such that the composition $Z \to \check{T} \to X$ is a central monomorphism. Then we may form the p-compact group X/Z [9, 8.3] with induced maximal torus $i/Z\colon T/Z \to X/Z$ [21, 4.6] that fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \kappa^{-1}(0) \arrow[r] \arrow[d, equal] & \kappa^{-1}(Z) \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_1(T) \arrow[r] \arrow[d] & \pi_1(T) \otimes \mathbf{Q} \arrow[r, "\kappa"] \arrow[d, "\cong"] & \check{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_1(T/Z) \arrow[r] & \pi_1(T/Z) \otimes \mathbf{Q} \arrow[r] & \check{T}/Z \arrow[r] & 0
\end{tikzcd}$$

with exact rows. From this we get an isomorphism

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(T) \arrow[r] \arrow[d, equal] & \pi_1(T/Z) \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \kappa^{-1}(0) \arrow[r] & \kappa^{-1}(Z) \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

of extensions of $W_T(X) = W_{T/Z}(X/Z)$-modules.

In particular, let $\Sigma_0(\mathbf{Z}_3^3) \subseteq \Sigma_0(\mathbf{Q}_3^3)$ be the free $\mathbf{Z}_3$-submodule with basis $e_1 = (1, -1, 0)$ and $e_2 = (0, 1, -1)$; this is the integral lattice for SU(3). Put $f = \frac{1}{3}(e_1 - e_2)$ and let $P\Sigma_0(\mathbf{Z}_3^3)$ be the free $\mathbf{Z}_3$-submodule of $\mathbf{Q}_3^3$ with basis $\{e_1, f\}$. Then there is an exact sequence

$$0 \to \Sigma_0(\mathbf{Z}_3^3) \xrightarrow{\iota} P\Sigma_0(\mathbf{Z}_3^3) \to \mathbf{Z}/3 \to 0$$

of $\mathbf{Z}_3[\Sigma_3]$-modules and $P\Sigma_0(\mathbf{Z}_3^3)$ corresponds to the maximal torus for PU(3).

Note that there is an extension, $B(u,v)$, of $A(u,v)$,

$$\begin{tikzcd}
\Sigma_0(\mathbf{Z}_3^3) \arrow[r, "\iota"] \arrow[d, "A(u,v)"'] & P\Sigma_0(\mathbf{Z}_3^3) \arrow[dl, "B(u,v)"] \\
L_4 &
\end{tikzcd}$$

if and only if $u+v$ is divisible by 3 and in that case the extension is unique and given by

$$B(u,v) = A(u,v)\begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} -u & -\frac{1}{3}(u+v) \\ u & \frac{1}{3}(2u-v) \\ 0 & -\frac{1}{3}(u+v) \\ -2v & -v \end{pmatrix}$$

where u and v are 3-adic integers and $u+v \in 3\mathbf{Z}_3$. Moreover, the inclusion ι is $W(\mathrm{SU}(3)) = W(\mathrm{PU}(3))$-equivariant and $B(u,v)$ is χ-equivariant where χ is the group homomorphism from (4.7).

Lemma 5.1. 1. *A $\mathbf{Z}_3$-linear map $B\colon P\Sigma_0(\mathbf{Z}_3^3) \to \mathbf{Z}_3^4$ is admissible with respect to $W(\mathrm{PU}(3))$ and $W(\mathrm{F}_4)$ is and only if $B \in W(\mathrm{F}_4)B(u,v)$ where u and v are 3-adic integers whose sum is divisible by 3.*

2. *$B(u,v)$ is split-injective when u and v are 3-adic units.*
3. *The map*

$$\begin{aligned} \langle(-1,-1)\rangle \backslash \{(u,v) \in (\mathbf{Z}_3)^2 | u+v \in 3\mathbf{Z}_3\} &\to W(\mathrm{F}_4)\backslash \mathrm{Hom}_{\mathbf{Z}_3}(P\Sigma_0(\mathbf{Z}_3^3), \mathbf{Z}_3^4) \\ \pm(u,v) &\to W(\mathrm{F}_4)B(u,v) \end{aligned}$$

is injective.

Proof. 1. B is admissible if and only if $B \circ \iota$ is, i.e. if and only if B is an extension of $A(u,v)$ (4.8.1) for some 3-adic integers, u and v.
2. If u and v are units then

$$\begin{pmatrix} -u^{-1} & 0 & u^{-1} & 0 \\ 2u^{-1} & 0 & 2u^{-1} & -v^{-1} \end{pmatrix}$$

is a left inverse of $B(u,v)$.

3. If $B(u_1,v_1) \in W(\mathrm{F}_4)B(u_2,v_2)$ then also $A(u_1,v_1) \in W(\mathrm{F}_4)A(u_2,v_2)$ and then (4.8.3) (u_1,v_1) and (u_2,v_2) are equal up to sign. □

When $u,v \in \mathbf{Z}_3^* \cup \{0\}$ with sum $u+v \in 3\mathbf{Z}_3$ there is a unique conjugacy class, $\overline{\psi}^{(u,v)}$, that makes the diagram

$$\begin{tikzcd} \mathrm{SU}(3) \arrow[r, "\psi^{(u,v)}"] \arrow[d] & \mathrm{SU}(3,3) \arrow[r, "e"] & \mathrm{F}_4 \\ \mathrm{PU}(3) \arrow[ur, "\overline{\psi}^{(u,v)}"'] & & \end{tikzcd}$$

commutes up to conjugation. By construction,

$$W(\mathrm{F}_4)\pi_1\big(T(e \circ \overline{\psi}^{(u,v)})\big) = W(\mathrm{F}_4)B(u,v)$$

in $W(\mathrm{F}_4)\backslash \mathrm{Hom}_{\mathbf{Z}_3}(P\Sigma_0(\mathbf{Z}_3^3), \mathbf{Z}_3^4)$.

Lemma 5.2. *Let u and v be 3-adic integers with sum $u+v \in 3\mathbf{Z}_3$ and let $B(u,v)\colon P\Sigma_0(\mathbf{Z}_3^3) \to \mathbf{Z}_3^4$ be the corresponding admissible homomorphism.*

1. *There exists a morphism* $f\colon \mathrm{PU}(3) \to \mathrm{F}_4$ *such that* $W(\mathrm{F}_4)\pi_1(T(f)) = W(\mathrm{F}_4)B(u,v)$ *if and only if* $u = 0 = v$ *or* $u, v \in \mathbf{Z}_3^*$.
2. *There exists a monomorphism* $f\colon \mathrm{PU}(3) \to \mathrm{F}_4$ *such that* $W(\mathrm{F}_4)\pi_1(T(f)) = W(\mathrm{F}_4)B(u,v)$ *if and only if* $u, v \in \mathbf{Z}_3^*$.

Proof. We have already seen that $W(\mathrm{F}_4)B(u,v)$ is realizable by a morphism $f\colon \mathrm{PU}(3) \to \mathrm{F}_4$ if $u = 0 = v$ or $u, v \in \mathbf{Z}_3^*$; if both u and v are non-zero then f is a monomorphism by (5.1.2). Conversely, if $W(\mathrm{F}_4)B(u,v)$ is realizable, so is $W(\mathrm{F}_4)A(u,v)$ and then (4.12) $u, v \in \mathbf{Z}_3^* \cup \{0\}$, $u + v \notin \mathbf{Z}_3^*$. $\square$

Alternatively, (5.2) says that any non-trivial morphism $\mathrm{PU}(3) \to \mathrm{F}_4$ is a monomorphism.

Proposition 5.3. *(Cf.* [1, 2.27.(ii)]*) Suppose that* u *and* v *are* 3*-adic units with* $u + v \in 3\mathbf{Z}_3$. *Then*

$$T(\mathrm{PU}(3)) \xrightarrow{B(u,v)} T(\mathrm{F}_4) \xrightarrow{i_2} \mathrm{F}_4$$

is toric if and only if $(u,v) \notin \mathbf{Z}_3^*(2,1) \cup \mathbf{Z}_3^*(1,-1)$.

Proof. Explicit (computer aided) computations of $W(\mathrm{F}_4)^{B(u,v)} = W(\mathrm{F}_4)^{A(u,v)}$. $\square$

The two generic non-3-toric morphisms

$$B(2,1) = \begin{pmatrix} -2 & -1 \\ 2 & 1 \\ 0 & -1 \\ -2 & -1 \end{pmatrix} \quad \text{and} \quad B(1,-1) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 2 & 1 \end{pmatrix}$$

are related by the equation $\varepsilon B(2,1) = 2B(1,-1)$ where

$$\varepsilon = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

is the admissible automorphism of $\mathbf{Z}_3^4$ corresponding to the exotic automorphism of F_4. (In general, $W(\mathrm{F}_4)(\varepsilon A(u,v)) = W(\mathrm{F}_4)(A(2v,-u))$, cf. [1, 2.11].)

Theorem 5.4. 1. $\mathrm{TRep}(\mathrm{PU}(3), \mathrm{F}_4)$ *is* T*-determined.*
2. *The map*

$$\langle(-1,-1)\rangle \backslash (\{(u,v) \in (\mathbf{Z}_3^*)^2 \mid u + v \in 3\mathbf{Z}_3\}) \setminus (\mathbf{Z}_3^*(2,1) \cup \mathbf{Z}_3^*(1,-1))) \to \mathrm{TRep}(\mathrm{PU}(3), \mathrm{F}_4)$$

taking $\pm(u,v)$ *to* $e \circ \overline{\psi}^{(u,v)}$, *is a bijection.*

Consider the set $\mathrm{Rep}(N(\mathrm{PU}(3)), N(\mathrm{F}_4))$ of conjugacy classes of maps from the maximal torus normalizer $N(PU(3))$ of $\mathrm{PU}(3)$ to the maximal torus normalizer $N(\mathrm{F}_4)$ of F_4. As we have seen (3.17), there is a map

$$\mathrm{Rep}(N(\mathrm{PU}(3)), N(\mathrm{F}_4)) \to W(\mathrm{F}_4) \backslash \mathrm{Hom}_{(W(\mathrm{PU}(3)), W(\mathrm{F}_4))}(T(\mathrm{PU}(3)), T(\mathrm{F}_4))$$

induced by the functors π_1 and π_2. It is easy to calculate directly that the cohomology group $H^2(\langle\chi(\sigma)\rangle\,;\pi_1(T(\mathrm{F}_4)))$ is trivial. Then also

$$H^2_\chi(W(\mathrm{PU}(3));\pi_1(T(\mathrm{F}_4)))=0 \tag{5.5}$$

for $\langle\sigma\rangle$ is a Sylow 3-subgroup of the Weyl group of PU(3) and we get

Lemma 5.6. *There is at most one element of* $\mathrm{Rep}(N(\mathrm{PU}(3)),N(\mathrm{F}_4))$ *corresponding to the orbit* $W(\mathrm{F}_4)(\chi,B(u,v))$, $(u,v)\in(\mathbf{Z}_3^*)^2$, $u+v\in 3\mathbf{Z}_3$.

Proof of Theorem 5.4. Let $f_1,f_2\in T\,\mathrm{Rep}(\mathrm{PU}(3),\mathrm{F}_4)$ be two toric representations and suppose that their restrictions to the maximal torus of PU(3) agree. Under the map

$$\begin{aligned}\mathrm{TRep}(\mathrm{PU}(3),\mathrm{F}_4)&\to\mathrm{TRep}(N(\mathrm{PU}(3)),N(\mathrm{F}_4))\\&\to W(\mathrm{F}_4)\backslash\mathrm{Hom}_{(W(\mathrm{PU}(3)),W(\mathrm{F}_4))}(T(\mathrm{PU}(3)),T(\mathrm{F}_4))\end{aligned}$$

f_1 and f_2 go to the same element of the target and it follows (5.6) that the lifts (3.5) $N(f_1)$ and $N(f_2)$ are conjugate, i.e. that f_1 and f_2 have conjugate restrictions to the maximal torus normalizer $N(\mathrm{PU}(3))$. In fact, $N(f_1)=B(u,v)\rtimes\chi=N(f_2)$ for some $(u,v)\in(\mathbf{Z}_3^*)^2\setminus(\mathbf{Z}_3^*(2,1)\cup\mathbf{Z}_3^*(1,-1))$.

We may approximate BPU(3) by a homotopy colimit over a category $\mathbf{I}=\mathbf{I}(\mathrm{SL}(2,\mathbf{F}_3),S_3)$ (a full subcategory of the Quillen category that may be described as formed from the inclusion of a Sylow 3-subgroup S_3 into the special linear group $\mathrm{SL}(2,\mathbf{F}_3)$) with just two objects, $\lambda\colon E^1\to\mathrm{PU}(3)$ and $\nu\colon E^2\to\mathrm{PU}(3)$, where E^1 and E^2 are elementary abelian groups of order 3 and 3^2, respectively [17, 6.8, 7,7]; see [24, §4] for the notation used here. Since f_1 and f_2 agree on the centralizers, $C_{\mathrm{PU}(3)}(\lambda E^1)=N_3(\mathrm{PU}(3))$ and $C_{\mathrm{PU}(3)}(\nu E^2)=E^2$, it only remains to compute the relevant Wojtkowiak obstruction groups [29]. For this we need information about the centralizer $C_{\mathrm{F}_4}(f_iE^2)$ and $C_{\mathrm{F}_4}(f_iN_3(\mathrm{PU}(3)))$.

We must have $f_1|E^2=e_5^2=f_2|E^2$ for only $e_5^2\in\mathrm{Mono}(E^2,\mathrm{F}_4)$ can contain in its automorphism group the automorphism group $\mathrm{SL}(2,\mathbf{F}_3)$ of (E^2,ν). Thus $C_{\mathrm{F}_4}(f_iE^2)$ is a p-compact toral group of maximal rank with E^1 as its component group (4.2).

The centralizer $C_{\mathrm{F}_4}(f_iN_3(\mathrm{PU}(3)))$ is (3.4) the p-compact toral group

$$C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(\check{T}(\mathrm{PU}(3))\rtimes\langle\sigma\rangle)=\check{T}(\mathrm{F}_4)^{\langle\chi(\sigma)\rangle}=t(\mathrm{F}_4)^{\langle\chi(\sigma)\rangle}=E^2$$

where $t(\mathrm{F}_4)\subset\check{T}(\mathrm{F}_4)$ denotes the maximal elementary abelian subgroup of the discrete approximation $\check{T}(\mathrm{F}_4)$ to $T(\mathrm{F}_4)$ and

The obstructions to a homotopy between the two maps $Bf_1,Bf_2\colon\mathrm{BPU}(3)\to\mathrm{BF}_4$ lie in the abelian groups $\lim^1_{\mathbf{I}}\underline{\pi}_1$ and $\lim^2_{\mathbf{I}}\underline{\pi}_2$ where $\underline{\pi}_1$ and $\underline{\pi}_2$ are the abelian $\mathbf{I}$-groups

$$\mathbf{Z}/2 \circlearrowright E^2 \xrightarrow{\mathrm{SL}(2,\mathbf{F}_3)/S_3} E^1 \circlearrowleft \mathrm{SL}(2,\mathbf{F}_3)$$

$$\mathbf{Z}/2 \circlearrowright 0 \xrightarrow{\mathrm{SL}(2,\mathbf{F}_3)/S_3} \mathbf{Z}_3^4 \circlearrowleft \mathrm{SL}(2,\mathbf{F}_3)$$

given by the homotopy groups of the above centralizers. The group $\mathrm{SL}(2,\mathbf{F}_3)$ has no normal subgroups of index two, so it necessarily acts trivially on E^1. It now follows from [24, 10.7.5] that both obstruction groups are trivial and we conclude that f_1 and f_2 are conjugate. This shows that $\mathrm{TRep}(\mathrm{PU}(3),\mathrm{F}_4)$ is T-determined.

Let now $f\colon \mathrm{PU}(3)\to\mathrm{F}_4$ be any toric monomorphism. Then there is (5.1.3, 5.3) a unique, up to sign, pair of units $(u,v)\in(\mathbf{Z}_3^*)^2$, $u+v\in 3\mathbf{Z}_3$, $(u,v)\notin\mathbf{Z}_3^*(2,1)\cup\mathbf{Z}_3^*(1,-1)$, such that $W(\mathrm{F}_4)\pi_1(T(f))=W(\mathrm{F}_4)B(u,v)$ and then $f=\overline{\psi}^{(u,v)}$ since the p-toric monomorphisms are T-determined. □

Lemma 5.7. *Let $(u,v)\in(\mathbf{Z}_3^*)^2$, $u+v\in 3\mathbf{Z}_3$, $(u,v)\notin\mathbf{Z}_3^*(2,1)\cup\mathbf{Z}_3^*(1,-1)$. Then*

$$C_{\mathrm{F}_4}(e\psi^{(u,v)}\mathrm{SU}(3))=\check{T}(\mathrm{F}_4)^{\chi(W(\mathrm{SU}(3)))}$$
$$C_{\mathrm{F}_4}(e\psi^{(u,v)}T(\mathrm{SU}(3)))=T(\mathrm{F}_4)$$

and $C_{\mathrm{F}_4}(\psi^{-1})=\psi^{-1}$ in both cases.

Proof. Since $e\psi^{(u,v)}$ is toric, the centralizer in F_4 of $e\psi^{(u,v)}T(\mathrm{SU}(3))$ equals the maximal torus of F_4. Proceed as in (4.16) to show that $C_{\mathrm{F}_4}(\psi^{-1})=\psi^{-1}$.

The centralizer $BC_{\mathrm{F}_4}(e\overline{\psi}^{(u,v)}\mathrm{PU}(3))$ is the homotopy colimit of the $\mathbf{I}$-space

$$\mathbf{Z}/2 \circlearrowright B(0)\xrightarrow{\mathrm{SL}(2,\mathbf{F}_3)/S_3}B(1)\circlearrowleft \mathrm{SL}(2,\mathbf{F}_3)$$

where $B(0)=B\check{T}(\mathrm{F}_4)^{\langle\chi(\sigma)\rangle}$ and $B(0)=BC_{\mathrm{F}_4}(e_5^2)$. We need to be more specific about the group actions that occur here.

The 3-normalizer $N_3(\mathrm{PU}(3))=C_{N(\mathrm{PU}(3))}(\check{T}(\mathrm{PU}(3))^{\langle\sigma\rangle})$ is the centralizer in $N(\mathrm{PU}(3))$ of $\check{T}(\mathrm{PU}(3))^{\langle\sigma\rangle}=E^1$. Since conjugation by $(0,\tau)$ restricts to the non-trivial automorphism of $\check{T}(\mathrm{PU}(3))^{\langle\sigma\rangle}$ we see that the induced action on $N_3(\mathrm{PU}(3))=T(\mathrm{PU}(3))\rtimes\langle\sigma\rangle$ is given by conjugation with $(0,\tau)\in\check{N}(\mathrm{PU}(3))=\check{T}(\mathrm{PU}(3))\rtimes W(\mathrm{PU}(3))$.

Since $\check{B}(u,v)\rtimes\chi\colon \check{N}_3(\mathrm{PU}(3))\to\check{N}(\mathrm{F}_4)$ is χ-equivariant with the Weyl groups acting by conjugation, we see (4.17) that $\mathbf{Z}/2$-acts on $\check{T}(\mathrm{F}_4)^{\langle\chi(\sigma)\rangle}=C_{\check{N}(\mathrm{F}_4)}(\check{N}_3(\mathrm{PU}(3)))$ as conjugation with $(0,\chi(\tau))$. With this information it is now easy to see, using [24, 10.7.5], that

$$\lim\nolimits^0_{\mathbf{I}}\pi_1=(\check{T}(\mathrm{F}_4)^{\langle\chi(\sigma)\rangle})^{\langle\chi(\tau)\rangle}=\check{T}(\mathrm{F}_4)^{\chi(W(\mathrm{SU}(3)))}$$

is the only non-trivial contribution from the $\mathbf{I}$-groups $\underline{\pi}_1$ and $\underline{\pi}_2$ to the Bousfield-Kan spectral sequence. This means that the morphisms

$$C_{\mathrm{F}_4}(e\overline{\psi}^{(u,v)}\mathrm{PU}(3))\to C_{\mathrm{F}_4}(N(e\overline{\psi}^{(u,v)})(N(\mathrm{PU}(3))))$$
$$\leftarrow C_{N(\mathrm{F}_4)}(N(e\overline{\psi}^{(u,v)})(N(\mathrm{PU}(3))))$$

are isomorphisms. Consider the corresponding group homomorphism $\mu\colon \check{T}(\mathrm{F}_4)^{\chi(W(\mathrm{SU}(3)))}\times\check{N}(\mathrm{SU}(3))\to\check{N}(\mathrm{F}_4)$ which is the inclusion on the first factor and equals $\check{N}(e\psi^{(u,v)})$ on the second factor. Since $\psi^{-1}\rtimes 1$ is inner on $\check{N}(\mathrm{F}_4)$, we have $\mu(1\times(\psi^{-1}\rtimes 1))=(\psi^{-1}\rtimes 1)\mu(1\times(\psi^{-1}\rtimes 1))=\mu(\psi^{-1}\times(1\rtimes 1))$ up to inner automorphism. This shows (4.17) that $C_{\mathrm{F}_4}(\psi^{-1}) = \psi^{-1}$ is the non-trivial automorphism of $C_{\mathrm{F}_4}(e\psi^{(u,v)}\mathrm{SU}(3)) = E^1$. $\square$

6. Non-toric morphisms of $\mathrm{PU}(3)$ to F_4

The non-toric morphisms of PU(3) to F_4 require special treatment. It is the object of this section to show that also the non-toric morphisms are T-determined, i.e. to complete the proof of the following theorem.

Theorem 6.1. 1. $\mathrm{Mono}(\mathrm{PU}(3), \mathrm{F}_4)$ *is T-determined.*

2. *The map*

$$\langle(-1,-1)\rangle \backslash \{(u,v) \in (\mathbf{Z}_3^*)^2 | u+v \in 3\mathbf{Z}_3\} \to \mathrm{Mono}(\mathrm{PU}(3), \mathrm{F}_4)$$
$$\pm(u,v) \to e\overline{\psi}^{(u,v)}$$

is a bijection.

Since the toric morphisms were dealt with in (5.4) only the non-toric ones need be considered in order to finish the proof of (6.1).

The first lemma, which is of a general nature, assures the existence of a kind of preferred lifts in certain situations.

Let G be a p-compact toral group sitting in short exact sequence $S \xrightarrow{i_1} G \to \pi_0(G)$ where S is a p-compact torus and $\pi_0(G)$ *cyclic* p-group. Let $j\colon N \to X$ be the maximal torus normalizer of a p-compact group, X, and let $i_2\colon T \to N$ be the inclusion of the identity component. Suppose that we are given a morphisms, B and f, such that the diagram

$$\begin{tikzcd} S \arrow[r, "B"] \arrow[d, "i_1"'] & T \arrow[d, "ji_2"] \\ G \arrow[r, "f"'] & X \end{tikzcd}$$

commutes up to conjugacy and B is admissible in the sense that for any $\xi \in \pi_0(G)$ there exists some w in the Weyl group for X such that $B\xi = wB$.

Lemma 6.2. *Assuming that the component group $\pi_0(G)$ is cyclic there is a unique representation $\phi \in \mathrm{Rep}(G, N)$ such that the diagram*

$$\begin{tikzcd} & & T \arrow[d, "i_2"] \\ & & N \arrow[d, "j"] \\ S \arrow[rruu, "B"] \arrow[r, "i_1"'] & G \arrow[ru, "\phi"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes up to conjugacy and such that the morphism

$$C_j\colon C_N(\phi G) \to C_X(fG),$$

induced by j, is a maximal torus normalizer for the centralizer $C_X(fG)$ of G in X.

Proof. The $\pi_0(G)$-map induced by j

$$\begin{tikzcd} BC_N(i_2BS)_{h\pi_0(G)} \arrow[rr] \arrow[dr] & & BC_X(ji_2BS)_{h\pi_0(G)} \arrow[dl] \\ & B\pi_0(G) & \end{tikzcd}$$

between the $\pi_0(G)$-spaces $BC_N(i_2BS) = \mathrm{map}(BS, BN)_{i_2B}$ and $BC_X(ji_2BS) = \mathrm{map}(BS, BX)_{ji_2B}$ is a maximal torus normalizer. There is an induced map

$$\begin{aligned}\mathrm{map}(BG, BN)_{i_1\to i_2B} = BC_N(i_2BS)^{h\pi_0(G)} \to BC_X(ji_2BS)^{h\pi_0(G)}\\ = \mathrm{map}(BG, BX)_{i_1\to ji_2B}\end{aligned} \tag{6.3}$$

of homotopy fixed point spaces.

According to [25, 4.6], the section $Bf \in BC_X(ji_2BS)^{h\pi_0(G)}$ admits, since $\pi_0(G)$ is assumed to be cyclic, a unique lift $B\phi \in BC_N(i_2BS)^{h\pi_0(G)}$ such that the restriction of (6.3) to the corresponding components,

$$BC_N(\phi G) = \mathrm{map}(BG, BN)_{B\phi} \to \mathrm{map}(BG, BX)_{Bf} = BC_X(fG)$$

is a maximal torus normalizer for the p-compact group $C_X(fG)$. $\square$

After these general and preparatory remarks, we now return to the discussion of non-toric morphisms from $\mathrm{PU}(3)$ to F_4.

Let $f\colon \mathrm{PU}(3) \to \mathrm{F}_4$ be a morphism of 3-compact groups such that $f|T(\mathrm{PU}(3)) = W(\mathrm{F}_4)B(2,1) \in [BT(\mathrm{PU}(3)), \mathrm{BF}_4]$. By (6.2), there is a unique $\phi(2,1) \in \mathrm{Rep}(N_3(\mathrm{PU}(3)), N(\mathrm{F}_4))$, extending $B(2,1)$, such that $C_{N(\mathrm{F}_4)}(\phi(2,1)N_3(\mathrm{PU}(3)))$ is a maximal torus normalizer for $C_{\mathrm{F}_4}(fN_3(\mathrm{PU}(3)))$. We shall now determine this map $\phi(2,1)$.

Let $\check{N}_3 = \check{T}_1 \rtimes \langle\sigma\rangle$ and $\check{N}_2 = \check{T}_2 \rtimes W_2$ be the discrete approximations to the the 3-normalizer $N_3(\mathrm{PU}(3))$ and the maximal torus normalizer $N(\mathrm{F}_4)$, respectively. Also, let $\check{B}(2,1)\colon \check{T}_1 \to \check{T}_2$ be a discrete approximation to $B(2,1)$. The stabilizer subgroup $W(\mathrm{F}_4)^{\check{B}(2,1)}$ at $\check{B}(2,1)$ for the action of $W(\mathrm{F}_4)$ on $\mathrm{Hom}(\check{T}_1, \check{T}_2)$ is isomorphic to the permutation group Σ_3 and generated by the two Weyl group elements

$$w_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of order 3 and 2, respectively.

Lemma 6.4. *The discrete approximation* $\check{\phi}(2,1)\colon \check{T}_1 \rtimes \langle\sigma\rangle \to \check{T}_2 \rtimes W(\mathrm{F}_4)$ *to* $\phi(2,1)$ *is conjugate to* $\check{B}(2,1) \rtimes \chi$.

Proof. For general reasons, the discrete approximation $\check{\phi}(2,1)$ to $\phi(2,1)$ has the form $\check{\phi}(2,1)(t,1) = (\check{B}(2,1)(t),1)$ and $\check{\phi}(2,1)(0,\sigma) = (a,\lambda(\sigma))$ where $\lambda\colon \langle\sigma\rangle \to W(\mathrm{F}_4)$ is a group homomorphism, $\check{B}(2,1)$ is λ-equivariant, and $a \in Z^1(\langle\lambda(\sigma)\rangle\,;\check{T}(\mathrm{F}_4))$ is a 1-cocycle.

Since the homomorphism $\check{B}(2,1)$ is χ-equivariant we know that $\lambda(\sigma)$ is an element of order 3 in the coset $\chi(\sigma)W_2^{\check{B}(2,1)}$. This leaves the three possibilities $\chi(\sigma), \chi(\sigma)w_1$, and $\chi(\sigma)w_1^2$ for $\lambda(\sigma)$. Since w_2 conjugates $\chi(\sigma)$ into $\chi(\sigma)w_1^2$ we can ignore the third possibility. We now rule out the second possibility.

Assume for the moment that $\lambda(\sigma) = \chi(\sigma)w_1$. Explicit computation shows that $H^0(\langle\chi(\sigma)w_1\rangle\,;\check{T}(\mathrm{F}_4))$ is a 3-discrete torus of rank 2 and that $H^0(\langle\chi(\sigma)w_1\rangle\,;\check{T}(\mathrm{F}_4))$ is cyclic of order 3 generated by the cohomology class of the 1-cocycle

$$a = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in t(\mathrm{F}_4) \subset \check{T}(\mathrm{F}_4)$$

which is fixed by $W(\mathrm{F}_4)^{\check{B}(2,1)}$. It follows that the centralizer

$$\begin{aligned}
&C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(\check{\phi}(2,1)\check{N}_3)\\
&\qquad = C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(\check{B}(2,1)\check{T}(\mathrm{PU}(3))) \cap C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(a,\chi(\sigma)w_1)\\
&\qquad = (\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)^{\check{B}(2,1)}) \cap C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(a,\chi(\sigma)w_1)\\
&\qquad = C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)^{\check{B}(2,1)}}(a,\chi(\sigma)w_1)\\
&\qquad = \check{T}(\mathrm{F}_4)^{\langle\chi(\sigma)w_1\rangle} \rtimes W(\mathrm{F}_4)^{\check{B}(2,1)}
\end{aligned}$$

is the (discrete) maximal torus normalizer for SU(3) and hence (6.2) that $C_{\mathrm{F}_4}(fN_3(\mathrm{PU}(3)))$ is isomorphic to the N-determined 3-compact group SU(3) [24, 1.2]. Thus $\phi(2,1)\colon N_3(\mathrm{PU}(3)) \to \mathrm{F}_4$ extends to a morphism $N_3(\mathrm{PU}(3))\times\mathrm{SU}(3) \to \mathrm{F}_4$ which is a non-toric monomorphism on the second factor and we get a factorization

$$N_3(\mathrm{PU}(3)) \to C_{\mathrm{F}_4}(\mathrm{SU}(3)) = \mathrm{SU}(3) \to \mathrm{F}_4$$

of $\phi(2,1)$ through another non-toric monomorphism of SU(3) to F_4. The restriction of this map to the maximal tori

$$T(\mathrm{PU}(3)) \to T(\mathrm{SU}(3)) \to T(\mathrm{F}_4)$$

provides a factorization, up to left action by $W(\mathrm{F}_4)$, of $B(2,1)$ as the composition of an isomorphism followed by $A(u,0)$ or $A(0,u)$, $u \in \mathbf{Z}_3^*$, and hence we have that the set

$$W(\mathrm{F}_4)\cdot A(2,1)\cdot \mathrm{GL}(\Sigma_0(\mathbf{Q}_3^3)) \subset \mathrm{Hom}_{\mathbf{Q}_3}(\Sigma_0(\mathbf{Q}_3^3),\mathbf{Q}_3^4)$$

contains $A(1,0)$ or $A(0,1)$. It is easy to verify, using a computer, that this is not the case, so we have arrived at a contradiction.

Thus $\lambda(\sigma) = \chi(\sigma)w_1$ can not occur and we are left with $\lambda(\sigma) = \chi(\sigma)$ as the only possibility. As $H^1(\langle\chi(\sigma)\rangle\,;\check{T}(\mathrm{F}_4)) = 0$ (5.5), $\check{\phi}(2,1) = \check{B}(2,1) \rtimes \chi$ is, up to conjugation, the only extension of the pair $(\check{B}(2,1),\chi)$ to a homomorphism $\check{T}_1 \rtimes \langle\sigma\rangle \to \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)$. □

A similar statement holds for the non-toric morphism $B(1,-1)$ which differs from $B(2,1)$ by an automorphism of F_4.

Proof of Theorem 6.1. It suffices to show that $f_1 \simeq f_2$ whenever $f_1, f_2 \colon \mathrm{PU}(3) \to \mathrm{F}_4$ are monomorphisms such that $f_1|T(\mathrm{PU}(3)) = W(\mathrm{F}_4)B(2,1) = f_2|T(\mathrm{PU}(3))$. We already know (6.4) that the two morphisms become conjugate when restricted to $N_3(\mathrm{PU}(3))$. Therefore, the situation is now exactly as in the proof of Theorem 5.4: In order to compute the relevant Wojtkowiak obstruction groups [29] we need information about the centralizer $C_{\mathrm{F}_4}(f_iE^2)$ and $C_{\mathrm{F}_4}(f_iN_3(\mathrm{PU}(3)))$.

Again, we must have $f_1|E^2 = e_5^2 = f_2|E^2$ and $C_{\mathrm{F}_4}(f_iE^2)$ is a p-compact toral group of maximal rank with $\mathbf{Z}/3$ as its component group (4.2).

Also, we know (6.2, 6.4) that the centralizer in $\check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)$ of $\check{\phi}(2,1)$ is the (discrete) maximal torus normalizer for $C_{\mathrm{F}_4}(f_iN_3(\mathrm{PU}(3)))$. Since

$$\begin{aligned} C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(\check{\phi}(2,1)\check{N}_3) &= C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(\check{B}(2,1)\check{T}_1) \cap C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(\chi(\sigma)) \\ &= (\check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)^{\check{B}(2,1)}) \cap (\check{T}(\mathrm{F}_4)^{\chi(\sigma)} \rtimes C_{W(\mathrm{F}_4)}(\chi(\sigma))) \\ &= \check{T}(\mathrm{F}_4)^{\chi(\sigma)} \rtimes C_{W(\mathrm{F}_4)^{\check{B}(2,1)}}(\chi(\sigma)) \\ &= t(\mathrm{F}_4)^{\chi(\sigma)} \rtimes \langle w_1\rangle \end{aligned}$$

is a finite group (of order 27 and with center of order 3) it follows that also $C_{\mathrm{F}_4}(f_iN_3(\mathrm{PU}(3)))$ is this finite, but non-abelian, 3-group.

The obstructions to a homotopy between the two maps $Bf_1, Bf_2 \colon B\mathrm{PU}(3) \to \mathrm{BF}_4$ lie in the set $\lim^1_{\mathbf{I}}\underline{\pi}_1$ and in the abelian group $\lim^2_{\mathbf{I}}\underline{\pi}_2$ where $\underline{\pi}_1$ and $\underline{\pi}_2$ are the $\mathbf{I}$-groups

$$\mathbf{Z}/2 \circlearrowright \pi \xrightarrow{\mathrm{SL}(2,\mathbf{F}_3)/S_3} E^1 \circlearrowleft \mathrm{SL}(2,\mathbf{F}_3)$$

$$\mathbf{Z}/2 \circlearrowright 0 \xrightarrow{\mathrm{SL}(2,\mathbf{F}_3)/S_3} \mathbf{Z}_3^4 \circlearrowleft \mathrm{SL}(2,\mathbf{F}_3)$$

given by the homotopy groups of the above centralizers, e.g. $\pi = t(\mathrm{F}_4)^{\chi(\sigma)} \rtimes \langle w_1\rangle$. The group $\lim^2_{\mathbf{I}}\underline{\pi}_2$ is trivial for general reasons [24, 10.7.5]. That also $\lim^1_{\mathbf{I}}\underline{\pi}_1 = *$ follows from (6.5) below since both the central $\mathbf{I}$-subgroup

$$\mathbf{Z}/2 \circlearrowright 0 \longrightarrow \mathbf{Z}/3 \circlearrowleft \mathrm{SL}(2,\mathbf{F}_3)$$

as well as the quotient **I**-group

$$\mathbf{Z}/2 \circlearrowright \pi \longrightarrow 0 \circlearrowleft \mathrm{SL}(2,\mathbf{F}_3)$$

where $\mathrm{SL}(2,\mathbf{F}_3)$ necessarily acts trivially, have vanishing $\lim^1$ by [24, 10.7] and (6.6). □

The following observations were used to compute the non-abelian $\lim^1$.

Let **I** be a small category. Define an **I**-group to be a functor from the category **I** to the category of groups. Let $A \to E \to G$ be a central extension of **I**-groups meaning that A, E, and G are **I**-groups, the arrows are natural transformations, and that the evaluation at each object of **I** yields a central extension of groups.

Lemma 6.5. *Any central extension of* **I***-groups* $A \to E \to G$ *induces an exact sequence*

$$* \to \lim^0_{\mathbf{I}} A \to \lim^0_{\mathbf{I}} E \to \lim^0_{\mathbf{I}} G \to \lim^1_{\mathbf{I}} A \to \lim^1_{\mathbf{I}} E \to \lim^1_{\mathbf{I}} G \to \lim^2_{\mathbf{I}} A$$

of sets. Moreover, the fibres of the map $\lim^1_{\mathbf{I}} E \to \lim^1_{\mathbf{I}} G$ *are precisely the orbits for an induced action of the abelian group* $\lim^1_{\mathbf{I}} A$ *on the set* $\lim^1_{\mathbf{I}} E$*.*

Corollary 6.6. *Let* **I** *be a finite group acting on a finite group* π*. If the* π *is a* p*-group and* p *does not divide the order of* **I***, then* $\lim^1_{\mathbf{I}} \pi = *$*.*

Proof. This follows, using the preceding lemma, by induction over the order of π since any non-trivial p-group has a non-trivial center. □

Proof of Theorem 1.1. Modulo the action of the Weyl group $W_{\mathrm{F}_4}(\mathrm{SU}(3,3))$ of order two (4.15), the sets

$$\mathrm{Rep}(\mathrm{SU}(3),\mathrm{SU}(3,3)) = \{0\} \cup \mathrm{Mono}(\mathrm{SU}(3),\mathrm{SU}(3,3)) \cup \mathrm{Mono}(\mathrm{PU}(3),\mathrm{SU}(3,3))$$

and

$$\mathrm{Rep}(\mathrm{SU}(3),\mathrm{F}_4) = \{0\} \cup \mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4) \cup \mathrm{Mono}(\mathrm{PU}(3),\mathrm{F}_4)$$

are (4.13, 6.1) in correspondence. □

Lemma 6.7. *Let* $(u,v) \in \mathbf{Z}_3^*(2,1) \cup \mathbf{Z}_3^*(1,-1)$*. Then*

$$C_{\mathrm{F}_4}(e\psi^{(u,v)}\mathrm{SU}(3)) = \check{T}(\mathrm{F}_4)^{\chi(W(\mathrm{SU}(3)))}$$
$$C_{\mathrm{F}_4}(e\psi^{(u,v)}T(\mathrm{SU}(3))) = T(\mathrm{SU}(3)) \times_{Z(\mathrm{SU}(3))} \mathrm{SU}(3)$$

and $C_{\mathrm{F}_4}(\psi^{-1}) = \psi^{-1}$ *in both cases.*

Proof. We shall apply the Bousfield-Kan spectral sequence [4] to $\mathrm{map}(\mathrm{BPU}(3),\mathrm{BF}_4)_{e\overline{\psi}^{(u,v)}}$ where $\mathrm{BPU}(3)$ is viewed as the homotopy colimit of the **I**-space

$$\mathbf{Z}/2 \circlearrowright B(0) \xrightarrow{\mathrm{SL}(2,\mathbf{F}_3)/S_3} B(1) \circlearrowleft \mathrm{SL}(2,\mathbf{F}_3) \tag{6.8}$$

where $B(0) = BC_{\mathrm{F}_4}(e\overline{\psi}^{(u,v)}N_3(\mathrm{PU}(3)))$ and $B(1) = BC_{\mathrm{F}_4}(e_5^2)$. It represents no loss of generality to assume that $(u,v) = (2,1)$.

As we saw in the proof of (5.7), $\mathbf{Z}/2$-acts on $\check{N}_3(\mathrm{PU}(3)) = \check{T}(\mathrm{PU}(3)) \rtimes \langle\sigma\rangle$ as conjugation with $(0,\tau) \in \check{N}(\mathrm{PU}(3)) = \check{T}(\mathrm{PU}(3)) \rtimes W(\mathrm{PU}(3))$. But this is again the restriction to

$$\check{\varphi}(2,1)(\check{N}_3(\mathrm{PU}(3))) \subset \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)$$

of conjugation by $(0,\chi(\tau))$. Thus (4.18) the $\mathbf{Z}/2$-action on

$$C_{\mathrm{F}_4}(e\overline{\psi}^{(2,1)}N_3(\mathrm{PU}(3))) = C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(\check{\varphi}(2,1)\check{N}_3) = \check{T}(\mathrm{F}_4)^{\langle\sigma\rangle} \rtimes \langle w_1\rangle$$

is through conjugation with $(0,\chi(\tau))$.

Note also that the multiplication map

$$\mu\colon C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(\check{\varphi}(2,1)\check{N}_3) \times \check{N}_3 \to \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)$$

satisfies

$$\mu(\psi^{-1}\times 1) = \psi^{-1}\mu(\psi^{-1}\times 1) = \mu(1\times N_3(\psi^{-1}))$$

up to inner automorphism. This means that the induced action on $C_{\mathrm{F}_4}(e\overline{\psi}^{(2,1)}N_3(\mathrm{PU}(3)))$ is $C_{\mathrm{F}_4}(N_3(\psi^{-1})) = \psi^{-1} \rtimes 1$.

Recall from [5] that there is an essentially unique monomorphism $\iota\colon \mathrm{DI}_2 \to \mathrm{F}_4$ inducing a monomorphism $t(\iota)\colon t(\mathrm{DI}_2) \to t(\mathrm{F}_4)$ and a group monomorphism $\chi\colon\ \mathrm{GL}(2,\mathbf{F}_3) = W(\mathrm{DI}_2) \to W(\mathrm{F}_4)$ extending (4.7). Now, $t(\iota)$ is isomorphic to e_5^2 and from the commutative diagram

$$\begin{tikzcd} t(\mathrm{DI}_2) \arrow[r, "w^{-1}"] \arrow[d, "t(\iota)"'] & t(\mathrm{DI}_2) \arrow[d, "t(\iota)"] \\ \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4) \arrow[r, "(0,\chi(w))"'] & \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4) \end{tikzcd}$$

we see (4.18) that $w \in \mathrm{GL}(2,\mathbf{F}_3)$ acts on $C_{\check{N}(\mathrm{F}_4)}(t(\mathrm{DI}_2)) = \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)^{t(\mathrm{DI}_2)}$ as conjugation with the element $(0,\chi(w))$ of the semi-direct product. The restriction to $\mathrm{SL}(2,\mathbf{F}_3)$ of this action gives the action on $C_{N(\mathrm{F}_4)}(t(\mathrm{DI}_2)) = C_{\mathrm{F}_4}(t(\mathrm{DI}_2))$ in (6.8).

The conclusion of this is that

$$\lim\nolimits^0_{\mathbf{I}}\underline{\pi}_1 = (\check{T}(\mathrm{F}_4)^{\langle\chi(\sigma)\rangle} \rtimes \langle w_1\rangle)^{\langle\chi(\tau)\rangle} = \check{T}(\mathrm{F}_4)^{\chi(W(\mathrm{SU}(3)))}$$

is the only non-trivial contribution from the groups $\lim^{-i}_{\mathbf{I}}\underline{\pi}_j$, $i+j \geq 0$, of the Bousfield-Kan spectral sequence. Consequently, $C_{\mathrm{F}_4}(e\psi^{(2,1)}\mathrm{SU}(3))$ is isomorphic to this group of order 3. The action of $C_{\mathrm{F}_4}(\psi^{-1})$, which is the restriction of the action of $C_{\mathrm{F}_4}(N_3(\psi^{-1}))$, is given by ψ^{-1}.

The centralizer

$$C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(e\psi^{(2,1)}\check{T}(\mathrm{SU}(3))) = \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)^{A(2,1)}$$

is the (discrete) maximal torus normalizer for $C_{\mathrm{F}_4}(e\psi^{(2,1)}\check{T}(\mathrm{SU}(3)))$ and the centralizer

$$C_{\check{T}(\mathrm{F}_4)\rtimes W(\mathrm{F}_4)}(e\psi^{(0,1)}\check{T}(\mathrm{SU}(3))) = \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)^{A(0,1)}$$

is the (discrete) maximal torus normalizer for $C_{\mathrm{F}_4}(e\psi^{(0,1)}\check{T}(\mathrm{SU}(3))) = \mathrm{SU}(3) \times_{Z(\mathrm{SU}(3))} T(\mathrm{SU}(3))$ [22, 3.4.3]. Since the two stabilizer subgroups $W(\mathrm{F}_4)^{A(2,1)}$ and $W(\mathrm{F}_4)^{A(0,1)}$ are conjugate in $W(\mathrm{F}_4)$, the two maximal torus normalizers are isomorphic and hence the two centralizers are isomorphic, too, by N-determinism [23] [24].

The group homomorphism $\mu\colon (\check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)^{A(2,1)} \to \check{T}(\mathrm{F}_4) \rtimes W(\mathrm{F}_4)$ which is the inclusion on the first factor and equals $A(2,1)$ on the second factor satisfies

$$\mu((1 \rtimes 1) \times \psi^{-1}) = (\psi^{-1} \rtimes 1)\mu((1 \rtimes 1) \times \psi^{-1}) = \mu((\psi^{-1} \rtimes 1) \times 1$$

up to inner automorphisms. This shows (4.17) that $C_{\mathrm{F}_4}(\psi^{-1}) = \psi^{-1}$. □

7. Morphisms from G_2 to F_4 at the prime $p = 3$

Using the Jackowski-McClure decomposition of $\mathrm{B}\,\mathrm{G}_2$ and the Bousfield-Kan spectral sequence we classify morphisms $\mathrm{G}_2 \to \mathrm{F}_4$ viewed as 3-compact groups and compute their centralizers.

The Weyl group of G_2, $W(\mathrm{G}_2) < \mathrm{GL}(\Sigma_0(\mathbf{Z}_3^3))$ is the product of the Weyl group $W(\mathrm{SU}(3)) = \langle \sigma, \tau \rangle$ of $\mathrm{SU}(3)$ and the central group $\langle -1 \rangle$ of order 2. The group morphism χ from (4.7) extends to a group homomorphism $\chi\colon W(\mathrm{G}_2) \to W(\mathrm{F}_4)$ simply by putting $\chi(-1) = -1$. Let $\mathbf{I} = \mathbf{I}(W(\mathrm{G}_2), W(\mathrm{SU}(3)))$ denote the category

$$\begin{tikzcd} 0 \arrow[loop left, "\langle -1 \rangle"] \arrow[r, "W(\mathrm{G}_2)/W(\mathrm{SU}(3))"] & 1 \arrow[loop right, "W(\mathrm{G}_2)"] \end{tikzcd}$$

of the central inclusion of $W(\mathrm{SU}(3))$ into $W(\mathrm{G}_2)$. Then $\mathrm{B}\,\mathrm{G}_2$ is [24, §7] $H^*\mathbf{F}_3$-equivalent to the homotopy colimit of an $\mathbf{I}^{\mathrm{op}}$-space

$$\begin{tikzcd} B(0) \arrow[loop left, "\langle \psi^{-1} \rangle"] & B(1) \arrow[l, "W(\mathrm{SU}(3))^{\mathrm{op}} \backslash W(\mathrm{G}_2)^{\mathrm{op}}"'] \arrow[loop right, "W(\mathrm{G}_2)^{\mathrm{op}}"] \end{tikzcd} \tag{7.1}$$

where $B(0) = \mathrm{BSU}(3)$ and $B(1) = BT(\mathrm{SU}(3))$.

Theorem 7.2. *The restriction map*

$$\mathrm{Rep}(\mathrm{G}_2, \mathrm{F}_4) \to \mathrm{Rep}(\mathrm{SU}(3), \mathrm{F}_4)$$

is bijective. The centralizer $C_{\mathrm{F}_4}(e\psi^{(u,v)}\mathrm{G}_2)$, $u, v \in \mathbf{Z}_3^* \cup \{0\}$, is isomorphic to $\mathrm{SU}(2)$ *if $uv = 0$ and trivial otherwise.*

Proof. We must show that any morphism $\mathrm{SU}(3) \to \mathrm{F}_4$ extends uniquely to G_2. Since this is true for the trivial morphism by [22, 6.7], we only need here to consider non-trivial morphisms.

Let $(u, v) \in (\mathbf{Z}_3^* \cup \{0\})^2$, $(u, v) \neq (0, 0)$. Since $e\psi^{(u,v)}\colon \mathrm{SU}(3) \to \mathrm{F}_4$ is invariant under ψ^{-1}, this map $e\psi^{(u,v)}$ and its restriction to the maximal torus form a homotopy coherent set of maps out of the $\mathbf{I}^{\mathrm{op}}$-space (7.1). Thus it suffices to show that $\lim_{\mathbf{I}}^{-i} \underline{\pi}_j(u, v) = 0$ for $i + j \geq -1$ where $\underline{\pi}_j(u, v)$ is the $\mathbf{I}$-group

$$\begin{tikzcd} \pi_j(0) \arrow[loop left, "\mathbf{Z}/2"] \arrow[r, "W(\mathrm{G}_2)/W(\mathrm{SU}(3))"] & \pi_j(1) \arrow[loop right, "W(\mathrm{G}_2)"] \end{tikzcd}$$

where the group $\pi_j(0) = \pi_j(u,v)(0) = \pi_j(BC_{\mathrm{F}_4}(e\psi^{(u,v)}\mathrm{SU}(3)))$ and the group $\pi_j(1) = \pi_j(u,v)(1) = \pi_j(BC_{\mathrm{F}_4}(e\psi^{(u,v)}T(\mathrm{SU}(3))))$. Since the abelian **I**-groups $\underline{\pi}_j(u,v)$ are in fact $\mathbf{Z}_3[\mathbf{I}]$-modules and $W(\mathrm{SU}(3))$ is normal in $W(\mathrm{G}_2)$, it follows from [24, 10.7.5] that $\lim^0_{\mathbf{I}}\underline{\pi}_j(u,v) = \pi_j(u,v)(0)^{\mathbf{Z}/2} = \pi_j(BC_{\mathrm{F}_4}(e\psi^{(u,v)}\mathrm{SU}(3)))^{\mathbf{Z}/2}$ is the subgroup that is invariant under the action of ψ^{-1} and that the higher limits are automatically trivial. By (4.16, 5.7, 6.7), $\pi_j(u,v)(0)^{\mathbf{Z}/2}$ is trivial except when either $u=0$ or $v=0$ when it equals the invariants $\pi_j(\mathrm{BSU}(3))^{\langle B\psi^{-1}\rangle}$.

We now examine the case $(u,v)=(0,1)$ more closely. According to Dynkin [12, 13] the Lie group F_4 contains a copy of (a central quotient of) $SU(2)\times\mathrm{G}_2$. The restriction to G_2 of this inclusion $\mathrm{SU}(2)\times\mathrm{G}_2\to\mathrm{F}_4$ equals, up to an automorphism of F_4, the map $e\psi^{(0,1)}$ for otherwise the restriction to the other factor, the inclusion of $\mathrm{SU}(2)$ into F_4, would factor through the trivial 3-compact group. The homotopy class of the restriction

$$\mathrm{BSU}(2)\times\mathrm{BSU}(3)\to\mathrm{BSU}(2)\times\mathrm{BG}_2\to\mathrm{BF}_4$$

to $\mathrm{SU}(2)\times\mathrm{SU}(3)$ is determined by its adjoint in $\pi_0(\mathrm{map}(\mathrm{BSU}(2),\mathrm{map}(\mathrm{BSU}(3),\mathrm{BF}_4)_{B(e\psi^{(0,1)})}) = \pi_0(\mathrm{map}(\mathrm{BSU}(2),\mathrm{BSU}(3))) = \mathrm{Rep}(\mathrm{SU}(2),\mathrm{SU}(3))$ so. Since $\mathrm{SU}(3)$ contains (7.3) an essentially unique copy of $\mathrm{SU}(2)$, we conclude that the diagram of 3-compact groups

$$\begin{tikzcd}
\mathrm{SU}(2)\times\mathrm{SU}(3) \arrow[r, "S\iota(2,3)\times 1"] \arrow[d] & \mathrm{SU}(3)\times\mathrm{SU}(3) \arrow[d] \\
\mathrm{SU}(2)\times\mathrm{G}_2 \arrow[r] & \mathrm{F}_4
\end{tikzcd}$$

commutes up to conjugacy. After taking adjoint maps we end up with

$$\begin{tikzcd}
\mathrm{BSU}(2) \arrow[r] \arrow[d, "BS\iota(2,3)"'] & \mathrm{map}(\mathrm{BG}_2,\mathrm{BF}_4)_{B(e\psi^{(0,1)})} \arrow[d] \\
\mathrm{BSU}(3) \arrow[r, "\simeq"] & \mathrm{map}(\mathrm{BSU}(3),\mathrm{BF}_4)_{B(e\psi^{(0,1)})}
\end{tikzcd}$$

which commutes up to homotopy and where the lower horizontal arrow represents (4.16) a homotopy equivalence homotopy equivariant under the action $\langle B\psi^{-1}\rangle$. By the above computations with the Bousfield-Kan spectral sequence,

$$\pi_*(\mathrm{map}(\mathrm{BG}_2,\mathrm{BF}_4),B(e\psi^{(0,1)})) = \pi_*(\mathrm{map}(\mathrm{BSU}(3),\mathrm{BF}_4),B(e\psi^{(0,1)}))^{\langle B\psi^{-1}\rangle},$$

and linked with (7.4) this shows that the upper horizontal map is a homotopy equivalence as well. □

The morphism $e\psi^{(u,v)}\colon \mathrm{G}_2\to\mathrm{F}_4$ where $u,v\in\mathbf{Z}_3^*$ with sum $u+v\in 3\mathbf{Z}_3$, is an example a non-trivial non-monomorphism defined on a center-free 3-compact group.

The following two results were needed for the proof of Theorem 7.2.

Lemma 7.3. *Let* $S\iota(2,3)\colon \mathrm{SU}(2) \to \mathrm{SU}(3)$ *be the canonical inclusion. The map*

$$\begin{aligned}\mathrm{Rep}(\mathrm{SU}(2),\mathrm{SU}(2)) &\to \mathrm{Rep}(\mathrm{SU}(2),\mathrm{SU}(3))\\ \psi^u &\to S\iota(2,3)\psi^u\end{aligned}$$

is a bijection that identifies $\mathrm{Out}(\mathrm{SU}(2)) = \mathbf{Z}_3^*/\langle -1\rangle$ *and* $\mathrm{Mono}(\mathrm{SU}(2),\mathrm{SU}(3))$.

Proof. This follows from (3.13) that identifies both $\mathrm{Rep}(\mathrm{SU}(2),\mathrm{SU}(2))$ and $\mathrm{Rep}(\mathrm{SU}(2),\mathrm{SU}(3))$ to $\mathbf{Z}_3/\langle -1\rangle$. □

Since $\psi^{-1}S\iota(2,3) = S\iota(2,3)\psi^{-1} = S\iota(2,3)$, the image of $\pi_*(\mathrm{BSU}(2))$ in $\pi_*(\mathrm{BSU}(3))$ is invariant under the action of the group $\langle B\psi^{-1}\rangle$.

Lemma 7.4. *There is an isomorphism, induced by* $S\iota(2,3)$*,*

$$\pi_*(\mathrm{BSU}(2)) \to \pi_*(\mathrm{BSU}(3))^{\langle B\psi^{-1}\rangle}$$

between the homotopy of $\mathrm{BSU}(2)$ *and the* $\langle B\psi^{-1}\rangle$*-invariant subgroup of the homotopy of* $\mathrm{BSU}(3)$.

Proof. There is a short exact sequence of homotopy groups

$$0 \to \pi_*(\mathrm{SU}(2)) \to \pi_*(\mathrm{SU}(3)) \to \pi_*(S^5) \to 0$$

of $\mathbf{F}_3$-complete spaces induced by the fibration of $\mathrm{SU}(3)$ onto S^5 with fibre $\mathrm{SU}(2)$. This fibration splits since $\pi_4(\mathrm{SU}(2))\otimes\mathbf{Z}_3 = 0$. The homomorphism ψ^{-1}, complex conjugation of matrices, restricts to the identity on the fibre and induces the degree -1-map on the base. Using that the 3-completion of S^5 is an H-space we see that the degree -1 self-map induces multiplication by -1 on the homotopy groups $\pi_*(S^5)\otimes\mathbf{Z}_3$ and the claim follows. □

8. Morphisms from $\mathrm{SU}(3)$ to G_2 at the prime $p = 3$

The classification of morphisms $\mathrm{SU}(3) \to \mathrm{G}_2$ of 3-compact groups proceeds very much like the classification of morphisms $\mathrm{SU}(3) \to \mathrm{F}_4$.

Lemma 8.1. *The set* $\mathrm{Mono}(E^1,\mathrm{G}_2)$ *contains two elements,* e^1_1*,* e^1_2*, with centralizer Weyl groups of order* 2, 6*, and Quillen automorphism groups of order* 2, 2*, respectively. The centralizer* $C_{\mathrm{G}_2}(e^1_2)$ *is isomorphic to* $\mathrm{SU}(3)$.

The set $\mathrm{Mono}(E^2,\mathrm{G}_2)/\mathrm{Aut}(E^2)$ *contains a unique element,* $e^2_2 = t(\mathrm{G}_2)$*, with Quillen automorphism group* $W(\mathrm{G}_2)$ *of order* 12.

Let $\chi_1\colon W(\mathrm{SU}(3)) \to W(\mathrm{G}_2)$ be the inclusion and $\chi_2\colon W(\mathrm{SU}(3)) \to W(\mathrm{G}_2)$ the injection given by $\chi_2(\sigma) = \sigma$ and $\chi_2(\tau) = -\tau$. Then the identity map $A_1\colon \Sigma_0(\mathbf{Z}_3^3) \to \Sigma_0(\mathbf{Z}_3^3)$ is χ_1-equivariant and the $\mathbf{Z}_3$-linear map $A_2\colon \Sigma_0(\mathbf{Z}_3^3) \to \Sigma_0(\mathbf{Z}_3^3)$ with matrix

$$A_2 = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

is χ_2-equivariant.

Lemma 8.2. *A $\mathbf{Z}_3$-linear map $\Sigma_0(\mathbf{Z}_3^3) \to \Sigma_0(\mathbf{Z}_3^3)$ is admissible with respect to $W(\mathrm{SU}(3))$ and $W(\mathrm{G}_2)$ is and only it belongs to $W(\mathrm{G}_2)(uA_1)$ or $W(\mathrm{G}_2)(uA_2)$ for some scalar $u \in \mathbf{Z}_3$.*

Proof. Computerized calculations show that any admissible homomorphism must, up to inner automorphisms, be either χ_1- or χ_2-equivariant. Next, one solves the two systems of linear equations $Aw = \chi_i(w)A$, $w \in W(\mathrm{SU}(3))$, $i = 1, 2$. □

Proposition 8.3. *Any non-trivial morphism $f\colon \mathrm{SU}(3) \to \mathrm{G}_2$ is a monomorphism.*

Proof. Let $f\colon \mathrm{SU}(3) \to \mathrm{G}_2$ be any non-trivial morphism and $T(f)\colon T(\mathrm{SU}(3)) \to T(\mathrm{G}_2)$ a lift of f to the maximal tori. Then $W(\mathrm{G}_2)\pi_1(T(f))$ equals $W(\mathrm{G}_2)(uA_1)$ or $W(\mathrm{G}_2)(uA_2)$ for some 3-adic integer, u. In fact, since the order of $W(\mathrm{SU}(3))$ is divisible by 3, u must be a unit (3.19). In the first case, $W(\mathrm{G}_2)\pi_1(T(f)) = W(\mathrm{G}_2)(uA_1)$, f is a monomorphism. And if $W(\mathrm{G}_2)\pi_1(T(f)) = W(\mathrm{G}_2)(uA_2)$, the kernel of $\check{T}(f)$ equals the center of $\mathrm{SU}(3)$ and f factors through a monomorphism $\overline{f}\colon \mathrm{PU}(3) \to \mathrm{G}_2$. However, such a monomorphism can not exist since the Quillen category of $\mathrm{PU}(3)$ contains an object $E^2 \rightarrowtail \mathrm{PU}(3)$ with Quillen automorphism group $\mathrm{SL}(2, \mathbf{F}_3)$ of order 24 exceeding the order of the Quillen automorphism group of $e_2^2 \in \mathrm{Mono}(E^2, \mathrm{G}_2)$. □

Consider now the diagram

$$\begin{tikzcd}
E^1 \arrow[d, "z"'] \arrow[dr, "z"'] \arrow[drr, "e_2^1"] & & \\
\mathrm{SU}(3) & \mathrm{SU}(3) \arrow[r, "e"'] & \mathrm{G}_2
\end{tikzcd}$$

where the $\mathrm{SU}(3)$ to the right stands for $C_{\mathrm{G}_2}(e_2^1)$ and z stands for center. Here, $e\psi^{-1} = e$ since $C_{\mathrm{G}_2}(\psi^{-1}) = \psi^{-1}$.

Lemma 8.4. *For any monomorphism $f\colon \mathrm{SU}(3) \to \mathrm{G}_2$, $fz = e_2^1$.*

Proof. Since $\pi_1(T(f)) = uA_1$, $u \in \mathbf{Z}_3^*$, the reduction mod 3, $t(f)\colon t(\mathrm{SU}(3)) \to t(\mathrm{G}_2)$, takes the center, $(1, -1)$, of $\mathrm{SU}(3)$ to the element $u(1, -1) \in t(\mathrm{G}_2)$ whose stabilizer subgroup is $W(\mathrm{SU}(3))$. □

It follows (3.9) that

$$\mathrm{Mono}(\mathrm{SU}(3), \mathrm{SU}(3))_{z \to z} = \mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2)_{z \to e_2^1} = \mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2)$$

or, alternatively, that the map

$$\begin{aligned} \langle -1 \rangle \backslash \mathbf{Z}_3^* &\to \mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2) \\ \pm u &\to e\psi^u \end{aligned}$$

is a bijection. Also, any monomorphism $f\colon \mathrm{SU}(3) \to \mathrm{G}_2$ is centric [7] in the sense that the map given by composition with Bf,

$$\mathrm{map}(B\mathrm{SU}(3), B\mathrm{SU}(3))_{B1} \to \mathrm{map}(B\mathrm{SU}(3), B\,\mathrm{G}_2)_{Bf}$$

is a homotopy equivalence. Clearly, f is toric as well (2.12).

Theorem 8.5. 1. $\mathrm{Rep}(\mathrm{SU}(3), \mathrm{G}_2) = \{0\} \cup \mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2)$ *is* T*-determined.*

2. *The action*

$$\mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2) \times \mathrm{Out}(\mathrm{SU}(3)) \to \mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2)$$

is transitive and the stabilizer at $f \in \mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2)$ *equals* $W_{\mathrm{G}_2}(f\mathrm{SU}(3)) = \langle \psi^{-1} \rangle$.

Proof. This is clear from the explicit description of the set $\mathrm{Rep}(\mathrm{SU}(3), \mathrm{G}_2)$. For instance, the restriction map

$$\mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2) \to \mathrm{Mono}(T(\mathrm{SU}(3)), \mathrm{G}_2)$$

can be identified to the map

$$\{u \in \mathbf{Z}_3^* \mid u \equiv 1 \bmod 3\} \to W(\mathrm{G}_2)(uA_1)$$

which clearly is injective. □

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