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Reviewed work(s):
Source: *Transactions of the American Mathematical Society*, Vol. 292, No. 2 (Dec., 1985), pp. 721-732
Published by: American Mathematical Society
Stable URL: http://www.jstor.org/stable/2000242

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 292, Number 2, December 1985

RATIONAL HOMOTOPY OF SPACES OF MAPS INTO SPHERES AND COMPLEX PROJECTIVE SPACES

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ABSTRACT. We investigate the rational homotopy classification problem for the components of some function spaces with S^n or $\mathbf{C}P^n$ as target space.

1. Introduction. For any pair of topological spaces X and Y, let $F(X,Y)$ be the space of all (free, continuous) maps of X into Y. In general, $F(X,Y)$ is a disconnected space; so for any map $f\colon X \to Y$, we let $F_f(X,Y) \subset F(X,Y)$ denote the (path-) component that contains f. A fundamental problem is to classify the components of $F(X,Y)$ up to homotopy type; initiated by the results of V. L. Hansen [**5**, **6**] on the case $Y = S^n$, this has been the subject of a number of papers [**1**, **10**, **11**, **15**] concerning the cases $Y = S^n$, $\mathbf{R}P^n$ or $\mathbf{C}P^n$. In this paper we discuss the homotopy classification problem from the point of view of rational homotopy theory restricting ourselves to the case where $Y = S^n$, $\mathbf{C}P^n$ is an n-sphere or complex projective n-space.

Our main result for the case where $Y = S^n$ is an n-sphere may be stated as follows.

THEOREM 1. *If n is odd, then all components of $F(X,S^n)$ are rationally homotopy equivalent.*

In the case where n is even, fix a generator $s_n \in H^n(S^n;\mathbf{Q})$ of $H^n(S^n;\mathbf{Z}) \subset H^n(S^n;\mathbf{Q})$. Let $f,g\colon X \to S^n$ be maps of X into S^n and let $f^*(s_n), g^*(s_n) \in H^n(X;\mathbf{Q})$ be the images of s_n in rational cohomology. Denote the constant map by $0\colon X \to S^n$.

THEOREM 2. *Assume that $n \geq 2$ is an even integer and that X is a connected and rationally $(2n-1)$-coconnected space, i.e. $\tilde{H}^0(X;\mathbf{Q}) = 0 = H^{\geq 2n-1}(X;\mathbf{Q})$. Then*

(1) *$F_f(X,S^n) \simeq_{\mathbf{Q}} F_0(X,S^n)$ if and only if $f^*(s_n) = 0$.*

(2) *$F_f(X,S^n) \simeq_{\mathbf{Q}} F_g(X,S^n)$ if there exist an algebra automorphism φ on $H^*(X;\mathbf{Q})$ and a rational number $t \neq 0$ such that $tf^*(s_n) = \varphi g^*(s_n)$.*

As a very special case we emphasize the following consequence which shows the connection with [**5**, **6**].

COROLLARY. *Let M^n be a closed n-manifold, $n \geq 1$. The components of $F(M^n,S^n)$ represent two rational homotopy types if M^n is orientable and n is even and one otherwise.*

Received by the editors March 24, 1984 and, in revised form, February 18, 1985.
1980 *Mathematics Subject Classification*. Primary 55P62, 55P15, 58D99.
Key words and phrases. Path-components of mapping spaces, minimal model, rational homotopy equivalence.

Actually, the investigations presented in this paper were prompted by the observation that the invariant used in [**5**, **6**, **15**] to distinguish between components associated to positive degrees is a torsion group, so that one might expect these components to be identical spaces in the rational category. The Corollary above is the affirmative answer to this conjecture.

We prove Theorem 2 by constructing (minimal) models for the components of $F(X,S^n)$ using a method of Haefliger's [**4**]. The applied technique works in fact whenever the target space Y is a 2-stage Postnikov tower. As an illustration of this, we consider in §3 spaces of maps into complex (and quaternionic) projective n-space (see Corollary 3.3 for an analogue of Theorem 2).

In order to assure e.g. continuity of evaluation maps, we shall work in the category of compactly generated spaces. Thus for any pair of compactly generated spaces X and Y, $F(X,Y)$ is equipped with the compactly generated topology associated with the compact-open topology (cf. [**18**, pp. 17–21]). Furthermore, we assume throughout that the domain space X is finite dimensional and of finite type; in particular, $\beta_i(x) = \dim_{\mathbf{Q}} H^i(X;\mathbf{Q}) < \infty$ for all $i \geq 0$. For short, we write $H^*(X)$ for $H^*(X;\mathbf{Q})$ and $H_*(X)$ for $H_*(X;\mathbf{Q})$. Our main references for rational homotopy theory are [**12**, **14**, **16**].

2. Minimal models for the components of $F(X,S^n)$. Note first that any component $F_f(X,S^n)$ of $F(X,S^n)$ is a nilpotent space by Theorem A of [**9**] and thus has a well-defined rational homotopy type. Moreover [**9**, Theorem B], its rationalization is given by

$$F_f(X,S^n)_{(0)} = F_{rf}(X,S^n_{(0)}),$$

where $r\colon S^n \to S^n_{(0)}$ is the rationalization map. Thus, Theorems 1 and 2 are immediate consequences of the following two theorems.

THEOREM 2.1. *If $n \geq 1$ is odd, then all components of $F(X,S^n_{(0)})$ are homotopy equivalent.*

THEOREM 2.2. *Assume that $n \geq 2$ is even and that X is a connected and rationally $(2n-1)$-coconnected space. Let $\lambda \in H^n(X)$ be an n-dimensional rational cohomology class. Then*

(1) $F_\lambda(X,S^n_{(0)}) \simeq_{\mathbf{Q}} F_0(X,S^n_{(0)})$ *if and only if* $\lambda = 0$.
(2) $F_\lambda(X,S^n_{(0)}) \simeq_{\mathbf{Q}} F_{t\lambda}(X,S^n_{(0)})$ *for any rational number* $t \neq 0$.
(3) $F_\lambda(X,S^n_{(0)}) \simeq_{\mathbf{Q}} F_{\varphi\lambda}(X,S^n_{(0)})$ *for any algebra automorphism* φ *on* $H^*(X)$.

In the formulation of Theorem 2.2 we write $F_\lambda(X,S^n_{(0)})$ for $F_f(X,S^n_{(0)})$ when $\lambda = f^*(s_n)$, using the fact that, under the assumptions made, the components of $F(X,S^n_{(0)})$ are classified by $H^n(X)$.

The proof of Theorem 2.1 is very easy. For if n is odd, then the rationalized n-sphere $S^n_{(0)} = K(\mathbf{Q},n)$ is an Eilenberg-Mac Lane space and hence the function space

$$F(X,S^n_{(0)}) = \prod_{i=0}^{n} K(H^{n-i}(X;\mathbf{Q}),i)$$

is a product of Eilenberg-Mac Lane spaces by a theorem of Thom [**17**, **4**] (see also Federer [**2**, **10**]). In particular, all the components of $F(X,S^n_{(0)})$ are homotopy equivalent. For the sake of completeness we mention that $F_f(X, S^n)$ for any map $f\colon X \to S^n$ has the minimal model

$$\prod_{i=1}^{n} L_i(H_{n-i}(X))$$

with trivial differential. For any rational vector space V, $L_i(V)$ here denotes the free commutative graded algebra generated by V considered as homogeneous of degree i. We also identify $\mathrm{Hom}(H^{n-i}(X), \mathbf{Q}) = H_{n-i}(X)$ so that $H^*(K(H^{n-i}(X), i)) = L_i(H_{n-i}(X))$ becomes the minimal model for the formal space $K(H_{n-i}(X), i)$ (see [**16**]).

For the rest of this section we concentrate on the even dimensional case. So assume now that $n \geq 2$ is an even integer and that X is a connected and rationally $(2n-1)$-coconnected space of finite type. In order to construct a model for $F_\lambda(X, S^n_{(0)})$ we apply the method of Haefliger [**4**].

Since n is even, the rationalized n-sphere $S^n_{(0)}$ is the mapping fibre of $k = \iota_n^2\colon K(\mathbf{Q}, n) \to K(\mathbf{Q}, 2n)$, where $\iota_n \in H^n(\mathbf{Q}, n; \mathbf{Q})$ is the characteristic class. As X is rationally $(2n-1)$-coconnected this implies (see e.g. [**4**]) that $F_\lambda(X, S^n_{(0)})$ is the mapping fibre of the map

$$\mathbf{k}\colon F_\lambda(X, K(\mathbf{Q}, n)) \to F(X, K(\mathbf{Q}, 2n))$$

obtained by composition with k. Write

$$\mathbf{k} = \prod_{i=2}^{2n} k_i\colon F_\lambda(X, K(\mathbf{Q}, n)) \to \prod_{i=2}^{2n} K(H^{2n-i}(X), i)$$

corresponding to the factorization of $F(X, K(\mathbf{Q}, 2n))$ given by Thom's theorem and let

$$k_i^*\colon H_{2n-i}(X) = H^i(H^{2n-i}(X), i) \to H^i(F_\lambda(X, K(\mathbf{Q}, n)))$$

be the homomorphism induced on cohomology. Now, the rational cohomology algebra

$$B := H^*(F_\lambda(X, K(\mathbf{Q}, n))) = \prod_{i=1}^{n} L_i(H_{n-i}(X))$$

is a (minimal) model for $F_\lambda(X, K(\mathbf{Q}, n))$ and hence $F_\lambda(X, S^n_{(0)})$, as the mapping fibre of $\mathbf{k} = \prod k_i$, has a model of the form (A_λ, d) where

$$A_\lambda = B \otimes \bigotimes_{i=2}^{2n} L_{i-1}(H_{2n-i}(X)),$$

$dB = 0$ and $dv = k_i^*(v)$ for $v \in H_{2n-i}(X)$.

We now aim at a more explicit description of the model (A_λ, d). Choose a vector space basis $\{a_{ij} | j \in J_i\}$ for $H^i(X)$, $0 < i < 2n-1$, $J_i = \varnothing$ if $H^i(X) = 0$ and $J_i = \{j | 1 \leq j \leq \beta_i\}$ if $\beta_i = \dim_{\mathbf{Q}} H^i(X) > 0$. If $\lambda \neq 0$, take $a_{n1} = \lambda$. Let $\{b_{n-i,j} | 1 \leq j \leq \beta_i\} \subset \mathrm{Hom}(H^i(X), \mathbf{Q}) = H_i(X)$ be the dual basis to $\{a_{ij}\}$, $0 < i < 2n-1$, and let $b_n \in \mathrm{Hom}(H^0(X), \mathbf{Q}) = H_0(X)$ be the generator dual to $1 \in H^0(X)$. Note that $b_n \in B^n$ and $b_{n-i,j} \in B^{n-i}$ for $0 < i < n$, $j \in J_i$; in fact the set $\{b_{n-i,j} | 0 < i < n,\ j \in J_i\} \cup \{b_n\}$ freely generates B.

According to [4, p. 614],

$$e_\lambda^*(\iota_n) = b_n \otimes 1 + \sum_{i=1}^{n-1} \sum_{j \in J_i} b_{n-i,j} \otimes a_{ij} + 1 \otimes \lambda \in B \otimes H^*(X),$$

where $e_\lambda\colon F_\lambda(X, K(\mathbf{Q}, n)) \times X \to K(\mathbf{Q}, n)$ is the evaluation map, $e_\lambda(f, x) = f(x)$, $f \in F_\lambda(X, K(\mathbf{Q}, n))$, $x \in X$. Therefore the adjoint of $\mathbf{k}$,

$$K_\lambda = k \circ e_\lambda\colon F_\lambda(X, K(\mathbf{Q}, n)) \times X \to K(\mathbf{Q}, 2n),$$

pulls back the characteristic element $\iota_{2n} \in H^{2n}(\mathbf{Q}, 2n; \mathbf{Q})$ to

$$\begin{aligned}
K_\lambda^*(\iota_{2n}) &= (k \circ e_\lambda)^*(\iota_{2n}) = e_\lambda^*(\iota_{2n})^2 \\
&= \left(b_n \otimes 1 + \sum_{i=1}^{n-1} \sum_{j \in J_i} b_{n-i,j} \otimes a_{ij} + 1 \otimes \lambda \right)^2 \\
&= b_n^2 \otimes 1 + \sum_{r=2}^{2n-2} s_r + 2 \sum_{i=1}^{n-1} \sum_{j \in J_i} (b_n b_{n-i,j} \otimes a_{ij} + b_{n-i,j} \otimes \lambda a_{ij}) + 2 b_n \otimes \lambda,
\end{aligned}$$

where

$$s_r = \sum (-1)^{ik} b_{n-i,j} b_{n-k,l} \otimes a_{ij} a_{kl}, \qquad 2 \le r \le 2n-2,$$

summation over all i, j, k, l such that $1 \le i,\ k \le n-1,\ i + k = r,\ j \in J_i,\ l \in J_k$.

For $0 < i < 2n-1$, finally let

$$\{v_{2n-i-1,j} | j \in J_i\} \subset \operatorname{Hom}(H^i(X), \mathbf{Q}) = H_i(X) \subset L_{2n-i-1}(H_i(X))$$

be the basis dual to $\{a_{ij} | j \in J_i\} \subset H^i(X)$ and let

$$v_{2n-1} \in \operatorname{Hom}(H^0(X), \mathbf{Q}) = H_0(X) \subset L_{2n-1}(H_0(X))$$

be the linear form dual to the generator $1 \in H^0(X)$.

According to [4, 1.2] we now have

$$\begin{aligned}
dv_{2n-i-1,j} &= k_{2n-i}^*(v_{2n-i-1,j}) = K_\lambda^*(\iota_{2n}) \cap b_{n-i,j}, \qquad 0 < i < 2n-1,\ j \in J_i, \\
dv_{2n-1} &= k_{2n}^*(v_{2n-1}) = K_\lambda^*(\iota_{2n}) \cap b_n,
\end{aligned}$$

where, as in [4], $(b \otimes a) \cap b_{n-i,j} = b \cdot (b_{n-i,j}(a))$ for $b \otimes a \in B \otimes H^*(X)$. Using the above expression for $K_\lambda^*(\iota_{2n})$ we arrive at the following explicit formula for the differential d of the model $A_\lambda = B \otimes \bigotimes_{i=2}^{2n} L_{i-1}(H_{2n-i}(X))$ for $F_\lambda(X, S_{(0)}^n)$:

$$db_n = 0, \qquad db_{n-i,j} = 0 \quad \text{for } 0 < i < n,$$

$$dv_{2n-i-1,j} = \left(s_i + 2 \sum_{j \in J_{i-n}} b_{2n-i,j} \otimes \lambda a_{i-n,j} \right) \cap b_{n-i,j},$$
$$n < i < 2n-1,\ j \in J_i,$$

$$dv_{n-1,1} = s_n \cap b_{01} + 2\delta(\lambda) b_n,$$

$$dv_{n-1,j} = s_n \cap b_{0j}, \qquad 1 < j \le \beta_n,$$

$$dv_{2n-i-1,j} = s_i \cap b_{n-i,j} + 2 b_n b_{n-i,j}, \qquad 1 < i < n,\ j \in J_i,$$

$$dv_{2n-2,j} = 2 b_n b_{n-1,j}, \qquad j \in J_1,$$

$$dv_{2n-1} = b_n^2,$$

where $\delta(\lambda) = 1$ if $\lambda \neq 0$ and $\delta(\lambda) = 0$ if $\lambda = 0$. We get immediately

PROPOSITION 2.3. *Let $n > 1$ be an even integer. Then $m_0 = (A_0, d)$ is a minimal model for the component $F_0(X, S^n)$ of the constant map. Furthermore,*

$$\dim_{\mathbf{Q}} \operatorname{Hom}(\pi_i(F_0(X, S^n)), \mathbf{Q}) = \beta_{n-i} + \beta_{2n-i-1}, \qquad i \geq 0,$$

and if $\beta_{n-1} = \beta_{2n-2} = 0$, then $F_0(X, S^n)$ is a simply connected coformal space (see [**16**]*).*

PROOF. Clearly, (A_0, d) is a minimal nilpotent DGA and thus a minimal model for $F_0(X, S^n)$. As a graded algebra, A_0 is freely generated by the set

$$\begin{aligned} S = &\{b_{n-i,j} | 0 < i < n,\ j \in J_i\} \cup \{b_n\} \\ &\cup \{v_{2n-i-1,j} | 0 < i < 2n-1,\ j \in J_i\} \cup \{v_{2n-1}\} \end{aligned}$$

which contains $\beta_{n-i}+\beta_{2n-i-1}$ elements of degree i. This yields the evaluation of the rank of the dual to the homotopy group $\pi_i(F(X, S^n), 0)$. As to the final assertion, we just note that the differential d on A_0 is quadratic (see e.g. [**3**, **16**]). $\square$

When $\lambda \neq 0$, the model (A_λ, d) is in general far from being minimal. However, if X is $(n+1)$-coconnected, then the model (A_λ, d) for $\lambda \neq 0$ simplifies to

$$\begin{aligned} &db_n = 0, \qquad db_{n-i,j} = 0 \quad \text{for } 0 < i < n, \\ &dv_{n-1,1} = s_n \cap b_{01} + 2b_n, \\ &dv_{n-1,j} = s_n \cap b_{0j}, \qquad 1 < j \leq \beta_n, \\ &dv_{2n-i-1,j} = s_i \cap b_{n-i,j} + 2b_n b_{n-i,j}, \qquad 1 < i < n,\ j \in J_i, \\ &dv_{2n-2,j} = 2b_n b_{n-1,j}, \qquad j \in J_1, \\ &dv_{2n-1} = b_n^2. \end{aligned}$$

Here the only deviation from minimality occurs in the relation for $dv_{n-1,1}$, so in this important special case it is easy to construct the minimal model for the nilpotent DGA A_λ.

PROPOSITION 2.4. *Assume furthermore, that X is $(n+1)$-coconnected, $n \geq 2$ even. Then the minimal model m_λ for $F_\lambda(X, S^n_{(0)})$, $\lambda \neq 0$, is the graded commutative algebra freely generated by the set $\{b_{n-i,j} | 0 < i < n,\ j \in J_i\} \cup \{v_{2n-i-1,j} | 0 < i < n,\ j \in J_i;\ i = n, j > 1\} \cup \{v_{2n-1}\}$ and with differential d given by*

$$\begin{aligned} &db_{n-i,j} = 0, \qquad 0 < i < n, \\ &dv_{n-1,j} = s_n \cap b_{01}, \qquad 1 < j \leq \beta_n, \\ &dv_{2n-i-1,j} = s_i \cap b_{n-i,j} - (s_n \cap b_{01}) b_{n-i,j}, \qquad 1 < i < n,\ j \in J_i, \\ &dv_{2n-2,j} = -(s_n \cap b_{01}) b_{n-1,j}, \qquad j \in J_1, \\ &dv_{2n-1} = \tfrac{1}{4}(s_n \cap b_{01})^2. \end{aligned}$$

Next follow two examples of the computation of the minimal models m_λ from Propositions 2.3 and 2.4.

EXAMPLE 2.5. (1) For any even integer $n \geq 2$,

$$m_0 = (S[b_n] \otimes E(v_{n-1}, v_{2n-1}); db_n = 0,\ dv_{n-1} = 0,\ dv_{2n-1} = b_n^2)$$

is a minimal model for $F_0(S^n, S^n)$ while $m_1 = (E(v_{2n-1}), d = 0)$ is a minimal model for $F_f(S^n, S^n)$, where $f\colon S^n \to S^n$ is any map of nonzero degree.

Using Thom's theorem [**17**] for the case of an odd n, we conclude that

$$F_f(S^n, S^n) \simeq_{\mathbf{Q}} \begin{cases} S^n \times S^{n-1}, & n \text{ even, } \deg f = 0, \\ S^{2n-1}, & n \text{ even, } \deg f \neq 0, \\ S^n, & n \text{ odd.} \end{cases}$$

In particular, the evaluation fibration $\mathrm{Ev}(S^n, S^n)$ from [**8**] is always rationally decomposable. (See V. L. Hansen [**5, 6, 7**] and W. A. Sutherland [**15**] for a discussion of the integral homotopy types represented by the components of $F(S^n, S^n)$.)

(2) The graded algebra

$$m_0 = S[b_2, b_4, \dots, b_{2m-2}, b_{2m}] \otimes E(v_{2m-1}, v_{2m+1}, \dots, v_{4m-1})$$

with differential d given by

$$\begin{aligned} db_{2i} &= 0, \qquad 1 \le i \le m, \\ dv_{4m-2i-1} &= \sum_{r+s=2m-i} b_{2r} b_{2s}, \qquad 0 \le i \le m, \end{aligned}$$

is the minimal model for $F_0(\mathbf{C}P^m, S^{2m})$.

The graded algebra $m_1 = S[b_2, b_4, \dots, b_{2m-2}] \otimes E(v_{2m+1}, v_{2m+3}, \dots, v_{4m-1})$ with differential d given by

$$\begin{aligned} db_{2i} &= 0, \qquad 1 \le i < m, \\ dv_{4m-2i-1} &= \sum_{r+s=2m-i} b_{2r} b_{2s} - b_{2m-2i}\Delta_m, \qquad 0 < i < m, \\ dv_{4m-1} &= \tfrac{1}{4}\Delta_m^2, \end{aligned}$$

with $\Delta_m = \sum_{r+s=m} b_{2r} b_{2s}$, is the minimal model for the component $F_f(\mathbf{C}P^m, S^{2m})$ for any map $f\colon \mathbf{C}P^m \to S^{2m}$ of nonzero degree.

We now turn to the

PROOF OF THEOREM 2.2. The fibration of rational spaces

$$F(X, K(\mathbf{Q}, 2n-1)) \to F_\lambda(X, S^n_{(0)}) \to F_\lambda(X, K(\mathbf{Q}, n))$$

has an associated long exact homotopy sequence of the form

$$\to H^{n-i-1}(X) \xrightarrow{\partial} H^{2n-i-1}(X) \to \pi_i(F_\lambda) \xrightarrow{\partial} H^{n-i}(X) \to H^{2n-i}(X) \to$$

in which the cohomology groups have rational coefficients and $F_\lambda(X, S^n_{(0)})$ has been abbreviated to F_λ. It is not hard to see, and it was proved by Federer [**2**], that ∂ is cup product with $\lambda \in H^n(X)$. We deduce e.g. that

$$\dim_{\mathbf{Q}} \pi_{n-1}(F_0) > \dim_{\mathbf{Q}} \pi_{n-1}(F_\lambda)$$

whenever $\lambda \neq 0$ (cf. [**6**, Lemma 2(i)]). This proves the first part of Theorem 2.2.

To prove the second part, we just note that multiplication by $t \neq 0$ on $\mathbf{Q} = H^n(S^n_{(0)}, \mathbf{Q})$ can be realized by a homotopy self equivalence M_t on $S^n_{(0)}$. By composition with M_t we get a homotopy equivalence $\mathbf{M}_t\colon F_\lambda(X, S^n_{(0)}) \to F_{t\lambda}(X, S^n_{(0)})$ between components.

The third and final part of Theorem 2.2 is a special case of the next theorem. □

Noting that the isomorphism type of the "pointed algebra" $(H^*(X), \lambda)$ determines the isomorphism type of the model (A_λ, d), as constructed above, we derive the following consequence:

THEOREM 2.6. *Let $n \geq 2$ be an even integer and let X and Y be connected and rationally $(2n-1)$-coconnected spaces of finite type. Assume that there exists an isomorphism $\varphi\colon H^*(X) \to H^*(Y)$ between the cohomology algebras. Then*

$$F_\lambda(X, S^n_{(0)}) \simeq_{\mathbf{Q}} F_{\varphi\lambda}(Y, S^n_{(0)})$$

for any $\lambda \in H^n(X)$. In particular, $F_f(X, S^n) \simeq_{\mathbf{Q}} F_g(Y, S^n)$ if $f\colon X \to S^n, g\colon Y \to S^n$ are maps such that $\varphi f^*(s_n) = tg^*(s_n)$ for some rational number $t \neq 0$.

We are now heading towards a partial converse to Theorem 2.2(3). We ask whether a rational homotopy equivalence $\alpha\colon F_\lambda(X, S^n) \to F_\mu(X, S^n)$ implies the existence of an algebra-automorphism φ on $H^*(X)$ such that $\varphi(\lambda) = \mu$. This is true under some restrictions—at least if X is $(n+1)$-coconnected.

Let V_1, V_2 be graded vector spaces, and $\Lambda V_1, \Lambda V_2$ the graded commutative algebras freely generated by them. An algebra map $\alpha\colon \Lambda v_1 \to \Lambda v_2$ is called *linearly generated* iff there is a graded linear map $\tilde{a}\colon V_1 \to V_2$ such that $\alpha = \Lambda\tilde{\alpha}$ (compare [**16**, p. 26]). Remark that one may construct a linearly generated rational isomorphism of models $\alpha\colon m_\mu \to m_\lambda$ in the situation of Theorem 2.2(3).

We investigate the following situation: Let n be even and X a simply-connected and $(n+1)$-coconnected CW-complex; furthermore $\lambda, \mu \in H^n(X)$ are linearly independent elements. Then, one may choose a basis $\{a_{ij}\}_{i=2,\dots,n}^{j\in J_i}$ of $H^+(X)$ such that $a_{n1} = \lambda$ and $a_{n2} = \mu$. These in turn allow us to construct a minimal m_λ of $F_\lambda(X, S^n_{(0)})$ as in Proposition 2.4; a minimal model m_μ of $F_\mu(X, S^n_{(0)})$ is obtained by replacing $v_{n-1,2}$ by $v_{n-1,1}$ and $s_n \cap b_{01}$ by $s_n \cap b_{02}$.

PROPOSITION 2.7. *Assume X and $\lambda, \mu \in H^n(X)$ given as above. If there is a linearly generated rational isomorphism $\alpha\colon m_\mu \to m_\lambda$, then there is an algebra automorphism φ on $H^*(X)$ and a rational number $t \neq 0$ such that $\varphi(\lambda) = t \cdot \mu$.*

REMARKS. (1) If λ and μ are linearly dependent, φ may be chosen as the identity map.

(2) We were unable to decide whether an arbitrary rational isomorphism $\alpha\colon m_\mu \to m_\lambda$ implies a cohomology isomorphism as in Proposition 2.7.

PROOF. As α is linearly generated, it restricts to vector space automorphisms

$$\begin{aligned}
&\alpha_{n-s} \text{ on } \langle b_{n-s,j}\rangle_{j\in J_s}, \qquad 2 \leq s < n;\\
&\alpha_{2n-s-1} \text{ on } \langle v_{2n-s-1,j}\rangle_{j\in J_s}, \qquad 2 \leq s < n;\\
&\alpha_{2n-1} \text{ on } \langle v_{2n-1}\rangle
\end{aligned}$$

and an isomorphism

$$\alpha_{n-1}\colon \langle v_{n-1,j}\rangle_{j\in J_n-\{2\}} \to \langle v_{n-1,j}\rangle_{j\in J_n-\{1\}}.$$

In particular, $\alpha_{2n-1}(v_{2n-1}) = k^2 \cdot v_{2n-1},\ k \in \mathbf{Q}^*$, because $d\alpha(v_{2n-1}) = \alpha d(v_{2n-1})$ is a square. Furthermore, $s_n \cap b_{01}$ and $s_n \cap b_{02}$ are either both zero or nonzero. In the first case, λ and μ are both undecomposable; then, φ may be chosen to be the identity map on $H^*(X)$, $* < n$ and on $\langle a_{n3}, \dots, a_{n,\beta_n}\rangle \subset H^n(X)$ and permuting $\lambda = a_{n1}$ and $\mu = a_{n2}$. From now on, we may assume that

$$s_n \cap b_{01} \text{ and } s_n \cap b_{02} \text{ are both nonzero.} \tag{1}$$

Before defining φ, we have to agree on some notation. The following vector space isomorphisms will be helpful:

$$\begin{aligned} \psi_{n-s}\colon \langle b_{n-s,j}\rangle_{j\in J_i} &\to \langle v_{2n-s-1,j}\rangle_{j\in J_i}, \qquad 2 \le s \le n,\\ b_{n-s,j} &\to v_{2n-s-1,j}. \end{aligned}$$

Let us now define $\varphi = \oplus\varphi_i\colon H^i(X) \to H^i(X)$ as a graded vector space isomorphism, and then verify that φ indeed can be made to an algebra homomorphism:

$$\begin{aligned} &\varphi_s = \alpha_s^*\colon H^s(X) \to H^s(X), \qquad s < n,\\ &\varphi_n(\lambda) = \varphi_n(a_{n1}) = a_{n2} = \mu \quad\text{and}\quad \varphi_n = (\psi_0^{-1}\circ\alpha_{n-1}\circ\psi_0)^* \text{ on } \langle a_{n2}, \dots, a_{n\beta(n)}\rangle. \end{aligned}$$

This is indeed a vector space isomorphism, as α is one. To make sure that α is a ring homomorphism, we have to run through the following calculations using that α is a chain map:

$$\begin{aligned} d\alpha(v_{2n-1}) &= k^2\cdot d_\lambda(v_{2n-1})\\ &= \Big(\sum k b_{01}(a_{ri}\cdot a_{n-r,j}) b_{n-r,i}\cdot b_{rj}\Big)^2,\\ \alpha d(v_{2n-1}) &= \Big(\sum b_{02}(a_{ri}\cdot a_{n-r,j})\alpha(b_{n-r,i})\cdot\alpha(b_{rj})\Big)^2\\ &= \Big(\sum b_{02}(\varphi a_{ri}\cdot\varphi a_{n-r,j}) b_{n-r,i}\cdot b_{rj}\Big)^2. \end{aligned}$$

We conclude

$$k\cdot b_{01}(x\cdot y) = b_{02}(\varphi x\cdot\varphi y), \qquad x\in H^r(X),\ y\in H^{n-r}(X),\ 0<r<n. \tag{2}$$

A similar calculation on $v_{2n-s-1,j}$ yields by comparison of coefficients:

$$(\psi^{-1}\alpha\psi b_{n-s})(x\cdot y) = b_{n-s}(\varphi x\cdot\varphi y), \tag{3}$$

where $b_{n-s}\in H_{n-s}(X)$, $x\in H^r(X)$ and $y\in H^{s-r}(X)$, $0<s<n$, and

$$\begin{aligned} &\sum b_{01}(a_{ri}\cdot a_{n-r,j})\cdot b_{n-r,i}\cdot b_{rj}\psi^{-1}\alpha\psi b_{n-r,j}\\ &\quad= \sum b_{02}(\varphi a_{ri}\cdot\varphi a_{n-r,j})\cdot b_{n-r,i}\cdot b_{rj}\cdot b_{n-r,j}. \end{aligned} \tag{4}$$

From (1), (2) and (4), we may conclude:

$$\psi^{-1}\alpha\psi = k\cdot\alpha \text{ in dimensions less than } n. \tag{5}$$

Using this in (3) yields

$$k\cdot(\varphi^* b_{n-s})(x\cdot y) = b_{n-s}(\varphi x\cdot\varphi y), \tag{6}$$

where $b_{n-s}\in H_{n-s}(X)$, $x\in H^r(X)$ and $y\in H^{s-r}(X)$, $0<s<n$.

Lastly a similar calculation for $v_{n-1,j}$ yields

$$k\cdot(\varphi^* z)(x\cdot y) = z(\varphi x\cdot\varphi y), \tag{7}$$

where $z \in \langle b_{0j} \rangle_{j \in J_n - \{2\}}$, $x \in H^r(X)$ and $y \in H^{n-r}(X)$, $0 < r < n$. Thus, collecting (2), (6), and (7), we get

(8) $$k \cdot (\varphi^* z)(x \cdot y) = z(\varphi x \cdot \varphi y), \qquad z \in H_*(X),\ x, y \in H^*(X).$$

For $\varphi' = t \cdot \varphi$, $t = k^{-1}$, this implies

$$((\varphi')^* z)(x \cdot y) = z(\varphi' x \cdot \varphi' y), \qquad z \in H_*(X),\ x, y \in H^*(X);$$

hence φ' is an algebra automorphism with $\varphi'(\lambda) = t \cdot \varphi(\lambda) = t \cdot \mu$. $\square$

3. Spaces of maps into $\mathbf{C}P^n_{(0)}$. In this section, we let X denote a connected and rationally $(2n+1)$-coconnected space with first Betti number $\beta_1(X) = 0$. The components of the space $F(X, \mathbf{C}P^n_{(0)})$ of maps of X into the rationalized complex projective n-space $\mathbf{C}P^n_{(0)}$, $n \geq 1$, are classified by $H^2(X)$. For any $\lambda \in H^2(X; \mathbf{Q})$, let

$$F_\lambda(X, \mathbf{C}P^n_{(0)}) = \{f \in F(X, \mathbf{C}P^n_{(0)}) | f^* c_1 = \lambda\}, \quad \cdot$$

where $c_1 \in H^2(\mathbf{C}P^n_{(0)}; \mathbf{Q})$ is the first rational Chern class, be the corresponding component of $F(X, \mathbf{C}P^n_{(0)})$.

We now proceed to construct the minimal model for $F_\lambda(X, \mathbf{C}P^n_{(0)})$. Since $\mathbf{C}P^n_{(0)}$ is the mapping fibre of $k = \iota_2^{n+1}\colon K(\mathbf{Q}, 2) \to K(\mathbf{Q}, 2n+2)$, where $\iota_2 \in H^2(\mathbf{Q}, 2; \mathbf{Q})$ is the characteristic class, $F_\lambda(X, \mathbf{C}P^n_{(0)})$ is the mapping fibre of the map

$$\mathbf{k}\colon K(H^0(X), 2) = F_\lambda(X, K(\mathbf{Q}, 2)) \to F(X, K(\mathbf{Q}, 2n+2))$$

obtained by composition with k. Hence $F_\lambda(X, \mathbf{C}P^n_{(0)})$ has a model whose underlying graded algebra is

$$A_\lambda = S[b] \otimes \bigotimes_{i=2}^{2n+2} L_{i-1}(H_{2n+2-i}(X)),$$

where $S[b] = H^*(F_\lambda(X, K(\mathbf{Q}, 2)); \mathbf{Q})$ is the symmetric algebra generated by an element b of degree 2. The differential d and A_λ is given by $db = 0$ and

$$\forall v \in H_{2n+2-i}(X)\colon dv = k_i^*(v),$$

where

$$\mathbf{k} = \prod_{i=2}^{2n+2} k_i\colon F_\lambda(X, K(\mathbf{Q}, 2)) \to F(X, K(\mathbf{Q}, 2n+2)) = \prod_{i=2}^{2n+2} K(H^{2n+2-i}(X), i)$$

and

$$k_i^*\colon H_{2n+2-i}(X) = H^i(H^{2n+2-i}(X), i; \mathbf{Q}) \to H^i(F_\lambda(X, K(\mathbf{Q}, 2)))$$

is the map induced on cohomology.

As in the previous section, we now offer a more explicit expression for the model A_λ. To $\lambda \in H^2(X)$ we associate its height, $h(\lambda)$, defined by

$$h(\lambda) = \begin{cases} \max\{i > 0 | \lambda^i \neq 0\} & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

Choose a vector space basis $\{a_{ij} | j \in J_i\}$ for $H^i(X)$, $2 \leq i \leq 2n$, $J_i = \{j | 1 \leq j \leq \beta_i\}$, such that $a_{2i,1} = \lambda^i$ when $1 \leq i \leq h(\lambda)$. Let

$$\{v_{2n+1-i,j}\} \subset \operatorname{Hom}(H^i(X), \mathbf{Q}) = H_i(X) \subset L_{2n+1-i}(H_i(X))$$

be the dual basis to $\{a_{ij}\} \subset H^i(X),\ 2 \le i \le 2n$, and let

$$v_{2n+1} \in \mathrm{Hom}(H^0(X), \mathbf{Q}) = H_0(X) \subset L_{2n+1}(H_0(X))$$

be the generator dual to $1 \in H^0(X)$. Then the set

$$\{v_{2n+1-i,j} | 2 \le i \le 2n,\ j \in J_i\} \cup \{v_{2n+1}\}$$

generates the graded algebra $\bigotimes_{i=2}^{2n+2} L_{i-1}(H_{2n+2-i}(X))$.

The evaluation map $e_\lambda\colon F_\lambda(X, K(\mathbf{Q},2)) \times X \to K(\mathbf{Q},2)$ is given by (see [**4**, **11**])

$$e_\lambda^*(\iota_2) = 1 \otimes \lambda + b \otimes 1 \in B \otimes H^*(X).$$

Thus the adjoint $K_\lambda = k \circ e_\lambda\colon F_\lambda(X, K(\mathbf{Q},2)) \times X \to K(\mathbf{Q}, 2n+2)$ of $\mathbf{k}$ satisfies

$$K_\lambda^*(\iota_{2n+2}) = e_\lambda^*(\iota_2)^{n+1} = \sum_{i=0}^{h(\lambda)} \binom{n+1}{i} b^{n+1-i} \otimes \lambda^i.$$

This means [**4**, 1.2] that

$$k_{2m+2-2i}^*(v_{2n+1-2i,1}) = \binom{n+1}{i} b^{n+1-i}, \qquad 0 < i \le h(\lambda),$$

$$k_{2n+2}^*(v_{2n+1}) = (n+1)b^{n+1}.$$

A model A_λ for $F_\lambda(X, \mathbf{C}P^n_{(0)})$ may be constructed as the commutative graded algebra freely generated by the set

$$S = \{b\} \cup \{v_{2n+1-i,j} | 2 \le i \le 2n,\ j \in J_i\} \cup \{v_{2n+1}\}$$

together with the differential $d\colon A_\lambda \to A_\lambda$ given by

$$ds = \begin{cases} b^{n+1-i} & \text{if } s = v_{2n+1-2i,1},\ 1 \le i \le h(\lambda), \\ b^{n+1} & \text{if } s = v_{2n+1}, \\ 0 & \text{otherwise,} \end{cases}$$

for any $s \in S$. Remark that the binomial coefficients can be omitted by appropriate choice of the v_{ij}. Then (A_λ, d) is a minimal nilpotent DGA when $h(\lambda) < n$. (A_λ, d) is not minimal when $h(\lambda) = n$, but one easily sees that the algebra m_λ freely generated by $S - \{b, v_{1,1}\}$ and equipped with trivial differential may serve as a minimal model in this case.

We collect the preceding remarks in

PROPOSITION 3.1. *The minimal model for the component $F_\lambda(X, \mathbf{C}P^n_{(0)})$ is (A_λ, d) if $h(\lambda) < n$, and $(m_\lambda, d = 0)$ if $h(\lambda) = n$.*

An immediate consequence is the following solution to the rational homotopy classification problem for the components of $F(X, \mathbf{C}P^n_{(0)})$.

THEOREM 3.2. *Let X be a connected and rationally $(2n+1)$-coconnected space of finite type with first Betti number $\beta_1(X) = 0$. Then*

$$F_\lambda(X, \mathbf{C}P^n_{(0)}) \simeq_{\mathbf{Q}} F_\mu(X, \mathbf{C}P^n_{(0)}) \Leftrightarrow h(\lambda) = h(\mu)$$

for any pair of cohomology classes $\lambda, \mu \in H^2(X)$.

Using [**9**, Theorem B], Theorem 3.2 can also be interpreted as a statement on the rational homotopy types represented by the components of the space $F(X, \mathbf{C}P^n)$ of maps of X into the unrationalized projective space $\mathbf{C}P^n$.

COROLLARY 3.3. *Let X be as in Theorem 3.2 and let $f, g\colon X \to \mathbf{C}P^n$ be two maps of X into $\mathbf{C}P^n$. Then $F_f(X, \mathbf{C}P^n) \simeq_{\mathbf{Q}} F_g(X, \mathbf{C}P^n) \Leftrightarrow h(f^*c_1) = h(g^*c_1)$, where $c_1 \in H^2(\mathbf{C}P^n; \mathbf{Q})$ is the first rational Chern class.*

The integral homotopy types represented by the components of $F(X, \mathbf{C}P^m)$ have been studied in [**1**, **10**, **11**] for some special domain spaces X.

We now conclude this paper by a few examples of the applications of the results in this section.

EXAMPLE 3.4. (1) For $1 \le m \le n$, let $M_f(m,n)$ denote the minimal model for the component $F_f(\mathbf{C}P^m, \mathbf{C}P^n)$ containing the map $f\colon \mathbf{C}P^m \to \mathbf{C}P^n$ between complex projective spaces. Let $\deg f \in \mathbf{Z}$ denote the degree of f; i.e. $\deg f = f_*(1)$, where $f_*\colon \mathbf{Z} = H_2(\mathbf{C}P^m; \mathbf{Z}) \to \mathbf{Z} = H_2(\mathbf{C}P^n; \mathbf{Z})$ is the homomorphism induced on integral homology. Then

$$M_f(m,n) = \begin{cases} (A(m,n), d_0), & 1 \le m \le n, \ \deg f = 0, \\ (A(m,n), d_1), & 1 \le m < n, \ \deg f \neq 0, \\ (E, d = 0), & 1 \le m = n, \ \deg f \neq 0. \end{cases}$$

Here,

$$A(m,n) = S[b] \otimes E(v_{2n-2m+1}, v_{2n-2m+3}, \dots, v_{2n-1}, v_{2n+1}),$$
$$d_0 b = 0, \quad d_0 v_{2n-2m+1} = d_0 v_{2n-2m+3} = \cdots = d_0 v_{2n-1} = 0, \quad d_0 v_{2n+1} = b^{n+1},$$
$$d_1 b = 0, \quad d_1 v_{2n-2m+1} = b^{n-m+1}, \quad d_1 v_{2n-2m+3} = b^{n-m+2}, \dots, d_1 v_{2n+1} = b^{n+1},$$

and E is the exterior algebra $E = E(v_3, v_5, \dots, v_{2n+1})$.

Thus the countably infinitely many integral homotopy types [**1**, **10**] represented by the components of $F(\mathbf{C}P^m, \mathbf{C}P^n)$ localize to only two distinct rational homotopy types, $F_0(\mathbf{C}P^m, \mathbf{C}P^n)$ and $F_i(\mathbf{C}P^m, \mathbf{C}P^n)$, $i\colon \mathbf{C}P^m \to \mathbf{C}P^n$ the inclusion map. For $m = n$ we get in particular

$$F_0(\mathbf{C}P^n, \mathbf{C}P^n) \simeq_{\mathbf{Q}} S^1 \times S^3 \times S^5 \times \cdots \times S^{2n-1} \times \mathbf{C}P^n,$$
$$F_i(\mathbf{C}P^n, \mathbf{C}P^n) \simeq_{\mathbf{Q}} S^3 \times S^5 \times \cdots \times S^{2n-1} \times S^{2n+1}.$$

By [**13**, Theorem 1.1(a)], the result for the component of the inclusion can also be interpreted as the determination of the rational homotopy type of a certain quotient space of the unitary group $U(n+1)$.

(2) It is obvious that the applied technique works just as well when the target space is quaternionic projective n-space $\mathbf{H}P^n$. By analogy with the preceding example we get in particular the rational homotopy equivalences

$$F_0(\mathbf{H}P^n, \mathbf{H}P^n) \simeq_{\mathbf{Q}} S^3 \times S^7 \times S^{11} \times \cdots \times S^{4n-1} \times \mathbf{H}P^n,$$
$$F_i(\mathbf{H}P^n, \mathbf{H}P^n) \simeq_{\mathbf{Q}} S^7 \times S^{11} \times \cdots \times S^{4n-1} \times S^{4n+3}$$

for the components of the space $F(\mathbf{H}P^n, \mathbf{H}P^n)$ of self maps on $\mathbf{H}P^n$. The result for the component of the identity improves some results from [**19**].

The second author would like to thank Deutsche Forschungsgemeinschaft for financial support and Matematisk Institut in Aarhus for kind hospitality while this paper was written.

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