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ON SPACES WITH THE SAME n-TYPE FOR ALL n

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Over thirty years ago J. H. C. Whitehead posed the following question: Suppose that X and Y are two spaces whose Postnikov approximations, $X^{(n)}$ and $Y^{(n)}$, are homotopy equivalent for each integer n. Does it follow that X and Y are homotopy equivalent?

Recall that $X^{(n)}$ can be obtained by attaching to X cells of dimension $n+2$ and higher to kill off the homotopy groups of X in dimensions above n. So it is easy to see that the answer to Whitehead's question is yes if X is either finite dimensional (use the cellular approximation theorem) or if X has only a finite number of nonzero homotopy groups. But in general, the answer is no. The first such example, found by Adams in 1957, [1], is essentially the following.

Example A. Let K be a 1-connected, finite, noncontractible CW-complex and let $X = \Pi_{n\geq 1} K^{(n)}$ with the direct limit topology. Then X and $X \times K$ have the same n-type for all n, but are not homotopy equivalent. □

Notice that this example does not have finite type; indeed, none of its nonzero homotopy groups are finitely generated. In 1966, Brayton Gray found one with finite type, [7]. Let $X = \Omega\Sigma(S^3 \vee \Sigma \mathbf{CP}^\infty)$. He took a nontrivial phantom map from $\mathbf{CP}^\infty$ to S^3, applied $\Omega\Sigma$ to its mapping cone and obtained a different homotopy type, Y, that had the same n-type as X for all n.

In 1975, C. Wilkerson classified, up to homotopy, those spaces Y having the same n-type as X for all n. In [18], he proved the following fundamental result.

Theorem 1. *Given a connected CW space X, let $SNT(X)$ denote the set of all homotopy types $[Y]$ such that $Y^{(n)} \simeq X^{(n)}$ for all n. Then there is a bijection of pointed sets,*

$$SNT(X) \approx \varprojlim{}^1 \, Aut\, X^{(n)}$$

where $Aut\, X^{(n)}$ is the group of homotopy classes of homotopy self-equivalences of $X^{(n)}$. □

Our interest in this topic began with the following question: How big is the set $SNT(X)$ for well known infinite dimensional spaces like the classifying space of a compact Lie group or the loop space of a finite complex? We have some answers. In Theorem 4, we show $SNT(BG)$ is almost always uncountably large when G is a compact connected Lie group. There are just a few exceptions and we list them. On the other hand, we have yet to find a

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finite complex K for which $SNT(\Omega K)$ has more than one member. In Theorem 5 we prove that $SNT(\Omega K) = [\Omega K]$ for a class of finite complexes that includes compact Lie groups and many Stiefel manifolds.

To get these answers we had to figure out a way to compute $\varprojlim^1 Aut X^{(n)}$ without knowing very much about the individual groups in the tower. As is well known, these automorphisms groups are next to impossible to calculate. The following algebraic result helped us get around this obstacle. Recall that a tower of groups $G_1 \leftarrow G_2 \leftarrow \dots$ is said to have the Mittag–Leffler property if there is a descending chain condition on the images in each G_n, of G_{n+k} as $k \to \infty$. To put it another way, if $G_n^{(k)} = image(G_n \leftarrow G_{n+k})$ then the tower $\{G_n\}$ is Mittag–Leffler if for each n, $\varprojlim G_n^{(k)} = G_n^{(N)}$ for some finite N. It is well known that if a tower $\{G_n\}$ has the Mittag–Leffler property then $\varprojlim^1 G_n = *$. Here is a partial converse.

THEOREM 2. *Let $G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \dots$ be a tower of countable groups. Then $\varprojlim^1 G_n = *$ if and only if the tower $\{G_n\}$ is Mittag–Leffler. Moreover, if $\varprojlim^1 G_n \neq *$, it is uncountably large.* □

Gray proved a result like this in ([7], p. 242). Theorem 2 is stronger than his result in two respects. First, it shows that his hypothesis, that each $image(G_n \leftarrow G_{n+k})$ be normal, is unnecessary. This is important because automorphism groups of Postnikov towers do not always have this normal image property. Secondly, Gray notes that if the groups in the tower are abelian, then the last sentence in Theorem 2 holds. We show it holds in the nonabelian case as well. After writing this paper we learned that the first part of this theorem is in the literature on shape theory, ([5], page 78).

One might wonder if, in Theorem 2, it is really necessary to require the groups in the tower to be countable. The following example shows that it is.

Example B. Let H_i, $i = 1, 2, \dots$ be a sequence of nontrivial groups and define $G_n = \Pi_{i \geq n} H_i$. The tower of inclusions $G_1 \leftarrow G_2 \leftarrow \dots$ is not Mittag–Leffler and yet $\varprojlim^1 G_n = *$. □

COROLLARY 2.1. *If X is a connected nilpotent space with finite type over some subring of the rationals, then either $SNT(X) = *$ or else it is uncountably large.* □

Of course, when we write $SNT(X) = *$, we mean that $[X]$ is its only member. The groups $Aut X^{(n)}$ are countable for those spaces that satisfy the hypothesis of 2.1 and so the corollary follows directly from Theorems 1 and 2.

The next result gives a homological criterion that determines whether or not $SNT(X)$ is trivial when X is an H_0-space. Recall that an H_0-space is one whose rationalization is an H-space. It's easy to spot an H_0-space with finite type; its rational cohomology is either an exterior algebra on odd degree generators or a polynomial algebra on generators of even degree, or a tensor product of the two. Familiar examples of H_0-spaces, include H-spaces, complex Stiefel manifolds, and the classifying spaces of compact connected Lie groups.

THEOREM 3. *Let X be a 1-connected, H_0-space with finite type over $\mathbf{Z}_P$ for some set of primes P. Then the following statements are equivalent*:

(i) $SNT(X) = *$
(ii) *the canonical map $Aut X \to Aut X^{(n)}$, has a finite cokernel for all n*
(iii) *the map $Aut X \xrightarrow{f \mapsto f^*} Aut H^{\leq n}(X; \mathbf{Z}_P)$ has a finite cokernel for all integers n.* □

There are some items in this theorem we should clarify. First, $\mathbf{Z}_P$ denotes the integers localized at the set of primes P, and the finite type hypothesis means that each $\pi_n X$ is a finitely generated $\mathbf{Z}_P$-module. Secondly, when we say a homomoprhism (or an anti-homomorphism) $f: A \to B$ has a finite cokernel we mean only that its image $f(A)$, has finite index in B. Finally, $Aut\, H^{\leq n}(X; \mathbf{Z}_P)$ denotes the group of all ring automorphisms that preserve the degrees of homogeneous elements in this graded ring.

The proof of Theorem 3 centers around the Postnikov system of X. Obviously, each stage of it satisfies condition (i). The main step is to show each $X^{(k)}$ also satisfies the other two conditions. When we pass to the limit we then exploit the finite cokernels in (ii) as well as the countability of the groups $Aut\, X^{(k)}$.

Is Theorem 3 the most general result possible? The next two examples provide some answers. The first deals with the hypothesis that X be an H_0-space. Perhaps this restriction could be relaxed a little, but the following example shows that it cannot be eliminated entirely.

Example C. Let $K = (S^2 \vee S^2)\bigcup_\varphi e^5$ where $\varphi = [[\iota_1, \iota_2], \iota_1]$. Then $SNT(K) = *$, of course, because K is a finite dimensional, but for all $n \geq 2$, the map $Aut\, K \to Aut\, H^{\leq n} K$ has an infinite cokernel. □

The example just given is clearly not an H_0-space. The next example deals with restrictions on the fundamental group. A glance at the proof of Theorem 3 reveals the condition $\pi_1 X = 0$ could be weakened to read: $\pi_1 X$ is nilpotent, it acts nilpotently on the higher homotopy groups of X, it has a finite commutator subgroup and its abelianization is a finitely generated $\mathbf{Z}_P$-module. Here is what happens, among finitely generated nilpotent groups, when we try to go any further in this direction.

Example D. Let $\pi = N \times Z$ where N is the free nilpotent group of class 2 on two generators. If $X = K(\pi, 1)$, then obviously, $SNT(X) = *$ and yet the cokernel of $Aut\, X \to Aut(H^1 X)$ is not finite. □

The next example may have been known to Adams but it came as a surprise to us. Indeed, recall Zabrodsky's result that the homotopy genus of BU is uncountably large ([21], page 190). It seems to suggest that you can't recognize BU from a long way off; you've got to get up close and localize to be sure. On the other hand, we found that if, at each stage in the Postnikov tower, a space looks like BU, then it must be BU. Since the classifying spaces BU and BSp play central roles in algebraic topology, it seems worthwhile being able to rule out counterfeits of these spaces.

Example E. If $X = BU$ or BSp, then $SNT(X) = *$. In fact, something stronger than Theorem 3 is true; namely $Aut\, X \to Aut\, X^{(2n)}$ is surjective for each n. □

When the rational homotopy groups of X vanish above a certain point, we get a criterion that is easier to check than the one given in Theorem 3.

COROLLARY 3.1. *Let X and P be as in Theorem 3 and suppose there exists an integer t such that $\pi_{>t} X \otimes \mathbf{Q} = 0$. Then $SNT(X)$ is trivial if and only if $Aut\, X \to Aut\, H^{\leq n}(X; \mathbf{Z}_P)$ has a finite cokernel in at least one of the following cases:*

(i) *for $n = t$,*
(ii) *for some integer $n \geq t$,*
(iii) *for all integers n.* □

We use this corollary and the nonexistence of essential maps from $K(\mathbf{Z}, n)$ to S^n, when $n > 1$, to get the following.

Example F. Let $X = S^{2n+1} \times K(\mathbf{Z}, 2n+1)$ where $n \geq 1$. Then for any nonempty set of primes P, $SNT(X_P) \neq *$ and $SNT(\Omega X_P) \neq *$. □

The case of ΩX_P when $n = 1$ confirms a conjecture Wilkerson made in [18], p. 283. The assertion that $SNT(X) \neq *$ contradicts a result of Zabrodsky (Example $E(a)$ in [22]). We will explain the connection and point out the error in his argument when we verify this example. Incidentally, Example F shows how badly $SNT(\ )$ behaves with respect to products, whereas the next result shows that it works well with connected covers.

COROLLARY 3.2. *Let X be an $n-1$ connected space that satisfies the hypothesis of Theorem 3. If $\tilde{X}$ denotes the n-connective cover of X, then $SNT(X) = *$ implies that $SNT(\tilde{X}) = *$.* □

Here is our main result on classifying spaces.

THEOREM 4. *Let G be a connected compact Lie group. Then $SNT(BG) = *$ if and only if $G = T^k$, $SU(n)$ or $PSU(n)$ when $k \geq 0$ and $n = 2$ or 3.* □

For 1-connected compact Lie groups, this means that $SNT(BG)$ is uncountably large in all but three cases: $G = *$, $SU(2)$, $SU(3)$. Incidentally, these three are the only cases where $Aut\, H^*BG$ is finite! In the remaining cases the abundance of nontrivial operations in cohomology and K-theory places severe restrictions on the possible self equivalences of BG. Indeed, in most cases the image of $Aut\, BG \to Aut\, H^*BG$ is finite while the range is not. We use Hubbuck's results on self maps of BG to show this when G is simple. In other cases we resort to direct calculations involving Weyl groups and the techniques pioneered by Adams and Mahmud, [2]. Of course, once we show the relevant cokernel is infinite, Theorem 4 then follows immediately from Theorem 3.

Let K be a finite complex. The homotopy theoretic nature of ΩK is quite different from that of the BG's in Theorem 4. The next result reflects this disparity. In it, K might be a finite H-space, or a complex Stiefel manifold, or a product of odd dimensional spheres.

THEOREM 5. *Let K be a 1-connected, finite CW-complex that is an H_0-space. Then for any set P of primes, $SNT(\Omega K_P) = *$. Moreover, the map $Aut\,\Omega K_{(p)} \to Aut\,\Omega K_{(p)}^{(n)}$, is surjective at almost all primes p and for almost all integers n.* □

At first we wondered if this result might be a special case of something much more general; namely—Is the obvious map $SNT(X) \to SNT(\Omega X)$ always surjective? The following example rules out this possibility.

Example G. Let $X = K(\mathbf{Z}, 2n) \times S^{2n}$ where $n \geq 2$. Then $SNT(X) = *$, but $SNT(\Omega X) \neq *$. □

Nevertheless, Theorem 5 and a growing list of the examples prompts us to make the following:

Conjecture. If K is a 1-connected finite CW complex, then $SNT(\Omega K) = *$.

The results in Theorems 3 and 5 suggest a possible connection between $SNT(X)$ and $SNT(X_P)$ in general, but its precise nature has so far eluded us. At first, we suspected that the obvious map $SNT(X) \to \Pi_p SNT(X_{(p)})$ was bijective but the following example shows that life is not that simple.

Example H. Let $X = BSU(3)$. Then $SNT(X) = *$ by Theorem 4, but $SNT(X_{(p)}) \neq *$ for $p > 3$. □

Our last result is an application of Theorem 5 and Zabrodsky's genus theorem, [20]. For a nilpotent space X, let $G(X)$ denote its homotopy theoretic genus set defined in terms of localization; that is, the set of all homotopy types $[Y]$ where Y is nilpotent, of finite type, and $Y_{(p)} \simeq X_{(p)}$ for all primes p. With regard to the cardinality of $G(X)$, the following is known. Wilkerson proved that $G(X)$ is a finite set when X is a 1-connected finite complex, [19]. Rector gave a complete description of $G(\mathbf{HP}^\infty)$ and showed it to be uncountably large, [16]. Zabrodsky proved for H_0-spaces X with only a finite number of nonzero homotopy (or homology) groups that $G(X)$ is isomorphic to a finite abelian cokernel determined by a certain monoid of self maps of X, [20]. As mentioned earlier, he also showed the genus of BU to be uncountably large in the last chapter of his book, [21]. Beyond this, not much is known about the genus of infinite dimensional spaces.

THEOREM 6. *Let K be a 1-connected finite CW complex that is an H_0-space. Then the map $G(K) \to G(\Omega K)$ that sends a homotopy type $[X]$ in $G(K)$ to $[\Omega X]$, is an epimorphism. In particular $G(\Omega K)$ inherits from $G(K)$ the structure of a finite abelian group.* □

The situation described in this theorem is very special one and, in general, the map $G(X) \to G(\Omega X)$ is not an epimorphism. Here is a relevant example.

Example I. Let X be the Grassmann manifold of complex n-planes in $\mathbf{C}^{n+k}$ where $n \geq 5$ and $2k > n^2$. Then $G(X) = *$ by Glover and Mislin, [6], but $G(\Omega X) \neq *$. □

This concludes the list of main results. Before starting the proofs, we want to acknowledge the help that Joe Neisendorfer has given us over the past year as these theorems and examples evolved. Thanks, Joe! We will now prove the theorems in the order they were presented and then we will work out the examples.

Proof of Theorem 2. The Mittag-Leffler property is well known to imply that $\varprojlim^1$ is trivial; see for example [4], page 256. Assume then that the tower $\{G_n\}$ is not Mittag-Leffler. This means that for some natural number α, the tower of nested images $\{H_n\}$, where

$$H_n = \text{image}\{G_\alpha \leftarrow G_n\}, \quad n \geq \alpha,$$

has infinitely many different H_n's. Let K_n denote the kernel of the epimorphism $G_n \to H_n$. The 6-term $\varprojlim - \varprojlim^1$ sequence, [4], page 252, applied to the short exact sequence of towers

$$* \to \{K_n\} \to \{G_n\} \to \{H_n\} \to *$$

ends with an epimorphism $\varprojlim^1 G_n \to \varprojlim^1 H_n$. Therefore it suffices to show that $\varprojlim^1 H_n$ is uncountably large. Our notation here does not indicate whether a tower starts at $n = 1$ or $n = \alpha$, because $\varprojlim^1$ of the tower is independent of the particular choice. To minimize subscripts, let $H = H_\alpha$. If each H_n were normal in H, we could form the short exact sequence of towers,

$$* \to \{H_n\} \to \{H\} \to \{H/H_n\} \to *,$$

apply the 6-term $\varprojlim - \varprojlim^1$ sequence and obtain

$$\to H \to \varprojlim H/H_n \to \varprojlim{}^1 H_n \to *.$$

We could then argue, as Gray did, that since H is countable and $\varprojlim H/H_n$ is not, it follows that $\varprojlim^1 H_n$ is also uncountable. We will now show that the same proof goes through almost unchanged without the assumption that each H_n is normal in H.

We denote by H/H_n the set of left cosets of H_n in H. The inclusion $H_n \subset H_{n+1}$ induces a surjection of coset spaces $H/H_n \to H/H_{n+1}$. We take the inverse limit of sets, $\varprojlim H/H_n$, and it is uncountable for the same reasons as before; namely, the H_n's become ever smaller as n increases. A typical element in $\varprojlim H/H_n$ is a nested sequence of left cosets $\{x_nH_n\}$. Let $\partial: \varprojlim H/H_n \to \varprojlim^1 H_n$ send $\{x_nH_n\}$ to the equivalence class $[\{x_n^{-1}x_{n+1}\}]$ in $\varprojlim^1 H_n$. Notice that $x_n^{-1}x_{n+1} \in H_n$, since $x_nH_n \supset x_{n+1}H_{n+1}$. Thus the definition of ∂ makes sense since $\varprojlim^1 H_n$ is the following quotient of $\Pi_\alpha^\infty H_n$. The group ΠH_n acts on the set ΠH_n in such a way that two classes, say $\{u_n\}$ and $\{v_n\}$, are in the same orbit under this action if and only if there is a class $\{w_n\} \in \Pi H_n$ such that $w_nu_n = v_nw_{n+1}$ for all $n \geq \alpha$.

To show that ∂ is well defined suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of representatives for the same sequence of cosets in H. Thus $x_nH_n = y_nH_n$ for each n. If we let $w_n = y_n^{-1}x_n$, it follows that $w_n \in H_n$ and

$$w_nx_n^{-1}x_{n+1} = y_n^{-1}y_{n+1}w_{n+1}.$$

Thus ∂ is well defined; it does not depend upon the particular choice of coset representatives.

To see that ∂ is surjective, take a sequence $\{u_n\} \in \Pi H_n$. Choose

$$\begin{aligned} x_\alpha &= 1 \\ x_{\alpha+1} &= u_\alpha \\ &\vdots \\ x_n &= u_\alpha u_{\alpha+1} \cdots u_{n-1} \\ &\vdots \end{aligned}$$

Then $\partial\{x_nH_n\} = [\{x_n^{-1}x_{n+1}\}] = [\{u_n\}]$ and so ∂ is surjective. There is an obvious action of H on $\varprojlim H/H_n$, namely $(h, \{x_nH_n\}) \mapsto \{hx_nH_n\}$, and it is easily checked that $\partial\{x_nH_n\} = \partial\{hx_nH_n\}$. On the other hand suppose $\{x_nH_n\} = \partial\{y_nH_n\}$. This implies there exists $\{z_n\} \in \Pi H_n$ such that

$$z_nx_n^{-1}x_{n+1} = y_n^{-1}y_{n+1}z_{n+1}$$

or

$$(x_nz_n^{-1})^{-1} \cdot (x_{n+1}z_{n+1}^{-1}) = y_n^{-1}y_{n+1}.$$

Now since $z_n \in H_n$ it follows that $x_nH_n = x_nz_n^{-1}H_n$. Hence, if $\partial\{x_nH_n\} = \partial\{y_nH_n\}$, we may assume that the coset representatives $\{x_n\}$ and $\{y_n\}$ were chosen so that $x_n^{-1}x_{n+1} = y_n^{-1}y_{n+1}$ for all $n \geq \alpha$. Then solving for x_n recursively we obtain

$$x_n = (x_\alpha y_\alpha^{-1})y_n \quad \text{for all } n \geq \alpha.$$

This implies that the surjection ∂ is at most countable-to-one. Therefore $\varprojlim^1 H_n$ must be uncountable and the theorem is proved.

Proof of Theorem 3. The Postnikov system of X is a sequence of principal fibrations,

$$K(\pi, n) \to X^{(n)} \to X^{(n-1)}$$

with certain well known properties. Since X is an H_0-space, the k-invariants, which classify these fibrations, have finite order. We show that for the automorphism groups involved, this has the following consequences.

LEMMA 3.1. *Let X be as in Theorem 3. Then for each integer n,*

(a) *the map $Aut\, X^{(n)} \to Aut\, H^{\leq n}(X^{(n)}; \mathbf{Z}_p)$ has a finite kernel and a finite cokernel, and*
(b) *the canonical map $Aut\, X^{(n+1)} \to Aut\, X^{(n)}$ has a finite cokernel.* □

Notice that the first part of this lemma implies the equivalence of conditions (ii) and (iii) in Theorem 3. To show conditions (i) and (ii) are equivalent, we need the following algebraic result.

LEMMA 3.2. *Let $G_1 \leftarrow G_2 \leftarrow \dots$ be a tower of countable groups in which each map $G_n \leftarrow G_{n+1}$ has a finite cokernel. Then the tower $\{G_k\}$ is Mittag-Leffler if and only if the canonical map $\varprojlim G_k \to G_n$ has a finite cokernel for each n.* □

Assume for the moment that these two lemmas are true. The hypothesis on X implies that $\{Aut\, X^{(k)}\}$ is a tower of countable groups. Therefore, by Theorems 1 and 2, $SNT(X) = *$ if and only if the tower $\{Aut\, X^{(k)}\}$ is Mittag-Leffler. By 3.1(b), this tower then satisfies the hypothesis of Lemma 3.2. Since the canonical map $Aut\, X \to \varprojlim Aut\, X^{(k)}$ is surjective, the equivalence of statements (i) and (ii) in Theorem 3 then follows from Lemma 3.2. The proof of Theorem 3 will be complete once we prove these two lemmas.

Proof of Lemma 3.1. We will first prove part (b) regarding the finite cokernels in the tower $\{Aut\, X^{(n)}\}$. Consider the diagram

$$\begin{tikzcd} X^{(n+1)} \arrow[r, dashed, "\bar{g}"] \arrow[d, "\pi"] & X^{(n+1)} \arrow[d, "\pi"] & \\ X^{(n)} \arrow[r, "g"] & X^{(n)} \arrow[r, "k"] & K(G, n+2). \end{tikzcd}$$

Here k is the k-invariant and so $k \circ \pi = 0$. Moreover since X is an H_0-space, this k-invariant has finite order. It is easy to see that there is a $\bar{g} \in Aut\, X^{(n+1)}$ such that $\bar{g}^{(n)} = g$, provided $k \circ g = k$.

Let $Aut\, X^{(n)}$ act on $H^{n+2}(X^{(n)}; G)$ and consider the orbit

$$\Theta = \{g^*[k] \mid g \in Aut\, X^{(n)}\}.$$

This orbit is finite because it is contained in the torsion subgroup of $H^{n+2}(X^{(n)}; G)$. The latter is finite because X has finite type over $\mathbf{Z}_p$. If N denotes the stabilizer of $[k]$, then there is a bijection between the orbit Θ and the set of left cosets of N in $Aut\, X^{(n)}$. Thus the index of N in $Aut\, X^{(n)}$ is finite. The subgroup N lies in the image of $Aut\, X^{(n+1)}$ and so this image must also have a finite index.

Consider now the map in 3.1(a),

$$Aut\, X^{(n)} \to Aut\, H^{\leq n} X^{(n)}.$$

Assume that all cohomology groups in this proof will have coefficients in $\mathbf{Z}_p$ unless noted otherwise. We will first show this map has a finite kernel. Since $X^{(n)}$ is an H_0-space, there is a commutative diagram,

$$\begin{tikzcd} Aut\, X^{(n)} \arrow[rr] \arrow[dr] & & Aut\, H^{\le n}(X^{(n)}; \mathbf{Q}) \arrow[r, "\approx"] & Aut\, X_0^{(n)} \\ & Aut\, H^{\le n} X^{(n)} \arrow[ur] & & \end{tikzcd}$$

in which the map across the top has a finite kernel, ([8], Cor. 2.2.a). Hence the same is true for the map in question.

We will now show this map has a finite cokernel for all n. Since X is simply connected, this is certainly true for $n = 1$. Assume now that $n \ge 2$, and that the result is true in dimensions less than n. Consider the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L \arrow[d] \\ Aut\, X^{(n)} \arrow[r] \arrow[d] & Aut\, H^{\le n} X^{(n)} \arrow[d] \\ Aut\, X^{(n-1)} \arrow[r] & Aut\, H^{\le n-1} X^{(n-1)}. \end{tikzcd}$$

Here K and L are the kernels of the vertical maps in the lower square. The map on the right is induced by restriction $H^{\le n} \to H^{\le n-1}$ followed by an isomorphism $H^{\le n-1} X^{(n)} \approx H^{\le n-1} X^{(n-1)}$. We assume by induction that the map along the bottom has a finite cokernel. We already know the same is true for the lower left map. Therefore to prove it for the middle map, it suffices to show $coker(K \to L)$ is finite. To do this we need better descriptions of the kernels K and L. We begin with K.

Each class $f \in K$ is represented by a self map of a principal fibration with the identity map on the base space.

$$\begin{tikzcd} K(\pi, n) \arrow[r] \arrow[d] & K(\pi, n) \arrow[d] \\ X^{(n)} \arrow[r, "f"] \arrow[d] & X^{(n)} \arrow[d] \\ X^{(n-1)} \arrow[r, "1"] & X^{(n-1)}. \end{tikzcd}$$

Let $k \in H^{n+1}(X^{(n-1)}, \pi)$ denote the k-invariant of this fibration. There is an obvious action of $Aut(\pi)$ on this cohomology group; let $Aut(\pi)_k$ denote the stabilizer of k. Since k has finite order, $Aut(\pi)_k$ has finite index in $Aut(\pi)$. According to [13], Theorems 2.1 and 2.9, there is an exact sequence

$$H^n(X^{(n-1)}; \pi) \to K \to Aut(\pi)_k \to 1.$$

The map out of K is induced by restriction to the fiber (or equivalently, by sending f to the automorphism it induces on $\pi_n X^{(n)}$). The map going into K occurs as follows. Given a principal fibration, say $F \to E \xrightarrow{p} B$, one can use the principal action $F \times E \to E$, together with maps from base to fiber, say $\varphi\ B \to F$, to create self equivalences of E by taking the composition

$$E \xrightarrow{\Delta} E \times E \xrightarrow{\varphi p \times 1} F \times E \to E$$

Since $[B, F] \approx H^n(X^{(n-1)}; \pi)$ for the fibration in question, this accounts for the map into K.

As for the other kernel L, recall it consists of automorphisms of $H^{\le n} X^{(n)}$ that restrict to the identity in degrees less than n. When we mod out decomposables,

$$DH^n X^{(n)} \to H^n X^{(n)} \to QH^n X^{(n)},$$

we get a sequence

$$Hom(QH^nX^{(n)}, DH^nX^{(n)}) \to L \to Aut(QH^nX^{(n)})$$

that is exact at L. The map $K \to L$ then induces a commutative diagram

$$\begin{tikzcd} H^n(X^{(n-1)}; \pi) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & Aut(\pi)_k \arrow[d] \\ Hom(QH^nX^{(n)}, DH^nX^{(n)}) \arrow[r] & L \arrow[r] & Aut(QH^nX^{(n)}) \end{tikzcd}$$

and to show $coker(K \to L)$ is finite, it suffices to show the other two vertical maps have this property.

Consider first the image of $Aut(\pi)_k$. The Hurewicz–Serre homomorphism $\pi_n X^{(n)} \to QH^nX^{(n)}$ becomes an isomorphism when tensored with $\mathbf{Q}$. Thus the image of this homomorphism is a $\mathbf{Z}_p$-submodule of finite index in $QH^nX^{(n)}$. Let A denote this image, let C denote the module of indecomposables, and let $Aut(C)_A = \{\psi \in Aut(C) | \psi(A) = A\}$. It is not difficult to see that the map $f \mapsto QH^n(f)$ takes $Aut(\pi)_k$ to a subgroup of finite index $Aut(C)_A$ and so it suffices to show the latter has finite index in $Aut(C)$.

Suppose that A has index d in C. There can only be a finite number of submodules of this index in C, because they all must contain dC which, since C is finitely generated, has finite index. $Aut(C)$ acts on this finite set of submodules (each of index d) and so the stabilizer of A, $Aut(C)_A$ has finite index in $Aut(C)$.

Thus we have shown that in the diagram induced by $K \to L$, the map on the right has a finite cokernel. We now consider the map on the left. The easiest way to see that it has a finite cokernel is to rationalize. Let $F \to E \to B$ denote the rationalized fibration. Then $E \simeq B \times F$ and the homomorphism $[B, F] \to Aut(B \times F)$ sends a map φ to the equivalence $(x, y) \mapsto (x, \varphi(x) \cdot y)$. The map in the diagram applies cohomology to this process. When tensored with $\mathbf{Q}$, the map on the left factors as

$$\begin{tikzcd} H^n(B; \pi) \otimes \mathbf{Q} \arrow[rr] \arrow[dr, "\approx"'] & & Hom(\pi^*, (H_nB)^*) \\ & Hom(H_nB, \pi \otimes \mathcal{Q}) \arrow[ur, "f \mapsto f^*"'] & \end{tikzcd}$$

Here we have identified $\pi^* = Hom(\pi, \mathbf{Q})$ with the rational indecomposables and $H^n(B; \mathbf{Q})$ with the corresponding decomposables. Since these rational maps are isomorphisms, the map in question has a finite cokernel and the proof of Lemma 3.1 is complete. □

Proof of Lemma 3.2. Fix n and recall the definitions $G_n^{(k)} = \text{image}(G_n \leftarrow G_{n+k})$ and $K_n^{(k)} = \text{kernel}(G_{n+k} \to G_n)$. We get a short exact sequence of towers,

$$* \to \{K_n^{(k)}\} \to \{G_{n+k}\} \to \{G_n^{(k)}\} \to *$$

Assume that $\{G_k\}$ is Mittag-Leffler and apply the 6-term $\varprojlim$–$\varprojlim^1$ sequence to the above. We get

$$\to \varprojlim G_k \to \varprojlim G_n^{(k)} \to \varprojlim{}^1 K_n^{(k)} \to * \to .$$

By Theorem 2, the third term $\varprojlim^1 K_n^{(k)}$ is either trivial or uncountably large. The second term, $\varprojlim G_n^{(k)} = \cap_k G_n^{(k)} \subseteq G_n$ and so it is countable. Since it maps onto the third, the latter must be trivial. By exactness then, the map $\varprojlim G_k \to \varprojlim G_n^{(k)}$ is surjective. The Mittag-Leffler property means that $\varprojlim G_n^{(k)} = G_n^{(N)}$ for some finite N and so the inclusion $\varprojlim G_n^{(k)} \to G_n$ has a finite cokernel. Composing this with the surjection from $\varprojlim G_k$ shows that the map in question, $\varprojlim G_k \to G_n$ has a finite cokernel.

The proof in the other direction is easier. If $\varprojlim G_k$ maps onto a subgroup I of finite index in G_n then it is easy to see that there can only be a finite number of distinct intermediate subgroups H with $I \leq H \leq G_n$. This forces $\varprojlim G_n^{(k)} = G_n^{(N)}$ for some finite N. Since this is true for each n, the tower is Mittag–Leffler. □

Proof of Corollary 3.1. The proof is a diagram chase using the results of Lemma 3.1. We will first show condition (ii) ⇒ (i). Assume (ii) holds for some $m \geq t$. Since

$$X^{(m)} \simeq X \cup_\alpha (e_\alpha^{m+2}) \cup (\text{higher cells}),$$

it follows that $H^{\leq m}(X^{(m)}; \mathbf{Z}_p) \approx H^{\leq m}(X; \mathbf{Z}_p)$. This accounts for the isomorphism on the right in the following diagram

$$\begin{array}{ccc} Aut\, X & \longrightarrow & Aut\, H^{\leq m}(X; \mathbf{Z}_p) \\ \downarrow & & \uparrow \approx \\ Aut\, X^{(m)} & \longrightarrow & Aut\, H^{\leq m}(X^{(m)}; \mathbf{Z}_p). \end{array}$$

We are assuming the map across the top has a finite cokernel. The map on the bottom has a finite kernel and a finite cokernel by Lemma 3.1a. This forces the map on the left side to have a finite cokernel. This fact, together with a second application of Lemma 3.1, implies that the composition

$$Aut\, X \to Aut\, X^{(m)} \to Aut\, X^{(t)} \to Aut\, H^{\leq t}(X^{(t)}; \mathbf{Z}_p) \xrightarrow{\approx} Aut\, H^{\leq t}(X; \mathbf{Z}_p)$$

has a finite cokernel. Thus condition (ii) implies (i).

For each integer $n > t$, $Aut\, X^{(n)} \to Aut\, X^{(t)}$ has a finite kernel, as well as a finite cokernel. To verify this claim about the kernel, notice the inclusion $X^{(n)} \to X^{(t)}$ is a rational equivalence and then consider the diagram

$$\begin{array}{ccc} Aut\, X^{(n)} & \longrightarrow & Aut\, X_0^{(n)} \\ \downarrow & & \downarrow \approx \\ Aut\, X^{(t)} & \longrightarrow & Aut\, X_0^{(t)}. \end{array}$$

Both horizontal maps in this diagram are finite-to-one, as noted earlier, and so the left side must likewise have a finite kernel. Therefore, the composition

$$Aut\, X \to Aut\, X^{(n)} \to Aut\, X^{(t)}$$

has a finite cokernel if and only if the first map does. The implication (i) ⇒ (iii) then follows, again using Lemma 3.1(a). The two reverse implications, (iii) ⇒ (ii) and (i) ⇒ (ii) are, of course, obvious. □

Proof of Corollary 3.2. Let $\pi = \pi_n X$. We have, up to homotopy, a fibration, $\tilde{X} \to X \to K(\pi, n)$, where the map into the Eilenberg–Mac Lane space is an n-equivalence. Given $f \in Aut\, X$, there is a unique $f^* \in Aut\, K(\pi, n)$ so that the square on the right

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & X & \longrightarrow & K(\pi, n) \\ \downarrow \tilde{f} & & \downarrow f & & \downarrow f_* \\ \tilde{X} & \longrightarrow & X & \longrightarrow & K(\pi, n) \end{array}$$

commutes. The map on the fiber, $\tilde{f}$, then exists to make the left square commute. This map must be an equivalence and it can be chosen so that the assignment $f \mapsto \tilde{f}$ defines a homomorphism $Aut\, X \to Aut\, \tilde{X}$. Take cohomology with coefficients in $\mathbf{Z}_p$, fix a positive integer $t \geq n$, and let E denote the image of $H^{\leq t}X$ in $H^{\leq t}\tilde{X}$. Since X is an H_0-space, $\tilde{X}$ is rationally a retract of X and so E has finite index in $H^{\leq t}\tilde{X}$. Our proof now centers about the following diagram.

$$\begin{array}{ccccc} Aut\,\tilde{X} & \xleftarrow{f\mapsto\tilde{f}} & Aut\,X & \xrightarrow{1} & Aut\,H^{\le t}(X)/\text{torsion} \\ {\scriptstyle 6}\downarrow & & {\scriptstyle 4}\downarrow & & {\scriptstyle 2}\downarrow \\ Aut\,H^{\le t}\tilde{X} & \xleftarrow{5} & Aut\,H^{\le t}\tilde{X}_E & \xrightarrow{3} & Aut(E/\text{torsion}). \end{array}$$

Most of the maps in this diagram are the obvious ones. For example, the center map, marked 4, sends $f \mapsto \tilde{f}^*$. Maps 1 and 6 likewise send $h \mapsto h^*$. We aim to show that *coker*(6) is finite. We begin by assuming $SNT(X) = *$. By Theorem 3 this implies *coker*(1) is finite. The kernel of $H^*X \to H^*\tilde{X}$ is the ideal generated by H^nX and possibly some torsion elements in higher degrees. It follows that every automorphism of $H^{\le t}(X)$/torsion preserves the larger ideal and passes to the quotient, E/torsion. This provides us with the map 2. We claim it is surjective. To see this, consider the composition $\tilde{X} \to X \to \Pi K(Z_P, n_i)$, where the map into the product of Eilenberg–Mac Lane spaces induces isomorphism on $QH^*(\ )$/torsion. It is not hard to see that each $\varphi \in Aut(E/\text{torsion})$ lifts to a self equivalence of $\Pi K(Z_P, n_i)$. This, of course, implies that 2 is surjective.

The domain of 3 is those automorphisms of $H^{\le t}\tilde{X}$ that take E to itself. Since E has finite index in $H^{\le t}\tilde{X}$, it follows easily that the kernel of 3 is finite. The commutativity of the right square then forces 4 to have a finite cokernel. The cokernel of 5 is also finite by essentially the same argument as the one used on $Aut(C)_A$ in the proof of Lemma 3.1. The commutativity of the left square then forces *coker* (6) to be finite. The conclusion then follows by Theorem 3. □

Proof of Theorem 4. The proof is a long one and so here is an outline of it. We first show that $SNT(BG)$ is trivial for the groups listed. This is the easy part. We are then left to show that $SNT(BG)$ is nontrivial for the remaining groups. We prove this first for the simple 1-connected Lie groups. Then we consider products of these groups using

LEMMA 4.1. *Let G and H be compact connected Lie groups. If either $SNT(BG)$ or $SNT(BH)$ is nontrivial, then likewise $SNT(B(G \times H)) \neq *$.* □

By the classification theorem we are then left (in the simply connected case) with showing

PROPOSITION 4.2. *$SNT(BG) \neq *$ for $G = SU(m) \times SU(n)$ where $(m, n) = (2,2)$, $(2,3)$, or $(3,3)$.* □

In the nonsimply connected case we use Corollary 3.2. It implies that $SNT(BG) \neq *$ provided the universal cover $\tilde{G}$ contains as a retract one of the simply connected groups H for which we have already shown $SNT(BH) \neq *$. Again, by the classification theorem, there are just a few groups left that don't fit this description.

PROPOSITION 4.3. *$SNT(BG) \neq *$ for $G = U(n)$ or $T^1 \times SU(n)$ where $n = 2$ or 3.* □

This, of course, is not a complete list of the groups G whose universal cover $\tilde{G}$ is homotopy equivalent to $SU(2)$ or $SU(3)$. However, the proof of Theorem 4 for those omitted, such as $T^k \times U(n)$ or $T^k \times PSU(n)$, follows easily from this last proposition using either Lemma 4.1 or Corollary 3.2.

This completes the outline of the proof. Let us begin. We start with those groups G for which $SNT(BG)$ is claimed to be trivial. For the torus T^k, this is clear since BT^k is an

Eilenberg–Mac Lane space. For the special unitary groups $G = SU(2)$ or $SU(3)$, $H^*BG = \mathbf{Z}[x_4]$ or $\mathbf{Z}[x_4, x_6]$. Thus $Aut(H^{\le t}BG) = \mathbf{Z}/2$ or $(\mathbf{Z}/2)^2$ for almost all values of t. In particular, these cohomology automorphisms groups are finite. This certainly forces the corresponding cokernel to be finite and so $SNT(BG) = *$ by Theorem 3. For the projective special unitary groups, $G = PSU(n)$, BG is rationally equivalent to $BSU(n)$ and, of course, it also has finite type. This implies that $Aut(H^{\le t}BG)$ is likewise finite and so, again by Theorem 3, $SNT(BG) = *$.

Assume now that G is simple and 1-connected. Hubbuck, with some help from Ishiguro, has shown that if f is a self map of BG, then there is an outer automorphism $\tau: G \to G$ such that the diagram

$$\begin{array}{ccc} BU & \xrightarrow{\Psi^k} & BU \\ \uparrow g & & \uparrow g \\ BG & \xrightarrow{f \cdot B_\tau} & BG \end{array}$$

commutes for some nonnegative integer k and for all g, [9], [10]. Recall that in cohomology, the Adams operation Ψ^k has the property $\Psi^k x = k^n x$ for all x of degree $2n$. Therefore when f is a self equivalence, the Ψ^k in the diagram must be Ψ^1, the identity map. Moreover since the group of outer automorphisms of G is always finite (the largest it can be is Σ_3 when $G = Spin(8)$), it follows that the image of $Aut(BG)$ is finite. On the other hand, the groups $Aut(H^{\le t}BG)$ are almost never finite. The only exceptions are the ones considered already: $*$, $SU(2)$, and $SU(3)$.

The reason behind this lies in the rational cohomology of BG and whether or not there exist polynomial generators and decomposable terms with the same degree. Take the groups $G = SU(n)$, $n \ge 4$ or $Sp(n)$, $n \ge 2$. For these it is easy to describe infinite families of automorphisms of H^*BG, by making use of the decomposable terms. For example, since $H^*BSU(n) = \mathbf{Z}[c_2, c_3, \ldots, c_n]$ where each c_i has degree $2i$, one could define

$$\varphi_\alpha(c_k) = \begin{cases} c_k & \text{if } k \neq 4 \\ c_4 + \alpha(c_2)^2 \end{cases}$$

where α could be any integer. This φ_α extends in a unique manner to an automorphism of H^*BG. Notice that $Aut(H^*BG) \approx Aut(H^{\le t}BG)$ in the special case where H^*BG is a polynomial algebra. The situation becomes more complicated, however, when H^*BG is not a polynomial algebra. Here we take a different approach to showing $Aut(H^{\le t}BG)$ is infinite. Consider the following diagram

$$\begin{array}{ccc} Aut(BG^{(n)}) & \to & Aut(H^{\le n}BG) \\ \vdots & & \\ Aut(K) & \to & Aut(H^*K) \end{array}$$

where K is a finite product of $K(\mathbf{Z}, n)$'s with the rational homotopy type of BG. By Lemma 3.1 the top map has a finite kernel and a finite cokernel. Likewise the bottom map is easily seen to be an epimorphism with a finite kernel. It follows from Wilkerson, [19], Theorem 2.3, the two groups joined by the dotted line are commensurable which, in particular, implies they have the same cardinality. Now it is a simple matter to show $Aut(H^*K)$ is infinite. One can pick generators and write down automorphisms in terms of them as was just done in the polynomial case. Finding an appropriate decomposable term with the same degree as one of the generators is a matter of simple arithmetic. It depends, of course, only on the rational type of G. The specific types (for example [3, 11, 15, 23] for F_4) can be found in Appendix A of Kane's book [11]. A quick glance through this list of types then confirms that for all of the remaining simple Lie groups G in question, $Aut(H^{\le t}BG)$ is infinite. Together with Hubbuck's Theorem and Theorem 3, this proves Theorem 4 for all simple 1-connected Lie groups.

Proof of Lemma 4.1. We can (and will) identify $B(G\times H)$ with $BG\times BH$. Assume that $SNT(BG\times BH)=*$. It follows from Theorem 2 that the tower $Aut(BG\times BH)^{(n)}$ is Mittag–Leffler as the groups in this tower are countable.

For each automorphism group, say A_n, in this tower let n' denote the smallest t such that

$$\operatorname{image}\{A_n \leftarrow A_t\} = \operatorname{image}\{A_n \leftarrow A_{t+k}\}$$

for all $k \ge 0$. We will show that the tower $Aut(BG)^{(n)}$ is also Mittag–Leffler. Take

$$f \in \operatorname{image}\{Aut(BG)^{(n)} \leftarrow Aut(BG)^{(n')}\}.$$

It then follows that

$$f\times 1 \in \operatorname{image}\{Aut(BG\times BH)^{(n)} \leftarrow Aut(BG\times BH)^{(n')}\}.$$

Here we have identified $(BG\times BH)^{(n)}$ with $BG^{(n)}\times BH^{(n)}$. Using this identification, if $f'\in Aut(BG)^{(n')}$ projects to f, then $f'\times 1$ projects to $f\times 1$. It follows just as in the proof of Lemma 3.2, that $f\times 1$ lies in the image of $\varprojlim Aut(BG\times BH)^{(n)}$. Consequently there is an equivalence $\varphi\in Aut(BG\times BH)$ with $\varphi^{(n)}=f\times 1$. The composition

$$BG \xrightarrow{i_1} BG\times BH \xrightarrow{\varphi} BG\times BH \xrightarrow{\pi_1} BG$$

is then a self map, say φ_1, of BG. Since the cohomology of BG is finitely generated, [15], we may assume that n was initially chosen larger than the degree of any of the generators. Then if $\varphi_1\colon BG \circlearrowleft$ is an n-equivalence, it is easily seen to induce a surjection (and hence an isomorphism) in cohomology and so, $\varphi_1\in Aut(BG)$. It follows, therefore that the tower $Aut(BG)^{(n)}$ is Mittag–Leffler and so the lemma is proved. □

Proof of Proposition 4.2. Assume that $G=SU(m+1)\times SU(n+1)$ where $n\ge m\ge 1$. Our theorem only requires three values of (m,n) but the proof of the general case is no harder. Since $QH^4(BG)\approx \mathbf{Z}\oplus\mathbf{Z}$ it follows that $GL(2,\mathbf{Z})$ is a subgroup of $Aut(H^*BG)$ and so, in particular, the automorphism group is infinitely large. Once again, we will show that the image of $Aut\,BG \to Aut\,H^*(BG)$ is finite. We need the following.

LEMMA 4.4. *Let G be a connected compact Lie group with maximal torus $T\xrightarrow{i} G$, and Weyl group W. Given $f\in Aut\,BG$, there exists $F\in Aut\,BT$ such that*

$$\begin{array}{ccc} BG & \xrightarrow{F} & BT \\ \downarrow{\scriptstyle Bi} & & \downarrow{\scriptstyle Bi} \\ BG & \xrightarrow{f} & BG \end{array}$$

commutes. In terms of the action of W on BT, for each $w\in W$, there exists $w'\in W$ such that $Fw=w'F$. □

This lemma is a direct consequence of a theorem of Notbohm, [14] that states that the function

$$Rep(T,G) \xrightarrow{\rho\mapsto B\rho} [BT,BG]$$

is a bijection. In more detail, his theorem implies that $f \cdot Bi = B\varphi$ for some homomorphism $\varphi: T \to G$. We may assume $\varphi(T) \subseteq T$, by composing it with an inner automorphism of G, if necessary. Since f is an automorphism, it follows that F is one also. The rest of the lemma is then immediate.

From now on we will regard the Weyl group W as a subgroup of $AutBT$. Since F is invertible we have $FwF^{-1} = w'$, and so F lies in the normalizer of W in $AutBT$. In the case at hand we have $W \approx \Sigma_{m+1} \times \Sigma_{n+1}$ and $AutBT \approx GL(m+n, \mathbb{Z})$. Moreover the cohomology of BG embeds in H^*BT and so F determines f, at least up to homology. Therefore, for the purpose of showing that the image of $Aut\,BG \to Aut(H^*BG)$ is finite it suffices to show that the normalizer of W in $GL(n+m, \mathbb{Z})$ is finite. We have an embedding,

$$N/C \to Aut(W),$$

where N and C are the normalizer and centralizer of W in $GL(m+n, \mathbb{Z})$, respectively. Since $Aut(W)$ is finite it suffices to show that the centralizer C is finite.

We claim that C is contained in the group of diagonal matrices in $GL(m+n, \mathbb{Z})$. The latter is certainly finite with order 2^{n+m}. To see this claim about C, recall that the symmetric group Σ_{n+1}, is generated by n transpositions $\sigma_i = (i, i+1)$, $i = 1, \ldots, n$. In its role as the Weyl group for $SU(n+1)$, Σ_{n+1} acts on a vector space U with a certain nonorthogonal basis $\{u_1, u_2, \ldots, u_n\}$ in such a way that each σ_i sends u_i to $-u_i$ and fixes pointwise the orthogonal complement of u_i. Moreover, the matrix representation of this action is defined over $\mathbb{Z}$. See ([11], p 156) for more details.

These properties hold as well for $\Sigma_{m+1} \times \Sigma_{n+1}$ acting as the Weyl group for $SU(m+1) \times SU(n+1)$ on a vector space V with an appropriate basis $\{v_1, v_2, \ldots, v_{m+n}\}$. We have $m+n$ reflections σ_i that generate this product group. Each σ_i sends v_i to $-v_i$ and fixes pointwise the orthogonal complement of v_i. The action is again defined over $\mathbb{Z}$.

Suppose that an element $g \in GL(V)$ centralizes W. Then $\sigma_i g = g\sigma_i$ for each reflection σ_i. Notice that

$$\sigma_i(g(v_i)) = g\sigma_i(v_i) = g(-v_i) = -g(v_i).$$

Since $\sigma_i(x) = -x$ if x is on the line spanned by v_i it follows that $g(v_i)$ is a multiple of v_i. Since this is true for each i, g must be diagonal with respect to the basis $\{v_1, \ldots, v_{m+n}\}$. Our claim, that the centralizer of W in $GL(m+n, \mathbb{Z})$ is finite, is now verified. □

The proof of Proposition 4.3 is similar to the one just given. Once again the general case, $1 < n < \infty$, is just as easy as the case $n = 2$ or 3. To show $SNT(BU(n)) \neq *$ for such n, we will show that the image of $Aut\,BU(n)$ in $Aut\,H^*BU(n)$ is finite. To this end it suffices to examine the Weyl group W of $U(n)$ and show that the centralizer of W in $Aut\,BT^n \approx GL(n, \mathbb{Z})$ is finite. Now the Weyl group of $U(n)$ is the symmetric group Σ_n and the representation $\Sigma_n \to GL(n, \mathbb{Z})$ in this case is that of the permutation matrices (with only zeroes and ones for entries). Let σ denote a permutation matrix and take $M \in GL(n, \mathbb{Z})$. Then $M \mapsto \sigma^{-1}M\sigma$ defines an action of Σ_n on the entries of M. This action is easily checked to be transitive on the diagonal entries of M and also transitive on the set of off-diagonal entries of M. Therefore M is in the centralizer of Σ_n if and only if it has the form (m_{ij}) where

$$m_{ij} = \begin{cases} a & \text{if } i = j \\ b & \text{if } i \neq j \end{cases}$$

for two integers a and b. This $n \times n$ matrix has determinant $(a-b)^{n-1}(a+(n-1)b)$. It is easy to check that the only integer solutions to the equation $\det(M) = \pm 1$ are $(a, b) = (\pm 1, 0)$ for $n \geq 2$ and $(0, \pm 1)$ when $n = 2$. Thus the centralizer in $GL(n, \mathbb{Z})$ of the Weyl group of $U(n)$ has order 4, when $n = 2$, and has order 2 for $2 < n < \infty$.

Let $W \subsetneq GL(n, \mathbf{Z})$ and $W' \subsetneq GL(n+1, \mathbf{Z})$ denote the Weyl groups of $SU(n)$ and $T^1 \times SU(n)$ respectively. Both groups are, of course, isomorphic to Σ_n and the embedding $GL(n, \mathbf{Z}) \to GL(n+1, \mathbf{Z})$, sending

$$M \mapsto \begin{pmatrix} 1 & \\ & M \end{pmatrix}$$

takes W onto W'. The basis vector e_1 is then fixed by every element of w' and, up to scalar multiples, is the only nonzero vector in $\mathbf{R}^{n+1}$ with this property. If g centralizes W', then $g(e_1) = w(g(e_1))$ for every $w \in W'$ and so $g(e_1) = \pm e_1$. Thus the centralizer of W' in $GL(n+1, \mathbf{Z})$ has order twice that of the centralizer of W in $GL(n, \mathbf{Z})$. The latter was shown to have order $\leq 2^n$ in Proposition 4.2. This completes the proof of 4.3 and also of Theorem 4.

Proof of Theorem 5. The hypothesis on K implies that it is p-equivalent to a finite product of odd dimensional spheres for all but a finite set, E, of exceptional primes. We will first prove the theorem for subsets $P \subseteq E$, and then for subsets in the complement of E. The general case will be shown to be an easy consequence of these results.

In the P-local part of the proof, all spaces will be assumed to be localized at P and all cohomology groups will have $\mathbf{Z}_P$ for coefficients unless noted otherwise. Take t to be the largest n for which $\pi_n \Omega K \otimes \mathbf{Q} \neq 0$. Let L denote the kernel of the map $Aut(H^{\leq t}\Omega K) \to Aut(QH^{\leq t}\Omega K)$. We will show that $Aut(\Omega K)$ maps onto a subgroup of finite index in $Aut(QH^{\leq t}\Omega K)$ and that $L \cap \{\text{image of } Aut(\Omega K)\}$ has finite index in L. This will give the result we seek; the order of $coker\{Aut(\Omega K) \to Aut(H^{\leq t}\Omega K)\}$ will be bounded above by the product of these two indexes.

Since ΩK has finite type over $\mathbf{Z}_P$, the graded $\mathbf{Z}_P$-module $QH^{\leq t}\Omega K$ is finitely generated and hence has a finite torsion subgroup. It follows that $Aut(\Omega K)$ maps onto a subgroup of finite index in $Aut(QH^{\leq t}\Omega K)$ if and only if it does so in $Aut(QH^{\leq t}(\Omega K)/\text{torsion})$. Now

$$Aut(QH^{\leq t}(\Omega K)/\text{torsion}) \approx \prod_{j=1}^{t} GL(r_j, \mathbf{Z}_P)$$

where r_j is the rank of $QH^j(\Omega K)/\text{torsion}$. Of course, $GL(0, \mathbf{Z}_P) = \{1\}$ and $GL(1, \mathbf{Z}_P) =$ the group of units in $\mathbf{Z}_P$.

We will construct self maps of ΩK using the following result

PROPOSITION 5.1. *Let K, as in Theorem 5, be localized at a set of primes P. Then there is a finite product of P-local, odd dimensional spheres and maps*

$$K \xrightarrow{a} \Pi S^{n_i} \xrightarrow{b} K$$

such that for some $N > 0$, ab and ba induce multiplication by N on indecomposables. This number N can be assumed to be relatively prime to all primes not in P. □

The proof will be given later. Suppose now that $r_j \geq 1$ for some j. This means there are exactly r_j spheres of dimension $j+1$ in the product ΠS^{n_i}. Let X denote the product of r_j copies of ΩS^{j+1} and write $\Omega \Pi S^{n_i} \simeq X \times Y$. For any $r_j \times r_j$-matrix M one can use the product multiplication on X to define a self map, say $\mathscr{F}(M)$: $X \to X$, with the property that $\mathscr{F}(M)$ induces multiplication by M on $\pi_j X$. In particular take U in $GL(r_j, \mathbf{Z}_P)$ with $U \equiv I \bmod N$. Let $\varphi: \Omega \Pi S^{n_i} \circlearrowleft$ be the product self map that restricts on X to $\mathscr{F}((U - I)/N)$ and is null homotopic on Y. Let 1 denote the identity on ΩK. Use the loop multiplication on ΩK to form the sum of self maps $1 + \Omega b \varphi \Omega a$. This is an equivalence. It projects to U in

$GL(r_j, \mathbf{Z}_P)$ and to the identity on the other factors. The congruence subgroup defined by $U \equiv I \mod N$ has finite index in $GL(r_j, \mathbf{Z}_P)$ and so it follows that the map $Aut(\Omega K) \to Aut(QH^{\leq t}\Omega K)$ has a finite cokernel.

To understand the kernel L, consider first the rational cohomology of ΩK. It is a polynomial algebra on a finite number of primitive generators $\{x_1, \ldots, x_r\}$ in even degrees. We will assume that $\deg(x_i) \leq \deg(x_{i+1})$. Let

$$L_0 = \{\varphi \in Aut(H^*(\Omega K; \mathbf{Q})) | \varphi \equiv 1 \text{ mod decomposables}\}.$$

It is clear that L_0 is generated by automorphisms of the form $1 + (x_j \to cm)$ that send

$$x_i \mapsto \begin{cases} x_i & \text{if } i \neq j \\ x_j + cm & \text{if } i = j, \end{cases}$$

where m is a monomial in the x_i's, of length ≥ 2, with the same degree as x_j and $c \in \mathbf{Q}$. There is an isomorphism from L_0 onto a certain subgroup of upper triangular matrices in $GL(d, \mathbf{Q})$. Here d is the sum of Betti numbers, $d = \Sigma_{i=1}^{t}$ rank $H^i(\Omega K; \mathbf{Q})$. To set up this isomorphism, order the monomials of degree $\leq t$ as follows: Start with $x_1 < x_2 < \ldots < x_r$ and then say $m < m'$ if $\deg(m) < \deg(m')$. For monomials of the same degree, say $m < m'$ if the fraction m/m', in lowest terms, has a bigger subscript in its denominator. Use these ordered monomials to label the rows and columns of a $d \times d$ matrix starting with the smallest in the upper left hand corner. Then assign to each $\varphi \in L_0$ the matrix of coefficients whose row m-column m' entry is the coefficient on m in $\varphi(m')$. This is the isomorphism alluded to earlier.

We seek the analogous representation from L to the corresponding subgroup of upper triangular matrices in $GL(d, \mathbf{Z}_P)$. This can be obtained from the rational one as follows. We can assume that the rational monomials m, lie in the image of the coefficient homomorphism induced by the inclusion $\mathbf{Z}_P \subset \mathbf{Q}$, and as such are not divisible by any prime in P. If this is not so, replace any offender by an appropriate multiple of itself. Then for each (possibly altered) m, let $[m]$ denote its preimage in $H^{\leq t}(\Omega K)$/torsion. The $[m]$'s form an additive basis for this $\mathbf{Z}_P$-module, and thus provided a framework for the representation we seek. This representation, call it $\rho: L \to GL(d, \mathbf{Z}_P)$ has a finite kernel since the torsion subgroup of $H^{\leq t}(\Omega K)$ is finite.

Let N^s be a sufficiently large power (to be determined later) of the number mentioned in Proposition 5.1 and take any $c \in N^s\mathbf{Z}_P$. We will show that each automorphism of the form $1 + ([x_j] \to c[m])$ in L is realizable by a self map of ΩK. Here, of course, $[x_j]$ and $[m]$ are assumed to have the same degree and $[m]$ is rationally decomposable. Once we have produced these maps it will follow that the index $(L : I \cap L)$ is finite, where I denotes the image of $Aut(\Omega K)$ in $Aut(H^{\leq t}\Omega K)$. To see this, let C denote the congruence subgroup defined by $x \equiv I$ mode N^s in $GL(d, \mathbf{Z}_P)$. C has finite index in this matrix group and so the inclusions

$$C \cap \rho(L) \subseteq \rho(I \cap L) \subseteq \rho(L)$$

together with the finite kernel of ρ imply that $(L: I \cap L)$ is finite.

Since the $[x_i]$'s are primitive, it is clear that cohomology automorphisms of the form $1 + ([x_j] \to c[m])$ are induced by maps of the form $1 + f: \Omega K \circlearrowleft$ where, as usual, 1 denotes the identity map, $+$ denotes loop addition, and f is a self map with the property:

$$f^*[x_i] = \begin{cases} 0 & \text{if } i \neq j \\ c[m] & \text{if } i = j. \end{cases}$$

Assume $[x_j]$ and $[m]$ have degree d and take f to be the composition

$$\Omega K \xrightarrow{\Omega u} \Omega\Pi S^{n_i} \xrightarrow{1} \Omega S^{d+1} \xrightarrow{2} \Omega S^{d+1} \xrightarrow{3} \Omega K$$

where, in cohomology,

1 sends a generator, say ι, to $(\Omega b)^*[m]$
2 sends ι to $(c/N^s)\iota$
3 sends $[x_j]$ to $N\iota$ and the other $[m]$'s to zero.

The existence of the first map follows from the decomposition of $\Sigma\Omega\Pi S^{n_i}$ as a wedge of spheres. The third map is a consequence of Proposition 5.1. If l denotes the rational length of $[m]$ then it follows $f^*[x_j] = cN^{l+1-s}[m]$. Consequently we can take the power s to be 1 plus the maximum length of products in $H^{\leq t}(\Omega K; \mathbf{Q})$.

This completes the proof that $Aut(\Omega K_P) \to Aut\, H^{\leq t}(\Omega K_P; \mathbf{Z}_P)$ has a finite cokernel, for subsets $P \subseteq E$ where E is the set of primes for which K is not p-equivalent to a product of spheres. For primes in the complement of E, the following result applies.

PROPOSITION 5.2. *Let $X \simeq S^{2n_1+1} \times \ldots \times S^{2n_r+1}$, or the localization of this product at some set of primes. Assume that $0 < n_1 \leq \ldots \leq n_r$. Then $Aut\,\Omega X \to Aut\, X^{(k)}$ is surjective for $k > 2n_r$.* □

The proof will be given later.

Given a set S of primes, write it as the disjoint union, $S = A \cup B$, where $A = S \cap E$. In the commutative diagram,

$$\begin{array}{ccc} Aut\Omega K_A & \to & Aut\, H^{\leq t}(\Omega K_A; \mathbf{Z}_A) \\ \downarrow & & \downarrow \\ Aut\Omega K_0 & \to & Aut\, H^{\leq t}(\Omega K_0; \mathbf{Q}) \\ \uparrow & & \uparrow \\ Aut\Omega K_B & \to & Aut\, H^{\leq t}(\Omega K_B; \mathbf{Z}_B) \end{array}$$

we have already seen that the map across the top has a finite cokernel. The map across the bottom does too, by Lemma 5.2 and Theorem 3. The canonical map from $Aut\Omega K_S$ to the pullback of the diagram

$$\begin{array}{ccc} & & Aut\Omega K_A \\ & & \downarrow \\ Aut\Omega K_B & \to & Aut\Omega K_0 \end{array}$$

is surjective by Theorem 4.1 of [17], while $Aut\, H^{\leq t}(\Omega K_S, \mathbf{Z}_S)$ is easily seen to be isomorphic to the pullback of

$$\begin{array}{ccc} & & Aut\, H^{\leq t}(\Omega K_A; \mathbf{Z}_A) \\ & & \downarrow \\ Aut\, H^{\leq t}(\Omega K_B, \mathbf{Z}_B) & \to & Aut\, H^{\leq t}(\Omega K_0, \mathbf{Q}). \end{array}$$

The map between these two pullbacks, induced by the commutative diagram that preceded them, has a finite cokernel since all three horizontal maps in the diagram do. Thus it follows $Aut\Omega K_S \to Aut\, H^{\leq t}(\Omega K_S, \mathbf{Z}_S)$ has a finite cokernel and so the theorem follows from Corollary 3.1.

We now prove the two propositions.

Proof of 5.1. All spaces in this proof will be assumed to be P-local although this notation won't reflect this. Let $K(\mathbf{Z}, \bar{n})$ denote the product Eilenberg–Mac Lane space $\Pi K(\mathbf{Z}, n_i)$. There is clearly a map $f: K \to K(\mathbf{Z}, \bar{n})$ that is a rational equivalence. Indeed we can choose f to induce an isomorphism on $QH^*(\ )$/torsion. Moreover, if $\infty > t > \max(n_i)$, there is a map $g: K(\mathbf{Z}, \bar{n}) \to K^{(t)}$ that is also a rational equivalence ([20], Proposition 1.7). Similar maps (f' and g') exist when K is replaced by ΠS^{n_i}. Take $t > \dim K$ and obtain the map $a: K \to \Pi S^{n_i}$ as follows, using the cellular approximation theorem:

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[drr, "a"'] & K(\mathbf{Z}, \bar{n}) \arrow[r, "g'"] & (\Pi S^{n_i})^{(t)} \\ & & \Pi S^{n_i} \arrow[u] \end{tikzcd}$$

Obtain the other map b in the same manner:

$$\begin{tikzcd} \Pi S^{n_i} \arrow[r, "f'"] \arrow[drr, "b"'] & K(\mathbf{Z}, \bar{n}) \arrow[r, "g"] & (K)^{(t)} \\ & & K \arrow[u] \end{tikzcd}$$

It is clear that a and b are rational equivalences. On $QH^*(\ )$/*torsion* they induce monomorphisms whose finite cokernels can be assumed to be P-primary. Therefore, if ab and ba do not induce multiplication by N on indecomposables this can be corrected by composing one of the maps, say a, with an appropriate self map of ΠS^{n_i}. □

Proof of 5.2. For each positive integer k, we have

$$\begin{aligned} Aut\,\Omega X^{(k)} &\subseteq [\Omega X, \Omega X^{(k)}] \\ &\approx [\Sigma\Omega X, X^{(k)}] \\ &\approx [\vee_\alpha S^{n_\alpha}, X^{(k)}] \text{ since } \Sigma\Omega X \simeq \vee_\alpha S^{n_\alpha} \\ &\approx [\vee_{n_\alpha \le k} S^{n_\alpha}, X^{(k)}] \\ &\approx [\vee_{n_\alpha \le k} S^{n_\alpha}, X] \\ &\subseteq [\Sigma\Omega X, X] \approx [\Omega X, \Omega X]. \end{aligned}$$

In other words, for each $f \in Aut\,\Omega X^{(k)}$ there is a self map $g: \Omega X \circlearrowleft$ with $g^{(k)} = f$. We claim that if $k > 2n_r$, then g must be in $Aut\,\Omega X$. To prove it we will show that if $g: \Omega X \circlearrowleft$ is a k-equivalence (where $k > 2n_r$) then $g^*: H^*(\Omega X; \mathbf{Z}) \circlearrowleft$ is an isomorphism. Of course, since this ring has finite type, it suffices to show that g^* is an epimorphism. Now $H^*(\Omega X; \mathbf{Z})$ is isomorphic to a product of divided power algebras and so to see that g^* is surjective it is enough to check that classes of the form $x^n/n!$ are in its image. Here x has degree $2n_i$ and is in the image of the map induced by a projection $\pi: \Omega X \to \Omega S^{2n_i+1}$. Since $g^{(k)}$ is an equivalence, it has an inverse. This inverse likewise has the form $\bar{g}^{(k)}$ for some $\bar{g}: \Omega X \circlearrowleft$. We may then assume that $x = g^* \bar{g}^* \pi^*(\iota)$ where ι is a suitable generator. It follows easily that $x^n/n!$ is in the image of this composition of maps and thus in the image of g^*. □

Proof of Theorem 6. For any nilpotent space X there is the obvious map

$$G(X) \xrightarrow{\Phi} \varprojlim G(X^{(n)})$$

that sends a homotopy type $[Y]$ from $G(X)$ to the coherent sequence of homotopy types, $([Y^{(1)}], [Y^{(2)}], \ldots, [Y^{(n)}], \ldots)$.

LEMMA 6.1. *The map Φ, just defined, is*

(a) *injective if $SNT(Y) = *$ for every $[Y] \in G(X)$, and*
(b) *surjective if $SNT(X_{(p)}) = *$ for each prime p.* □

Assume for now this lemma is true. Using it and Theorem 5 we show

PROPOSITION 6.2. *The map $\Phi: G(\Omega K) \to \varprojlim G(\Omega K^{(n)})$ is a bijection for K as in Theorem* 6. □

Now recall Zabrodsky's genus theorem [20]. Given a connected nilpotent H_0-space X with finite type and with only a finite number of nonzero homotopy groups, he showed that for any t sufficiently large (in the sense of the prime powers that divide it) there is an exact sequence

$$\mathscr{E}_t(X) \xrightarrow{\det} \prod^{k} ((\mathbb{Z}/t)^*/\pm 1) \to G(X) \to 0.$$

The first term here is the monoid, under composition, of homotopy classes of self maps of X, that are equivalences at each prime divisor of t. The integer k, in the middle term is the number of n's for which $QH^n(X, \mathbb{Z})/\text{torsion} \neq 0$. Given $f \in \mathscr{E}_t(X)$, the determinant of f^* on this quotient module will be a unit in $\mathbb{Z}/t$. The map *det* assigns to f the sequence of these determinants, modulo the indicated $\mathbb{Z}/2$ action on each coordinate. Although the domain of *det* is not a group, its image is; it coincides with the kernel of the map onto $G(X)$.

Now take n larger than the dimension of K. There is a commutative diagram

$$\begin{array}{ccccc}
\mathscr{E}_t(K^{(n)}) & \xrightarrow{\det} & \Pi^k((\mathbb{Z}/t)^*/\pm 1) & \to & G(K^{(n)}) \\
\downarrow & & \downarrow = & & \downarrow \\
\mathscr{E}_t(\Omega K^{(n)}) & \xrightarrow{\det} & \Pi^k((\mathbb{Z}/t)^*/\pm 1) & \to & G(\Omega K^{(n)})
\end{array}$$

where the map on the left sends f to Ωf, and the map on the right sends X to ΩX. It follows that the map on the right is an epimorphism with domain isomorphic to $G(K)$. Together with Proposition 6.2, this implies the theorem.

Proof of Lemma 6.1. To say $\Phi(Y) = \Phi(Z)$ means $Y^{(n)} \simeq Z^{(n)}$ for each n. Thus if $SNT(Y) = *$, then $Y \simeq Z$ and Φ is injective.

Take a coherent sequence $\{[Y_n]\}$ in $\varprojlim G(X^{(n)})$. Since $[Y_n] \in G(X^{(n)})$, $\pi_q Y_n = 0$ for $q > n$. For each n choose an n-equivalence $f_n: Y_{n+1} \to Y_n$. Using simplicial methods, [4], one can produce a space

$$Y = \varprojlim (Y_1 \xleftarrow{f_1} Y_2 \xleftarrow{f_2} \dots)$$

with the property $Y^{(n)} \simeq Y_n$ for each n. It follows that $Y_{(p)}^{(n)} \simeq X_{(p)}^{(n)}$ for each n and for each prime p. Assuming $SNT(X_{(p)}) = *$, it follows that $Y_{(p)} \simeq X_{(p)}$. Thus $Y \in G(X)$, and so Φ is surjective.

Proof of Proposition 6.2. Let Y represent a homotopy type in $G(\Omega K)$. Take P to be the set of all primes p for which the map $Aut\ Y_{(p)} \to Aut\ Y_{(p)}^{(t)}$ is surjective. Here, as usual, t is a finite dimension above which the rational homotopy groups of Y vanish. It follows that the corresponding map between pullbacks

$$\text{pullback}\left\{\begin{array}{c}\Pi_p Aut\, Y_{(p)}\\ \downarrow\\ Aut\, Y_0\end{array}\right\} \longrightarrow \text{pullback}\left\{\begin{array}{c}\Pi_p Aut\, Y_{(p)}^{(t)}\\ \downarrow\\ Aut\, Y_0^{(t)}\end{array}\right\}$$

is also surjective. The canonical map from $Aut\, Y_P$ to the first pullback is surjective, by [17], while the second pullback is isomorphic to $Aut\, Y_P^{(t)}$. This combined with Lemma 3.1 implies the map

$$Aut\, Y_P \longrightarrow Aut\, H^{\le t}(Y; \mathbf{Z}_P)$$

has a finite cokernel. From Theorem 5 we know that P is cofinite. If l is a prime not in P, and $P' = P \cup \{l\}$, then the corresponding map

$$Aut\, Y_{P'} \longrightarrow Aut\, H^{\le t}(Y; \mathbf{Z}_{P'})$$

has a finite cokernel because it is the composition of three maps similarly endowed. The first is the surjection from $Aut\, Y_{P'}$ to the pullback of the left column in the following diagram

$$\begin{array}{ccc}
Aut\, Y_P & \to & Aut\, H^{\le t}(Y; \mathbf{Z}_P)\\
\downarrow & & \downarrow\\
Aut\, Y_0 & \to & Aut\, H^{\le t}(Y; \mathbf{Q})\\
\uparrow & & \uparrow\\
Aut\, Y_{(l)} & \to & Aut\, H^{\le t}(Y; \mathbf{Z}_{(l)}).
\end{array}$$

The second is the map of pullbacks induced by this diagram. It has a finite cokernel because all three horizontal maps in the diagram do. The third is an isomorphism between $Aut\, H^{\le t}(Y, \mathbf{Z}_{P'})$ and the pullback of the right column of the diagram. After repeating this argument of a finite number of times (once for each prime not in P) we see that $Aut\, Y \to Aut\, H^{\le t}(Y; \mathbf{Z})$ has a finite cokernel for each $t < \infty$ and so $SNT(Y) = *$ by Theorem 3. Thus $\Phi: G(\Omega K) \to \varprojlim G(\Omega K^{(n)})$ is injective by 6.1. That it is surjective follows directly from 6.1 and Theorem 5. □

THE EXAMPLES

Example A. Adams took K to be a sphere but his proof adapts easily. Here is how it goes. It is easy to see that X and $X \times K$ have the same n-type; namely the weak product of $K^{(1)}, K^{(2)}, \ldots, K^{(n-1)}$ and $\aleph_0$ copies of $K^{(n)}$. Now suppose that X and $X \times K$ were homotopy equivalent. It would follow that K is a retract of X. Since K is compact, its image under the associated inclusion $i: K \to X$ would be compact. Since X has the direct limit topology, this implies that the inclusion of K factors through a finite product $K^{(1)} \times \ldots \times K^{(m)}$ and thus K would be a retract of this finite product. Consequently $\pi_* K$ would be a direct summand of the homotopy groups of this finite product. This is impossible, because the homotopy groups of the finite product vanish above a certain finite dimension whereas, by Serre's theorem, those of K are nonzero infinitely often.

Example B. Since $\varprojlim G_n = 0$, it is clear that this tower is not Mittag-Leffler. Let $\{G_1\}$ denote the constant tower in which each group is G_1 and each map is the identity map. Use the inclusions $G_n \subsetneq G_1$ to form a short exact sequence of towers

$$* \to \{G_n\} \to \{G_1\} \to \{G_1/G_n\} \to *$$

Notice that $G_1/G_n \approx H_1 \times \cdots \times H_{n-1}$. It follows that $\varprojlim\{G_1/G_n\} \approx \Pi_{n\geq 1} H_n = G_1 = \varprojlim\{G_1\}$ and that the quotient map in the sequence above induces an isomorphism between these two limits. Since the constant tower is certainly Mittag-Leffler we conclude that $\varprojlim^1 G_n = *$, by using the 6 term $\varprojlim$–$\varprojlim^1$ sequence.

Example C. The ring structure in H^*K is trivial for dimensional reasons and so it follows that $Aut(H^{\leq n}K)$ contains as a direct summand, $Aut(H^2K)$ which is isomorphic to $GL(2, \mathbf{Z})$. We will show that the image of $Aut(K)$ in $GL(2, \mathbf{Z})$ is the subgroup of upper triangular matrices and hence has infinite index. Take $f \in Aut(S^2 \vee S^2)$ and notice that f is the restriction of a class from $Aut(K)$ if and only if $f_\#(\varphi) = n\varphi$ in $\pi_4(S^2 \vee S^2)$, for some integer n. Suppose that

$$f_\#(\iota_1) = a\iota_1 + c\iota_2$$
$$f_\#(\iota_2) = b\iota_1 + d\iota_2.$$

Since $\varphi = [[\iota_1, \iota_2], \iota_1]$, a straightforward calculation yields

$$f_\#(\varphi) = a(ad - bc)\varphi - c(ad - bc)[[\iota_1, \iota_2], \iota_2].$$

This last Whitehead product is not a multiple of φ. Since the determinant $ad - bc = \pm 1$, the entry c must be zero for f to extend to a self equivalence of K. Of course, the Hurewicz homomorphism faithfully records this in H^2K. Thus the image of $Aut(K)$ in $GL(2, \mathbf{Z})$ is the upper triangular matrices.

Example D. Each automorphism of π preserves the commutator subgroup and so there is an obvious homomorphism,

$$Aut(\pi) \to Aut(\pi_{ab}).$$

It suffices to show the cokernel here is infinite. Let x, y, and z be generators of π where x and y generate the subgroup N, and z generates the infinite cyclic subgroup $\mathbf{Z}$. Then the commutator $[x, y]$ generates $[N, N]$ while the images $\bar{x}$ and $\bar{y}$ generate the quotient $N_{ab} \approx \mathbf{Z} \times \mathbf{Z}$. The center $\mathscr{Z}(\pi)$ is clearly isomorphic to $[N, N] \times \mathbf{Z}$. Since every automorphism must also preserve the center, it follows that if $\varphi \in Aut(\pi_{ab})$ lifts to an automorphism of π, then $\varphi(\bar{z}) = \pm \bar{z}$. Since $Aut(\pi_{ab}) \approx GL(3, \mathbf{Z})$, where the isomorphism is given in terms of the basis $\{\bar{x}, \bar{y}, \bar{z}\}$, we see that the image of $Aut(\pi)$ is contained in the subgroup of those matrices whose righthand columns have zeros in their top and middle entries. The index of this subgroup is, of course, infinite.

Example E. Take a finite type CW-decomposition of BU with cells in even dimensions only. Given $f \in Aut(BU^{(2n)})$, restrict it to the $2n$-skeleton to get a map $f_n : BU_{(2n)} \to BU$. Since $\pi_q BU = 0$ for q odd, there are no obstructions to extending this f_n to a self map, f_∞ of BU. The extension, f_∞ is not unique. It may or may not be a self equivalence of BU. With care, it can be chosen to be in $Aut(BU)$. To see this, assume first that $f_\#$ has degree 1 on $\pi_q BU$ for $q \leq 2n$ and suppose that f_∞ has degree $\lambda \neq 1$ for the first time on $\pi_{2m}BU$ for some $m > n$. Using G. Segal's splitting $Q(\mathbf{CP}^\infty) \simeq BU \times F$, where F has finite homotopy groups, there is a connection between stable self maps of $\mathbf{CP}^\infty$ and self maps of BU. Exploiting this connection, as was done in [12], one can show that $\lambda - 1$ is a multiple of $(m - 1)!$ and that there is a self map, say g, of BU that has degree $1 - \lambda$ on $\pi_{2m}BU$ and is null homotopic on lower skeleta. We can replace f_∞ by $g + f_\infty$ and repeat this argument if necessary to achieve an extension of f_n that is in $Aut(BU)$. If f does not induce the identity on $\pi_* BU^{(2n)}$, there are three other possibilities: the $degree(f_n)$ on π_q could be $(-1)^q$, $(-1)^{q+1}$ or -1. In each case there is a self equivalence of $\Sigma^\infty \mathbf{CP}^\infty$ and of BU that induces this homomorphism and the argument just used adapts to these cases as well.

The proof for BSp is virtually the same. J. Becker's splitting, $Q(\mathbf{HP}^\infty) \simeq BSp \times F'$, and the results of [12] on stable self maps of $\mathbf{HP}^\infty$ imply that every $4n$-equivalence $BSp_{(4n)} \to BSp$ extends to a self equivalence of BSp.

Example F. Let P denote a nonempty set of primes and assume that all spaces associated with this example are localized at P. By Corollary 3.1, it suffices to consider the map

$$Aut\, X \to Aut\, H^{2n+1}(X, \mathbf{Z}_P).$$

Choose a basis for $H^{2n+1}(X, \mathbf{Z}_P)$ using the projections of X onto its two factors. We can then identify the automorphisms of this cohomology group with matrices in $GL(2, \mathbf{Z}_P)$. We claim that the image of $Aut\, X$ in this matrix group lies in the subgroup of upper triangular matrices. As such, it would, of course, have infinite index and so by 3.1, the result $SNT(X) \neq *$ would follow.

Let $f \in Aut(X)$ and consider the composition

$$K(\mathbf{Z}, 2n+1) \xrightarrow{i_2} X \xrightarrow{f} X \xrightarrow{\pi_1} S^{2n+1}.$$

Zabrodsky has extended Miller's solution to the Sullivan conjecture to show that this composition must induce the trivial map in cohomology. In fact, in [22], Theorem D, he shows that if Z is a 1-connected space with finite type that has only finitely many nonzero homotopy groups and Y is a finite complex, then the only maps from Z to either Y or ΩY are phantom; that is, they must restrict to a null homotopic map on every finite skeleton of the domain. Consequently, if f^*, from the composition above, is represented by a 2×2 matrix acting on the left, then that matrix must be upper triangular. The same argument applies to the loopspace to show $SNT(\Omega X) \neq *$.

We mentioned earlier that this example contradicts a result of Zabrodsky. To be precise, in the same paper (p. 135) he considers fibrations of the form

$$S^n \to E \to K(\mathbf{Z}, m), \quad m > 2$$

and claims that if n is odd, then such a fibration must be trivial. With this in mind, take $Y \in SNT(S^3 \times K(\mathbf{Z}, 3))$. Take a generator of $\pi_3 Y$, say $g: S^3 \to Y$, and let $f: Y \to K(\mathbf{Z}, 3)$ represent in $H^3 Y$, a generator of the kernel of g^*. It follows that, up to homotopy, f is a fibration with fiber S^3. But if f is the trivial fibration, then its total space, Y, must be homotopy equivalent to the product of the fiber and base. This, of course, implies $SNT(S^3 \times K(\mathbf{Z}, 3)) = *$, contradicting our result.

Zabrodsky's claim was based on a computation of the rational homotopy groups of the function space, $aut_1 S^n$, which is in error, (p. 142). Unfortunately, he is no longer around to correct it, and so we take this opportunity to do so. The function space in question consists of all *unbased* self-maps of S^n of degree 1. The evaluation map gives us a fibration

$$(\Omega^n S^n)_1 \to aut_1 S^n \to S^n$$

and, rationally, this fibration has a section when n is odd. His error was the omission of $\pi_n(aut_1 S^n) \otimes \mathbf{Q}$ when n is odd.

Speaking of errors, Example F also contradicts a claim we made in an AMS abstract announcing a preliminary version of Theorem 3. We claimed there that $SNT(X)$ was trivial if the map $Aut\, X \to Aut(H^*(X, \mathbf{Z})/\text{torsion})$ had a finite cokernel. This is not true! A glaring counterexample to this claim is $\Omega S^3 \times \mathbf{CP}^\infty$. For this example, the map in question is surjective!

Example G. Let $Im(k) = \text{image}\{Aut\,X^{(k)} \to Aut\,X_0^{(k)}\}$. Wilkerson showed in ([18], Prop. III) that

$$\varprojlim{}^1 Im(k) \approx \varprojlim{}^1 Aut(X^{(k)})$$

and so this gives us another way to compute $SNT(X)$. In particular for $X = S^{2n} \times K(\mathbf{Z}, 2n)$, we will show that the tower $\{Im(k)\}$ is almost constant and thus $\varprojlim^1$ of it is trivial. More precisely we will show that the maps in the tower, $Im(k) \leftarrow Im(k+1)$, are isomorphisms for all $k > 4n$. Choose a basis for $H^{2n}(X^{(k)}, \mathbf{Z})$ using the projections of X onto its two factors. We claim that

$$Im(k) \approx \text{image}\{Aut\,X^{(k)} \to Aut\,H^{2n}(X^{(k)}, \mathbf{Z})\}$$

and when the target is identified with $GL(2, \mathbf{Z})$, $Im(k)$ coincides with the subgroup of all lower triangular matrices. To see this, write $X = S \times K$ where S is the sphere and K is the Eilenberg–Mac Lane space. Then for $k > 4n$, we have

$$\begin{aligned} Im(k) &\subseteq [X^{(k)}, X_0^{(k)}] \\ &\approx [X, X_0] \\ &\approx [S \times K, X_0] \\ &\subseteq [S \vee K, X_0] \\ &\approx [S, X_0] \times [K, X_0]. \end{aligned}$$

It is not difficult to see that the last mapping sets mentioned are faithfully detected by cohomology in degree $2n$. The only relation listed above that isn't obvious is the fourth one. It can be verified using the commutative square

$$\begin{array}{ccc} [S \vee K, S \times K] & \leftarrow & [S \times K, S \times K] \\ \downarrow & & \downarrow \\ [S \vee K, K \times K] & \leftarrow & [S \times K, K \times K] \end{array}$$

in which the bottom map is an isomorphism of groups and the vertical maps are induced by a principal fibration and are injective. Thus we have shown that $Im(k)$ can be detected by $H^{2n}(\ ,\mathbf{Z})$. To determine those cohomology automorphisms that are realizable, let $f \in Aut(X^{(k)})$ where again $k > 4n$. This choice of k insures that for any $z \in H^{2n}((S^{2n})^{(k)}, \mathbf{Z})$, that $z^2 = 0$. It follows that the composition,

$$K(\mathbf{Z}, 2n) \xrightarrow{i_2} X^{(k)} \xrightarrow{f} X^{(k)} \xrightarrow{\pi_1} (S^{2n})^{(k)}$$

must induce the zero map in $H^{2n}(\ ,\mathbf{Z})$ because it induces the zero map in $H^{4n}(\ ,\mathbf{Q})$. Thus $Im(k)$ is isomorphic to a subgroup of lower triangular matrices. We claim it contains all of them. Indeed, take any $f \in Aut(S)$, $g \in Aut(K)$, and $h \in [S, K]$. We then have a map,

$$\begin{gathered} S \times K \to S \times K \\ (s, k) \mapsto (f(s), h(s) \cdot g(k)). \end{gathered}$$

Since any lower triangular matrix in $GL(2, \mathbf{Z})$ can be realized by an appropriate choice of f, g, and h the claim follows.

Thus we have shown $SNT(X) = *$. The proof that $SNT(\Omega X) \neq *$ is much easier. Since ΩX is an H_0 space with just two nonzero rational homotopy groups, it suffices to consider the cokernel of the following composition

$$Aut(\Omega X) \to Aut(H^{\leq 4n-2}\Omega X) \to Aut(QH^{2n-1}\Omega X) \to GL(2, \mathbf{Z}).$$

Here the last map is an isomorphism and the middle map has a finite cokernel. Since the only map from $K(\mathbf{Z}, 2n-1)$ to ΩS^{2n} is the trivial one, the image of this composition consists again of the lower triangular matrices. The cokernel is thus infinite and by Corollary 3.1, the result follows.

Example H. Let $\mathscr{U}$ denote the group of units in $\mathbf{Z}_{(p)}$. Since $H^*(X, \mathbf{Z}_{(p)}) \approx \mathbf{Z}_{(p)}[x_4, x_6]$, it is easy to see that when $n \geq 6$,

$$Aut\, H^{\leq n}(X, \mathbf{Z}_{(p)}) \approx \mathscr{U} \times \mathscr{U}.$$

Let I denote the image of $Aut X$ in $Aut\, H^{\leq n}(X, \mathbf{Z}_{(p)})$ and assume for the moment that

$$I = \{(u^2, u^3) \mid u \in \mathscr{U}\}.$$

Then I has infinite index in $\mathscr{U} \times \mathscr{U}$. If this is not obvious, consider the element $(1, p+1)$. It is in $\mathscr{U} \times \mathscr{U}$, but no nonzero power of it is in I. Thus it projects to an element of infinite order in the quotient $(\mathscr{U} \times \mathscr{U})/I$. So the cokernel is infinite and it follows that $SNT(X) \neq *$.

We will sketch the computation for the image, I. It seems easier to work with $U(3)$ rather than $SU(3)$ and so we will, as follows . . . Take $f \in Aut\, X$. Since $BU(3)$ is p-equivalent to $X \times \mathbf{CP}^{\infty}$, replace f on X, by $g = f \times 1$ on $BU(3)$. We would like to restrict g to a maximal torus as was done in Lemma 4.4. While p-local versions of 4.4 are false, (this can be checked, for example, on $BSU(2)$), there is a p-complete version due to Adams and Wojtkowiak, [3]. So from here on, take all spaces given to be p-complete and let $\mathbf{Z}_p$ denote the p-adic integers. Then by the result just cited, g is covered by a map $\bar{g} \in Aut BT^3$. We identify this automorphism group with $GL(3, \mathbf{Z}_p)$ and the Weyl group of $U(3)$ with the subgroup W, of permutation matrices. The class $\bar{g}$ lies in $N(W)$, the normalizer of W in $GL(3, \mathbf{Z}_{(p)})$. One can show that $N(W) \approx W \times C(W)$ where the centralizer $C(W)$, consists of all matrices M_{ab} in $GL(3, \mathbf{Z}_p)$ of the form,

$$M_{ab} = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}.$$

This description of the centralizer was established in the proof of 4.3. To determine the cohomology automorphism induced by the map f, it suffices to let $\bar{g} = M_{ab}$ and consider its effect on the standard invariant forms:

$$\sigma_1 = t_1 + t_2 + t_3$$
$$\sigma_2 = t_1 t_2 + t_1 t_3 + t_2 t_3$$
$$\sigma_3 = t_1 t_2 t_3.$$

Here the t_i's form a basis for $H^2(BT^3, \mathbf{Z}_p)$ and M_{ab} sends

$$t_1 \mapsto a t_1 + b t_2 + b t_3$$
$$t_2 \mapsto b t_1 + a t_2 + b t_3$$
$$t_3 \mapsto b t_1 + b t_2 + a t_3.$$

A long calculation (omitted from this sketch) then shows that, modulo decomposables,

$$\bar{g}^* \sigma_1 = (a + 2b)\sigma_1$$
$$\bar{g}^* \sigma_2 = (a - b)^2 \sigma_2$$
$$\bar{g}^* \sigma_3 = (a - b)^3 \sigma_3.$$

The term $u = a - b$ is a p-adic unit because $det(M_{ab}) = (a + 2b)(a - b)^2$. Since u^2 and u^3 are evidently p-local, so is u, and the description of the image, I, given above follows.

Example I. Let W denote the Stiefel manifold of n-frames in $\mathbf{C}^{n+k}$. Consider the principal $U(n)$ fibration,

$$U(n) \to W \to X,$$

obtained by sending an n-frame of orthonormal vectors to the n-plane they span. Since the dimension of $U(n)$ is n^2 and the connectivity of W is $2k$, the condition $n^2 < 2k$ insures that the inclusion of the fiber, $U(n) \to W$, is nullhomotopic. This, in turn, forces $\Omega X \simeq \Omega W \times U(n)$. It is then easy to see that $G(U(n)) \subset G(\Omega X)$. Since Zabrodsky has shown that $G(U(n)) \neq *$ for $n \geq 5$, ([21] p. 152), the claim follows.

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