Math. Z. 241, 761–799 (2002) **Mathematische Zeitschrift**
DOI: 10.1007/s00209-002-0445-3

Rational equivalences between classifying spaces

Jesper Michael Møller

Matematisk Institut, Universitetsparken 5, 2100 København, Denmark
(e-mail: moller@math.ku.dk)

Received: 22 June 1992; in final form: 30 August 1993/
Published online: 6 August 2002 –

Abstract. The paper contains a homotopy classification of rational equivalences between classifying spaces of compact connected Lie groups with an application to genus sets of such spaces.

Mathematics Subject Classification (1991): 55R35, 55P10, 55P60, 55S37

0. Introduction

Let G_1 and G_2 be two connected compact Lie groups with classifying spaces BG_1 and BG_2. Write $G_1 \geq G_2$ if there exists a Lie group epimorphism $G_1 \to G_2$ with finite kernel and write $BG_1 \geq BG_2$ if there exists a rational equivalence, i.e. a map whose rationalization is a homotopy equivalence, from BG_1 to BG_2. The purpose of this paper is to investigate the transitive relation $\geq$ on Lie groups as well as on classifying spaces.

Consider, for some fixed connected compact Lie group G, the finite set of all isomorphism classes of connected compact Lie groups locally isomorphic to G. This set, equipped with the covering group relation $\geq$, is called the local isomorphism system of G. Its structure was analyzed by Baum [Ba]: Classical Lie group theory tells us that the local isomorphism system of G contains a (uniquely determined) maximal element of the form $H \times T$ where H is the simply connected compact Lie group homotopy equivalent to the universal covering group of G and T is a torus of rank ≥ 0 (isomorphic to the connected component of the center of G.) Any element of the local isomorphism system of G is [Ba, Proposition 2] the quotient of $H \times T$ by a

I thank C.U. Jensen and S.Jøndrup for supplying the proof of Lemma 3.2 and K. Ishiguro and D. Notbohm for many entertaining discussions

special subgroup – a special subgroup, denoted (K, φ), is [Ba, 1] the graph of some homomorphism $\varphi : K \to T$ defined on a central subgroup K of H. Thus we may write

$$(*) \qquad G = H \times T \big/ (K, \varphi)$$

for some (essentially unique) special subgroup (K, φ).

Letting $\mathrm{Epi}(G_1, G_2)$ denote the set of Lie epimorphisms of G_1 onto G_2, the main technical advantage of Baum's approach is expressed in

Theorem 0.1. [Ba, Proposition 5, Corollary 6] *Assume that*

$$G_1 = H \times T \big/ (K_1, \varphi_1) \textit{ and } G_2 = H \times T \big/ (K_2, \varphi_2)$$

are two connected compact Lie groups, locally isomorphic to G, expressed as quotients of $H \times T$.

1. $\mathrm{Epi}(G_1, G_2)$ *corresponds bijectively to the set of all pairs*

   $$(\alpha, \beta) \in \mathrm{Aut}(H) \times \mathrm{Epi}(T)$$

   for which $\alpha(K_1) \subset K_2$ and the diagram

   $$\begin{tikzcd} K_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha|K_1"'] & T \arrow[d, two heads, "\beta"] \\ K_2 \arrow[r, "\varphi_2"'] & T \end{tikzcd}$$

   commutes.
2. *The pair (α, β), satisfying the conditions in (1), represents a Lie group isomorphism $G_1 \xrightarrow{\cong} G_2$ if and only if $\beta \in \mathrm{Aut}(T)$ and $\alpha(K_1) = K_2$.*

Here, $\mathrm{Epi}(T)$ denotes the monoid of of epimorphisms of T onto itself and $\mathrm{Aut}(H)$ the group of Lie automorphisms of H.

The bijection in point (1) associates to the pair (α, β) the epimorphism $G_1 \twoheadrightarrow G_2$ covered by $\alpha \times \beta : H \times T \twoheadrightarrow H \times T$. If G is presented as in $(*)$, any other presentation of G has by point (2) the form

$$G \cong H \times T \big/ (\alpha(K),\ \beta \circ \varphi \circ (\alpha|K)^{-1})$$

for $(\alpha, \beta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(T)$.

As K_1 is central in H, the restriction of α to K_1 only depends on the image of α in the group $\mathrm{Out}(H)$ of outer automorphisms of H. And since $\mathrm{Out}(H)$ is a finite group, isomorphic to the automorphism group of the Dynkin diagram of H, Baum's method makes it possible to do actual calculations – a graphic display of the local isomorphism system of $U(30)$ can be found in [Ba].

A surprising result of Baum's analysis is that two connected compact Lie groups can cover each other without being isomorphic. (The simplest example of this occurs in the local isomorphism system of $U(5)$.) Here is a systematic way of constructing such examples. Suppose that G is presented as in formula $(*)$ above and let $[G, G]$ denote the set of isomorphism classes of connected compact Lie groups L with $G \geq L \geq G$. For any epimorphism $R\colon T \twoheadrightarrow T$ which is injective on the subgroup $\varphi(K) \subset T$, define

$$F_R G = H \times T/(K, R\varphi)$$

where (inspired by [N-S]) the 'F' stands for 'fake'. The commutative diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, equal] & T \arrow[d, two heads, "R"] \\ K \arrow[r, "R\varphi"'] & T \end{tikzcd}$$

shows (Theorem 0.1) that $G \geq F_R G$.

The isomorphism type of $F_R G$ depends on the presentation $(*)$. However,

$$F_R\left(H \times T/(K, \varphi)\right) \cong F_{R\circ\beta^{-1}}\left(H \times T/(\alpha(K), \beta\circ\varphi\circ(\alpha|K)^{-1})\right)$$

for any choice of $(\alpha, \beta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(T)$. Hence the set of isomorphism types $\{F_R G\}$, with R running through the allowable epimorphisms, is presentation independent.

It turns out that also $F_R G \geq G$ so that

$$G \geq F_R(G) \geq G$$

or, equivalently, $F_R G \in [G, G]$, whenever the group in the middle is defined. Actually, any group in $[G, G]$ has this form.

Proposition 0.2. [Proposition 3.1] $[G, G]$ *equals the set of isomorphism classes of Lie groups of the form* $F_R G$ *where* $R \in \mathrm{Epi}(T)$ *is injective on* $\varphi(K) \subset T$.

A proof of this statement can be found in Section 3. The material of Section 3 also shows that $[G, G]$ is parameterized by the quotient of $\mathrm{Aut}(\varphi(K))$ by the subgroup of those group automorphisms of $\varphi(K)$ that extend to automorphisms of T. For example, $[U(n), U(n)]$, $n > 2$ contains $\frac{1}{2}\varphi(n)$ elements where φ is Euler's φ-function. (These Lie groups are denoted $FU_k(n)$ in [N-S].)

If φ is trivial, i.e. $G = (H/K) \times T$, or if G is simple, $[G, G] = \{G\}$.

Following the pioneers [R–S], we now consider Baum's method from a homotopy point of view.

Recall that $BG_1 \geq BG_2$ means that there exists a rational equivalence $BG_1 \to BG_2$. Of course, $G_1 \geq G_2$ implies $BG_1 \geq BG_2$ but the converse implication is not true in general (see below). However, $BG_1 \geq BG_2$ does imply (Proposition 2.7) that G_1 and G_2 are locally isomorphic so we are in a situation where Baum's method applies.

In the following homotopy version of Theorem 0.1, $\varepsilon_{\mathbb{Q}}(BG_1, BG_2) \subset [BG_1, BG_2]$ denotes the set of homotopy classes of rational equivalences of BG_1 to BG_2 and $\varepsilon_{\mathbb{Q}}(BH) \subset [BH, BH]$ the monoid of homotopy classes of rational self-equivalences of BH. The invertible elements in $\varepsilon_{\mathbb{Q}}(BH)$ form the group $\mathrm{Aut}(BH)$ of homotopy classes of homotopy self-equivalences of BH.

Theorem 0.3. *Assume that*

$$G_1 = H \times T\big/(K_1, \varphi_1) \text{ and } G_2 = H \times T\big/(K_2, \varphi_2)$$

are two connected compact Lie groups, locally isomorphic to G, expressed as quotients of $H \times T$.

1. *$\varepsilon_{\mathbb{Q}}(BG_1, BG_2)$ corresponds bijectively to the set of all pairs*

$$(f_H, \varphi) \in \varepsilon_{\mathbb{Q}}(BH) \times \mathrm{Epi}(T)$$

for which $\eta(f_H)(K_1) \subset K_2$ and the diagram

$$\begin{tikzcd} K_1 \arrow[r, "\varphi_1"] \arrow[d, tail, "\eta(f_H)|K_1"'] & T \arrow[d, two heads, "\varphi"] \\ K_2 \arrow[r, "\varphi_2"'] & T \end{tikzcd}$$

commutes.

2. *The pair (f_H, φ), satisfying the conditions in (1), represents a homotopy equivalence $BG_1 \xrightarrow{\simeq} BG_2$ if and only if $f_H \in \mathrm{Aut}(BH)$, $\varphi \in \mathrm{Aut}(T)$, and $\eta(f_H)(K_1) = K_2$.*

In the above theorem, $\eta : \varepsilon_{\mathbb{Q}}(BH) \to \mathrm{Aut}(Z_H)$, where Z_H is the center of H, is a certain homomorphism defined in Section 2. See Theorem 2.3 for a p-complete version of Theorem 0.3.

The bijection in point (1) associates to (f_H, φ) a rational equivalence $BG_1 \to BG_2$ covered by $f_H \times \varphi : BH \times BT \to BH \times BT$.

The monoid $\varepsilon_{\mathbb{Q}}(BH)$ and the function η are completely known, rather simple, algebraic structures, see [J-M-O$_1$], ([Mø], Theorem 3.1) and Example 2.2, and this theoretically complete determination of all rational equivalences between classifying spaces also appears to be a very user-friendly algorithm in concrete applications; see Example 2.6 containing e.g.

$$\varepsilon_{\mathbb{Q}}\left(BU(n)\right)=\{\psi^{\lambda}\times\psi^{\kappa}\mid(\lambda,\kappa)\in\mathbb{Z}\times\mathbb{Z},\,(\lambda,n!)=1,\,\lambda\equiv\kappa \bmod n\},\quad n>2,$$

together with a few other explicit formulas.

Jackowski, McClure & Oliver's foundational method leads to an alternative description [J-M-O_2] of rational self-equivalences in terms of admissible homomorphisms and root systems. Notbohm [N_1] offers a K-theoretic classification of rational self-equivalences.

Bearing in mind that $\varepsilon_{\mathbb{Q}}(BH)$ in general contains more than just induced maps [J-M-O_1,Mø], a comparison of Theorem 0.1 and Theorem 0.3 indicates why there exist rational equivalences $BG_1\to BG_2$ that are not induced by Lie group epimorphisms $G_1\twoheadrightarrow G_2$. In greater detail, suppose $H=H_1\times\cdots\times H_t$ is the factorization of the simply connected compact Lie group H into simple factors. Then $\varepsilon_{\mathbb{Q}}(BH)$ contains products of unstable Adams operations of the form $f=\psi^{n_1}\times\cdots\times\psi^{n_t}$ where the exponent n_i is prime to the Weyl group order of H_i. We have $\eta(f)=\psi^{n_1}\times\cdots\times\psi^{n_t}$ where $\psi^{n_i}(z)=z^{n_i}$ for any z in the center of H_i. Assuming that G is given as in the above formula $(*)$, define

$$G[n_1,\dots,n_t]=H\times T\Big/\left(\left(\prod\psi^{n_i}\right)(K),\,\varphi\circ\left(\prod\psi^{n_i}|Z_H\right)^{-1}\right)$$

and note the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "(\prod\psi^{n_i})|K"', "\cong"] & T \arrow[d, equal] \\ \left(\prod\psi^{n_i}\right)(K) \arrow[r, "\varphi\circ(\prod\psi^{n_i}|Z_H)^{-1}"'] & T \end{tikzcd}$$

showing (Theorem 0.3) that $BG\geq BG[n_1,\dots,n_t]$. But, generally speaking, no Lie group epimorphism takes G onto $G[n_1,\dots,n_t]$ as (Theorem 0.1) no Lie group automorphism takes K isomorphically onto $\left(\prod\psi^{n_i}\right)(K)$. (Thus $G[n_1,\dots,n_t]\to G$ is a 'finite covering map of finite loop spaces' but not necessarily a finite covering map of compact Lie groups.) The isomorphism type of $G[n_1,\dots,n_t]$ is independent of the chosen presentation $(*)$.

Using this construction to make up for the lack of unstable Adams operations in the Lie category we obtain

Proposition 0.4. *Let G_1 and G_2 be as in Theorem 0.1 and Theorem 0.3.*

1. *$BG_1\geq BG_2$ if and only if $G_1[n_1,\dots,n_t]\geq G_2$ for some natural numbers n_i with n_i prime to the Weyl group order of the simple factor H_i.*

2. $BG_1 \simeq BG_2$ *if and only if* $G_1 \cong G_2$ *as Lie groups.*

Point (2) in Proposition 0.4 was proved by H. Scheerer [S] in case of simply connected compact Lie groups; A. Osse [O] recently published an independent and quite different proof also for connected compact Lie groups, and D. Notbohm [N_3] obtained the same result for compact Lie groups in general.

Actually also $BG[n_1, \dots, n_t] \geq BG$ for $BG[n_1, \dots, n_t][m_1, \dots, m_t]$ when $n_i m_i \equiv 1$ modulo the Weyl group order of H_i. Thus

$$BG \geq BF_R G_1[n_1, \dots, n_t] \geq BG$$

whenever the middle classifying space is defined. This observation leads to a homotopy version of Proposition 0.2.

Let $\langle G, G\rangle$ denote the set of isomorphism classes of connected compact Lie groups L for which $BG \geq BL \geq BG$; thus $F_R G[n_1, \dots, n_t] \in \langle G, G\rangle$.

Proposition 0.5. [Proposition 3.5] $\langle G, G\rangle$ *equals the set of isomorphism classes of Lie groups of the form* $F_R G[n_1, \dots, n_t]$ *where* $R \in \mathrm{Epi}(T)$ *is injective on* $\varphi(K) \subset T$ *and* $n_1, \dots, n_t$ *are natural numbers with* n_i *prime to the Weyl group order of the simple factor* H_i*.*

Of course, $[G, G] \subset \langle G, G\rangle$; however, these two sets are not in general identical (see Example 3.6). If K is trivial, i.e. $G = H \times T$, or G is simple, $\langle G, G\rangle = \{G\}$.

Now follows an application of Theorem 0.3 and its p-complete analog, Theorem 2.3, to genus sets of Lie group classifying spaces.

Let B be a connected nilpotent space of finite type. Denote by $G_0^\wedge(B)$ the genus set consisting of all connected nilpotent homotopy types X of finite type with completion $X^\wedge \simeq B^\wedge$ and rationalization $X_{(0)} \simeq B_{(0)}$.

The perhaps most famous example of a genus set is $G_0^\wedge(Sp(2)) = \{Sp(2), E_5\}$ where E_5 is called the Hilton–Roitberg criminal [H–R]. Another striking result is Zabrodsky's estimate [Z_1, Example 4.6.7]

$$|G_0^\wedge(SU(n))| \geq \prod_{m=2}^{n} \frac{\varphi((m-1)!)}{2}$$

of the size of the genus set of $SU(n)$. On the other hand compact Lie groups modulo maximal tori are generically rigid, i.e. their genus sets are singletons [G–M, P]. The finiteness of these genus sets is not accidental. Indeed, $G_0^\wedge(B)$ is finite, by Wilkerson [W, Theorem C], for any simply connected finite CW-complex B. The situation changes drastically when turning to infinite complexes. Rector [R], with help from McGibbon [McG] at the prime $p = 2$, found an invariant classifying the elements of the genus set of $BSU(2)$ and used it to show that $G_0^\wedge(BSU(2))$ is uncountably large;

in fact $G_0^\wedge(BG)$ is uncountably large [Mø, Theorem 2.3] for any non-abelian connected compact Lie group G. Notbohm & Smith investigated the genus set of $BU(n)$; their FUN Proposition [N-S, Proposition 2.1] asserts that $B([U(n), U(n)]) \subset G_0^\wedge(BU(n))$, i.e. that the classifying space of any Lie group covering and covered by $U(n)$ is of the same genus as $BU(n)$.

The following construction may be seen as an attempt to extract the essence of the FUN Proposition. Assume that G is a connected compact Lie group presented as in formula $(*)$ above. For any prime p define

$$G^{(p)} = H \times T/(K_p, \varphi|K_p)$$

where K_p is the p-primary part of the abelian group $K \subset Z_H$. ($G^{(p)} = H \times T$ for almost all p.) The isomorphism type of $G^{(p)}$ is independent of the choice of special subgroup (K, φ).

Proposition 0.6. *[Proposition 2.7, Proposition 4.1] Let L be a connected compact Lie group. Then $BL \in G_0^\wedge(BG)$ if and only if $L^{(p)} \in \langle G^{(p)}, G^{(p)} \rangle$ for all primes p.*

Corollary 0.7. *For any connected compact Lie group G, $B(\langle G, G \rangle) \subset G_0^\wedge(BG)$.*

If G is simple or of the form $G = H \times T$, G itself is the only connected compact Lie group whose classifying space is of the same genus as BG. (I do not know a general expression for the cardinality of the sets $\langle G, G \rangle \subset \{L \mid BL \in G_0^\wedge(BG)\}$.)

The final theorem of this paper, a kind of a converse to Corollary 0.7, grew out of a suggestion by C.A. McGibbon to consider rational equivalences between spaces of the same genus as BG.

Theorem 0.8. *Let G be a connected compact Lie group and X a space of the same genus as BG. If there exists a rational equivalence between X and BG, then $X \in B(\langle G, G \rangle)$.*

When G is simple a much stronger statement holds.

Corollary 0.9. *Let G be a simple Lie group and X a space of the same genus as BG. If there exists an essential map between X and BG, then $X \simeq BG$.*

This implies in particular, as pointed out to me by K.-I. Maruyama and by the referee, that no nontrivial map connects $BSp(2)$ and BE_5. See also [I-M-N] for related results.

1. Roots and covering homomorphisms

This section establishes the terminology pertaining to roots and covering homomorphism of Lie groups to be used in the next section.

Let G be a compact connected Lie group with maximal torus T_G. The adjoint representation of T_G in the complex vector space $\mathbb{C} \otimes_{\mathbb{R}} LG$ is completely determined by its irreducible characters $T_G \to S^1$. A root of G is a homomorphism $\alpha : LT_G \to \mathbb{R}$ fitting into a commutative diagram

$$\begin{tikzcd} LT_G \arrow[r, "\alpha"] \arrow[d, "\exp"'] & \mathbb{R} \arrow[d, "e(t)=e^{2\pi i t}"] \\ T_G \arrow[r, "\chi_\alpha"'] & S^1 \end{tikzcd}$$

for some nontrivial irreducible character χ_α contained in the adjoint representation. The root system of G is the set $\Phi_G \subset \mathrm{Hom}\,(LT_G, \mathbb{R})$ of all roots of G.

The kernel of the universal covering space homomorphism $\exp : LT_G \to T_G$ is naturally isomorphic to the fundamental group $\pi_1(T_G)$ so any root $\alpha : LT_G \to \mathbb{R}$ restricts to a homomorphism of $\ker \exp = \pi_1(T_G)$ into $\ker e = \mathbb{Z}$; i.e.

$$\Phi_G \subset \mathrm{Hom}\,\left(\pi_1(T_G)\,,\, \mathbb{Z}\right) \subset \mathrm{Hom}\,\left(\pi_1(T_G)^\wedge_p\,,\, \mathbb{Z}^\wedge_p\right)$$

where $\mathbb{Z}^\wedge_p$ denotes the p-adic integers and $\pi_1(T_G)^\wedge_p \cong \pi_1(T_G) \otimes \mathbb{Z}^\wedge_p$ is the completion at the prime p of the fundamental group.

The compact connected Lie group G is finitely covered by a compact Lie group of the form $H \times T$ where H is a simply connected compact simple Lie group and T is a torus. According to Baum [Ba] we may even arrange that G as a quotient of $H \times T$ has the special form

$$G = H \times T \big/ (K, \varphi)$$

where $(K, \varphi) \subset H \times T$ is the graph of a homomorphism $Z_H \supset K \xrightarrow{\varphi} T$ defined on a central subgroup K of H. Thus we have principal K-bundles

$$\begin{array}{ccccc} K & \stackrel{j}{\rightarrowtail} & H \times T & \stackrel{q}{\twoheadrightarrow} & G \\ \| & & \cup & & \cup \\ K & \stackrel{j}{\rightarrowtail} & T_H \times T & \stackrel{q}{\twoheadrightarrow} & T_G \end{array}$$

where $j(h) = (h, \varphi(h))$, $h \in K$, and q is the projection homomorphism. Consider the short exact sequence

$$0 \longrightarrow \pi_1(T_H) \times \pi_1(T) \xrightarrow{\pi_1(q)} \pi_1(T_G) \longrightarrow K \longrightarrow 1$$

classifying the principal K-bundle $q|T_H \times T$ over T_G.

Lemma 1.1. *Let $\Phi_H \subset \mathrm{Hom}\,(\pi_1(T_H), \mathbb{Z})$ and $\Phi_G \subset \mathrm{Hom}\,(\pi_1(T_G), \mathbb{Z})$ be the root systems of H and G, respectively. Then*

1. $\Phi_H = \Phi_G \circ \pi_1(q)$
2. $\bigcap_{\alpha \in \Phi_G} \ker \alpha = \pi_1(q)(\pi_1(T))$.

Proof. (1) The linear isomorphism $Lq : LH \times LT \xrightarrow{\cong} LG$ is equivariant in the sense that

$$Lq \circ \mathrm{Ad}\,(h) = \mathrm{Ad}\,(q(h)) \circ Lq$$

for all $h \in H \subset H \times T$. Thus composition with $q : T_H \times T \longrightarrow T_G$ determines a bijection between the irreducible characters of the T_G-representation $\mathbb{C} \otimes_{\mathbb{R}} LG$ and those of the $(T_H \times T)$-representation $\mathbb{C} \otimes_{\mathbb{R}} (LH \times LT)$. The commutative diagram

$$\begin{tikzcd}
LT_H \times LT \arrow[r, "Lg"] \arrow[d, "\exp"'] & LT_G \arrow[r, "\alpha"] \arrow[d, "\exp"'] & \mathbb{R} \arrow[d, "e"] \\
T_H \times T \arrow[r, "q"'] & T_G \arrow[r, "\chi_\alpha"'] & S^1
\end{tikzcd}$$

now shows that the roots of $H \times T$ precisely are the homomorphism of the form $\alpha \circ Lq$ for $\alpha \in \Phi_G$.

(2) Let $\alpha \in \Phi_G$ be a root. In the basic situation

$$\begin{tikzcd}
LT_G \arrow[r, "\alpha"] \arrow[d, "\exp"'] & \mathbb{R} \arrow[d, "e"] \\
T_G \arrow[r, "\chi_\alpha"'] & S^1
\end{tikzcd}$$

the kernel, U_α, of the character χ_α is a closed codimension 1 subgroup of T_G with Lie algebra $LU_\alpha = \ker \alpha$. (The hyperplanes LU_α, $\alpha \in \Phi_G$, form the walls of the Weyl chambers ([B-tD], Definition V.2.11).) As the intersection $\bigcap_\alpha U_\alpha$ is the center Z_G of G ([B-tD], Proposition V.2.3) we see that $LZ_G = \bigcap_\alpha \ker \alpha \subset LT_G$. Restricting to the integral lattice gives

$$\pi_1(Z_G) = \bigcap_\alpha \ker \alpha \subset \pi_1(T_G) \ .$$

Finally, since $q : H \times T \longrightarrow G$ is the quotient map with respect to the special subgroup $(K, \varphi) \subset H \times T$, q maps $T \subset H \times T$ isomorphically onto the identity component Z_G^0 of Z_G; see the proof of ([Ba], Proposition 5). In particular, $\pi_1(q)(\pi_1(T)) = \pi_1(Z_G)$. $\square$

The proof of the first part of Lemma 1.1 does not use that $q : H\times T \longrightarrow G$ is of the special form but only that q is a surjective homomorphism of compact connected Lie groups of the same dimension.

The p-completion of the short exact sequence for the principal K-bundle $q|T_H \times T$ has the form

$$0 \longrightarrow \pi_1(T_H)^\wedge_p \times \pi_1(T)^\wedge_p \xrightarrow{\pi_1(q)} \pi_1(T_G)^\wedge_p \longrightarrow K_p \longrightarrow 1$$

where K_p is the p-primary subgroup of K. Because p-completion is an exact functor of finitely generated abelian groups, Lemma 1.1 immediately translates into a p-adic version as well. In particular

$$\pi_1(q)\left(\pi_1(T)^\wedge_p\right) = \bigcap_\alpha \ker\alpha \subset \pi_1(T_G)^\wedge_p$$

where the intersection is taken over all $\alpha \in \varPhi_G \subset \mathrm{Hom}\,(\pi_1(T_G)^\wedge_p, \mathbb{Z}^\wedge_p)$.

2. Rational equivalences

The aim of this section is, for any pair, G_1 and G_2, of locally isomorphic compact connected Lie groups, to describe an injection

$$\varepsilon_{\mathbb{Q}}\left((BG_1)^\wedge_p\,,\,(BG_2)^\wedge_p\right) \subset \varepsilon_{\mathbb{Q}}(BH^\wedge_p) \times \varepsilon_{\mathbb{Q}}(BT^\wedge_p)$$

where $H\times T$, H simply connected, T a torus, is a compact Lie group that covers G_1 and G_2. Here, and in the following, $\varepsilon_{\mathbb{Q}}(X,Y) \subset [X,Y]$ denotes the set of homotopy classes of rational equivalences $X \longrightarrow Y$ and $\varepsilon_{\mathbb{Q}}(X) \subset [X,X]$ the monoid of rational self-equivalences of X.

First, a few remarks about the exponential exact sequence of a torus, T. The integral exponential exact sequence,

$$0 \longrightarrow \pi_1(T) \longrightarrow LT \xrightarrow{\exp} T \longrightarrow 1$$

relates the torus, its Lie algebra and its integral lattice and it yields an isomorphism of abelian groups

$$\eta\colon\ \mathrm{Hom}\,(\pi_1(T)\,,\,\pi_1(T)) \xrightarrow{\cong} \mathrm{Hom}\,(T,T)$$

taking $\omega\colon \pi_1(T) \to \pi_1(T)$ to the homomorphism $\eta(\omega)\colon T \to T$ covered by $\omega\otimes 1_{\mathbb{R}}\colon LT \to LT$. The p-adic exponential exact sequence [J-M-O$_2$]

$$0 \longrightarrow \pi_1(T)^\wedge_p \longrightarrow \pi_1(T)^\wedge_p \otimes \mathbb{Q} \longrightarrow T_{p^\infty} \longrightarrow 1$$

relates the subgroup $T_{p^\infty} \subset T$ of elements of p-power order and the p-adic lattice $\pi_1(T)^\wedge_p\colon = \pi_1(T)\otimes\mathbb{Z}^\wedge_p$. It is obtained by tensoring

$$0 \longrightarrow \mathbb{Z}^\wedge_p \longrightarrow \mathbb{Q}_p \longrightarrow \mathbb{Z}/p^\infty \longrightarrow 0$$

with $\pi_1(T)$ and exploiting the natural isomorphism $\operatorname{Tor}(T, \mathbb{Z}/p^\infty) \cong \pi_1(T) \otimes \mathbb{Z}/p^\infty$ coming from $- \otimes \mathbb{Z}/p^\infty$ applied to the integral exponential exact sequence. The p-adic exponential exact sequence yields an isomorphism of abelian groups

$$\eta_p \colon \operatorname{Hom}\left(\pi_1(T)^\wedge_p\,,\ \pi_1(T)^\wedge_p\right) \xrightarrow{\cong} \operatorname{Hom}(T_{p^\infty}\,,\ T_{p^\infty})$$

taking $\omega\colon \pi_1(T)^\wedge_p \to \pi_1(T)^\wedge_p$ to the homomorphism $\eta(\omega)\colon T_{p^\infty} \to T_{p^\infty}$ covered by $\omega \otimes 1_{\mathbb{Q}}$. (Note ([F], Example 5, p. 181) that any abelian group homomorphism between free finitely generated $\mathbb{Z}^\wedge_p$-modules automatically is $\mathbb{Z}^\wedge_p$-linear.)

Also for any pair, T_1 and T_2, of tori, we define an isomorphism

$$\eta_p \colon \operatorname{Hom}\left(\pi_1(T_1)^\wedge_p\,,\ \pi_1(T_2)^\wedge_p\right) \longrightarrow \operatorname{Hom}\left((T_1)_{p^\infty}\,,\ (T_2)_{p^\infty}\right)$$

by associating to any homomorphism ω the homomorphism $\eta_p(\omega)$ such that

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(T_1)^\wedge_p \arrow[r] \arrow[d, "\omega"'] & \pi_1(T_1)^\wedge_p \otimes \mathbb{Q} \arrow[r] \arrow[d, "\omega\otimes 1_{\mathbb{Q}}"'] & (T_1)_{p^\infty} \arrow[r] \arrow[d, "\eta_p(\omega)"'] & 1 \\
0 \arrow[r] & \pi_1(T_2)^\wedge_p \arrow[r] & \pi_1(T_2)^\wedge_p \otimes \mathbb{Q} \arrow[r] & (T_2)_{p^\infty} \arrow[r] & 1
\end{tikzcd}$$

is a homomorphism of p-adic exponential exact sequences. Standard homological algebra shows

Lemma 2.1. *Let* $\omega \in \operatorname{Hom}\left(\pi_1(T_1)^\wedge_p\,,\ \pi_1(T_2)^\wedge_p\right)$.

1. ω *is an epimorphism* $\Rightarrow \eta_p(\omega)$ *is an epimorphism.*
2. $\eta_p(\omega)$ *is a monomorphism* $\Rightarrow \omega$ *is a monomorphism.*
3. *If* $\dim T_1 = \dim T_2$, *then*

$$\begin{aligned} \omega \text{ is a monomorphism } &\Leftrightarrow \omega \otimes 1_{\mathbb{Q}} \text{ is an isomorphism} \\ &\Leftrightarrow \eta_p(\omega) \text{ is an epimorphism} \end{aligned}$$

and

$$\omega \text{ is an isomorphism } \Leftrightarrow \eta_p(\omega) \text{ is an isomorphism}$$

In particular, η_p restricts to an isomorphism of monoids

$$\eta_p \colon \varepsilon_{\mathbb{Q}}(BT^\wedge_p) \cong \operatorname{Mono}\left(\pi_1(T)^\wedge_p\right) \xrightarrow{\cong} \operatorname{Epi}(T_{p^\infty})$$

where $\operatorname{Mono}(-)(\operatorname{Epi}(-))$ denotes the monoid of injective (surjective) endomorphisms.

There is a parallel construction when T is replaced by a simply connected compact Lie group H. Let Z_H denote the center of H and $(Z_H)_p = Z_H \cap$

$(T_H)_{p^\infty}$ the subgroup of elements of p-power order in Z_H; here, T_H is the maximal torus of H. Define

$$\eta_p\colon \varepsilon_{\mathbb{Q}}(BH_p^\wedge) \longrightarrow \operatorname{Aut}\left((Z_H)_p\right)$$

by $\eta_p(f_H) = \eta_p(\omega_H) \mid (Z_H)_p$ where the rational equivalences $f_H \in \varepsilon_{\mathbb{Q}}(BH_p^\wedge)$ and $\omega_H \in \varepsilon_{\mathbb{Q}}((BT_H)_p^\wedge)$ are related by the homotopy commutative diagram

$$\begin{tikzcd} (BT_H)_p^\wedge \arrow[r, "\omega_H"] \arrow[d] & (BT_H)_p^\wedge \arrow[d] \\ BH_p^\wedge \arrow[r, "f_H"] & BG_p^\wedge \end{tikzcd}$$

The map ω_H exists by [A-W] and is a rational equivalence by ([J-M-O$_2$], Proposition 1.2). Since ω_H is unique up to left W_H-action [A-W], $\eta_p(\omega_H)$ is well-defined on the subgroup $(Z_H)_p$ of central elements. It remains to be seen that $\eta_p(\omega_H)$ maps $(Z_H)_p$ isomorphically to itself. The isomorphism ([Mø], Theorem 3.1)

$$\varepsilon_{\mathbb{Q}}(BH_p^\wedge) \cong \prod_{i=1}^{s} \varepsilon_{\mathbb{Q}}\left((BH_i)_p^\wedge\right) \wr \Sigma_{n_i} ,$$

$H = \prod_{i=1}^s H_i^{n_i}$, H_i simple and $H_i \neq H_j$ for $i \neq j$ (with the convention that $\mathrm{Sp}(n)$ and $\mathrm{Spin}\,(2n+1)$, $n > 2$, are considered to be distinct only if $p = 2$) shows that η_p is determined by its values for simple Lie groups. If H is simple and $(Z_H)_p \neq 1$, then $p \mid |W_H|$ and [J-M-O$_1$] $\varepsilon_{\mathbb{Q}}(BH_p^\wedge) = \operatorname{Aut}(BH_p^\wedge)$ is generated by $B\alpha$, $\alpha \in \operatorname{Out}(H)$, and unstable Adams operations ψ^λ, $\lambda \in (\mathbb{Z}_p^\wedge)^*$. Clearly, $\eta_p(B\alpha) = \alpha|(Z_H)_p$ and $\eta_p(\psi^\lambda)(t) = t^\lambda$, $t \in (T_H)_{p^\infty}$, take $(Z_H)_p$ isomorphically to itself.

Here is an explicit list of η_p for the simply connected compact simple Lie groups.

Example 2.2. See Bourbaki ([Bo], Chp. VI, §4) for the action of $\operatorname{Out}(H)$ on Z_H.

i) The center of $SU(n)$, $n = p^j m \geq 2$, $j \geq 1$, $(p, m) = 1$, is cyclic of order n, $Z_{SU(n)} = \{\omega^i E \mid 0 \leq i < n\}$, $\omega = e^{2\pi i/n}$, and

$$\eta_p\colon \varepsilon_{\mathbb{Q}}\left(BSU(n)_p^\wedge\right) \longrightarrow (Z_{SU(n)})_p = \{\omega^{mi} \mid 0 \leq i < p^j\}$$

is given by $\eta_p(\psi^\lambda)(\omega^m) = \omega^{m\cdot\lambda \bmod p^j}$ where $\lambda \bmod p^j \in (\mathbb{Z}/p^j)^*$ is the reduction modulo p^j of $\lambda \in (\mathbb{Z}_p^\wedge)^*$.

ii) $\mathrm{Spin}\,(8)$ has center $Z \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and η_2 maps $\operatorname{Out}(\mathrm{Spin}\,(8)) \cong \Sigma_3$ isomorphically to $\operatorname{Aut}(Z)$ and all unstable Adams operations to the identity. $\mathrm{Spin}\,(2n), n > 4$ even, has center $Z \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\eta_2(\alpha)$, $1 \neq \alpha \in$

$\mathrm{Out}(\mathrm{Spin}\,(2n)) \cong \Sigma_2$, is the nontrivial automorphism of Z that fixes the kernel of the double covering $\mathrm{Spin}\,(2n) \longrightarrow SO(2n)$; $\eta_2(\psi^\lambda)(t) = t$ for all $\lambda \in (\mathbb{Z}_2^\wedge)^*$ and $t \in Z_2 \subset T_{2\infty}$. $\mathrm{Spin}\,(2n)$, $n \geq 5$ odd, has center $Z \cong \mathbb{Z}/4$ and η_2 maps $\mathrm{Out}(\mathrm{Spin}\,(2n)) \cong \Sigma_2$ isomorphically to $\mathrm{Aut}(Z)$; $\eta_2(\psi^\lambda)(t) = t^{\lambda \bmod 4}$ for all $\lambda \in (\mathbb{Z}_2^\wedge)^*$ and $t \in Z$.

iii) E_6 has center $Z \cong \mathbb{Z}/3$ and η_3 maps $\mathrm{Out}(E_6) \cong \Sigma_2$ isomorphically to $\mathrm{Aut}(Z)$; $\eta_2(\psi^\lambda)(t) = t^{\lambda \bmod 3}$ for all $\lambda \in (\mathbb{Z}_3^\wedge)^*$ and $t \in Z$.

iv) In all other cases, η_p is trivial.

The locally isomorphic compact connected Lie groups G_1 and G_2 may be assumed given on the form

$$G_i = H \times T \big/ (K_i, \varphi_i)\ , \quad i = 1, 2\ ,$$

where T is a torus, H is a simply connected compact Lie group and $\varphi_i \colon K_i \to T$ a homomorphism defined on a subgroup $K_i \subset Z_H$. The principal K_i-bundles

$$K_i \xrightarrow{j_i} H \times T \xrightarrow{q_i} G_i\ , \quad i = 1, 2$$

where $j_i(k) = (k, \varphi_i(k))\ ,\ k \in K_i$, induce fibrations

$$BK_i \xrightarrow{Bj_i} BH \times BT \xrightarrow{Bq_i} BG_i\ , \quad i = 1, 2\ ,$$

of integral spaces and similar fibrations of p-completed spaces. The following theorem describes the set $\varepsilon_{\mathbb{Q}}((BG_1)_p^\wedge, (BG_2)_p^\wedge)$ of rational equivalences $(BG_1)_p^\wedge \to (BG_2)_p^\wedge$.

Theorem 2.3. *For a given prime p, let $(K_i)_p \subset K_i$, $i = 1, 2$, denote the subgroup of K_i of elements of p-power order.*

(1) The product maps

$$\begin{array}{ll} \varepsilon_{\mathbb{Q}}(BH_p^\wedge) \times \varepsilon_{\mathbb{Q}}(BT_p^\wedge) & \longrightarrow \varepsilon_{\mathbb{Q}}(BH_p^\wedge \times BT_p^\wedge) \\ \mathrm{Aut}(BH_p^\wedge) \times \mathrm{Aut}(BT_p^\wedge) & \longrightarrow \mathrm{Aut}(BH_p^\wedge \times BT_p^\wedge) \end{array}$$

are monoid isomorphisms.

(2) Let $f \colon (BG_1)_p^\wedge \longrightarrow (BG_2)_p^\wedge$ be a (rational) equivalence. Then there exist (rational) equivalences $f_H \colon BH_p^\wedge \longrightarrow BH_p^\wedge$, $f_T \colon BT_p^\wedge \longrightarrow BT_p^\wedge$ and a monomorphism (an isomorphism) $\alpha \colon (K_1)_p \longrightarrow (K_2)_p$ such that the diagram

$$\begin{tikzcd} B(K_1)_p \arrow[r, "B\alpha"] \arrow[d, "Bj_1"'] & B(K_2)_p \arrow[d, "Bj_2"] \\ BH_p^\wedge \times BT_p^\wedge \arrow[r, "f_H \times f_T"] \arrow[d, "Bq_1"'] & BH_p^\wedge \times BT_p^\wedge \arrow[d, "Bq_2"] \\ (BG_1)_p^\wedge \arrow[r, "f"] & (BG_2)_p^\wedge \end{tikzcd}$$

commutes up to homotopy. The (rational) equivalences f_H and f_T are unique up to homotopy and the monomorphism α is unique.

(3) Let $f_H\colon BH_p^\wedge \longrightarrow BH_p^\wedge$ and $f_T\colon BT_p^\wedge \longrightarrow BT_p^\wedge$ be (rational) equivalences. The equation $f \circ Bq_1 \simeq Bq_2 \circ (f_H \times f_T)$, has a solution for some (rational) equivalence $f\colon (BG_1)_p^\wedge \longrightarrow (BG_2)_p^\wedge$ if and only if the commutative diagram

$$\begin{tikzcd}
(Z_H)_p \arrow[d, "\eta_p(f_H)"', "\cong"] & (K_1)_p \arrow[l, hook'] \arrow[r, "\varphi_1"] \arrow[d, dashed, "\alpha"'] & T_{p^\infty} \arrow[d, "\eta_p(f_T)"] \\
(Z_H)_p & (K_2)_p \arrow[l, hook'] \arrow[r, "\varphi_2"] & T_{p^\infty}
\end{tikzcd}$$

can be completed by some (monomorphism) isomorphism $\alpha\colon (K_1)_p \longrightarrow (K_2)_p$.

Proof. (1) It suffices to prove surjectivity as the two maps in point (1) are monomorphisms by general principles.

Let g be a rational self-equivalence of $BH_p^\wedge \times BT_p^\wedge$. Choose [A-W], [J-M-O$_2$, Proposition 1.2] $\varphi \in \mathrm{Epi}((T_H)_{p^\infty} \times T_{p^\infty})$, where T_H is the maximal torus of H, such that

$$\begin{tikzcd}
(BT_H)_p^\wedge \times BT_p^\wedge \arrow[r, "(B\varphi)_p^\wedge"] \arrow[d] & (BT_H)_p^\wedge \times BT_p^\wedge \arrow[d] \\
BH_p^\wedge \times BT_p^\wedge \arrow[r, "g"'] & BH_p^\wedge \times BT_p^\wedge
\end{tikzcd}$$

commutes up to homotopy. The induced map

$$\begin{aligned}
\pi_2((B\varphi)_p^\wedge) &\in \mathrm{Mono}(\pi_1((T_H)_p^\wedge) \times \pi_1(T_p^\wedge)) \\
&\subset \mathrm{Aut}_{\mathbb{Q}_p}((\pi_1(T_H) \times \pi_1(T)) \otimes \mathbb{Q}_p)
\end{aligned}$$

belongs to the normalizer $N(W_{H\times T})$ of the Weyl group of $H \times T$ in the general linear group of the vector space $(\pi_1(T_H) \times \pi_1(T)) \otimes \mathbb{Q}_p$. An elementary linear algebra calculation, using the fact that the inverse roots of H spans $\pi_1(T_H) \otimes \mathbb{Q}_p$, shows that

$$N(W_{H\times T}) = N(W_H) \times \mathrm{Aut}_{\mathbb{Q}_p}(\pi_1(T) \otimes \mathbb{Q}_p)$$

is the product of the normalizer $N(W_H)$ of the Weyl group of H in the general linear group of $\pi_1(T_H) \otimes \mathbb{Q}_p$ and the full general linear group of $\pi_1(T) \otimes \mathbb{Q}_p$. Hence

$$\varphi = \varphi_H \times \varphi_T$$

for some $\varphi_H \in \mathrm{Epi}((T_H)_{p^\infty})$ and some $\varphi_T \in \mathrm{Epi}(T_{p^\infty})$.

Now define f_H to be the composition

$$BH_p^\wedge \longrightarrow BH_p^\wedge \times BT_p^\wedge \xrightarrow{g} BH_p^\wedge \times BT_p^\wedge \longrightarrow BH_p^\wedge$$

of g and the obvious inclusion and projection maps. A look at the homotopy groups $\pi_*(BH_p^\wedge) \otimes \mathbb{Q} \subset \pi_*(BH_p^\wedge \times BT_p^\wedge) \otimes \mathbb{Q}$ reveals that f_H is a rational equivalence of $BH_p^\wedge$. Observe that the above commutative square remains commutative if the bottom map g is replaced by $f_H \times (B\varphi_T)_p^\wedge$. Since rational self-equivalences are determined by their restrictions to the maximal torus [J-M-O$_2$, Corollary 1.10], it follows that $g \simeq f_H \times (B\varphi_T)_p^\wedge$. This proves surjectivity.

If $g \in \mathrm{Aut}(BH_p^\wedge \times BT_p^\wedge)$ is a homotopy self-equivalence, $\pi_2((B\varphi_T)_p^\wedge) = \pi_2(g)$ is an automorphism of $\pi_2(BH_p^\wedge \times BT_p^\wedge) = \pi_2(BT_p^\wedge)$ and $\pi_*(f_H) = \pi_{\geq 4}(g)$ an automorphism of $\pi_*(BH_p^\wedge) = \pi_{\geq 4}(BH_p^\wedge \times BT_p^\wedge)$. Thus $f_H \in \mathrm{Aut}(BH_p^\wedge)$ and $\varphi_T \in \mathrm{Aut}(T_{p^\infty})$ in this case.

(2) Let $T_H \subset H$ be a maximal torus in H and let $T_1 = T_H \times T/(K_1, \varphi_1)$, $T_2 = T_H \times T/(K_2, \varphi_2)$ serve as maximal tori for G_1, G_2. There exists [A-W] a homotopy commutative diagram

$$\begin{tikzcd}
(BT_1)_p^\wedge \arrow[r] \arrow[d] & (BT_2)_p^\wedge \arrow[d] \\
(BG_1)_p^\wedge \arrow[r, "f"] & (BG_2)_p^\wedge
\end{tikzcd}$$

The map between the classifying spaces of the maximal tori is a rational equivalence since f is one, see ([J-M-O$_2$], Proposition 1.2), so it induces on homotopy some $\omega\colon \pi_1(T_1)_p^\wedge \to \pi_1(T_2)_p^\wedge$ such that $\omega \otimes 1_{\mathbb{Q}}$ is an isomorphism. Define the isomorphism ω' by the commutative diagram

$$\begin{tikzcd}
\pi_1(T_1) \otimes \mathbb{Q}_p \arrow[rrr, "\omega \otimes 1_{\mathbb{Q}}", "\cong"'] & & & \pi_1(T_2) \otimes \mathbb{Q}_p \\
& \pi_1(T_1)_p^\wedge \arrow[ul] \arrow[r, hook] & \pi_1(T_2)_p^\wedge \arrow[ur] & \\
& \pi_1(T_H \times T)_p^\wedge \arrow[u, "(q_1)_*"] \arrow[dl] & \pi_1(T_H \times T)_p^\wedge \arrow[u, "(q_2)_*"'] \arrow[dr] & \\
\pi_1(T_H \times T) \otimes \mathbb{Q}_p \arrow[uuu, "(q_1)_*", "\cong"'] \arrow[rrr, dashed, "\omega'"] & & & \pi_1(T_H \times T) \otimes \mathbb{Q}_p \arrow[uuu, "\cong", "(q_2)_*"']
\end{tikzcd}$$

where $q_i\colon (H \times T\,,\, T_H \times T) \longrightarrow (G_i\,,\, T_i)\,,\; i = 1, 2$, are the projections. The isomorphism $\omega \otimes \mathbb{Q}$ is admissible [A-W] in the sense that for any $w_1 \in N(T_H) \subset H \subset H \times T$

$$(\omega \otimes \mathbb{Q}) \circ \mathrm{Ad}\;(q_1(w_1)) = \mathrm{Ad}\;(q_2(w_2)) \circ (\omega \otimes \mathbb{Q})$$

for some $w_2 \in N(T_H) \subset H \subset H \times T$. This implies that

$$\forall w_1 \in N(T_H) \exists\, w_2 \in N(T_H) \colon \omega' \circ \mathrm{Ad}\,(w_1) = \mathrm{Ad}\,(w_2) \circ \omega'$$

i.e. that also ω' is an admissible automorphism. Admissibility implies, see [Mø] or [J-M-O2], that for any root α_2' of $H \times T$ there exist another root α_1' of $H \times T$ such that

$$\alpha_2' \circ \omega' = \lambda \alpha_1'$$

for some nonzero scalar $\lambda \in \mathbb{Q}_p$. By Lemma 1.1, $\alpha_1' = \alpha_1 \circ (q_1)_*$ and $\alpha_2' = \alpha_2 \circ (q_2)_*$ for some roots $\alpha_i \colon \pi_1(T_i)_p^\wedge \longrightarrow \mathbb{Z}_p^\wedge$ of G_i, $i = 1, 2$. Hence, for any root α_2 of G_2, there exists a root α_1 of G_1 such that

$$\alpha_2 \circ (\omega \otimes \mathbb{Q}) \circ (q_1)_* = \alpha_2 \circ (q_2)_* \circ \omega' = \alpha_2' \circ \omega' = \lambda \alpha_1' = \lambda \alpha_1 \circ (q_1)_*\ ,$$

i.e. such that $\alpha_2 \circ (\omega \otimes \mathbb{Q}) = \lambda \alpha_1$. Thus $\omega(\ker \alpha_1) \subset \ker \alpha_2$. As α_2 runs through the root system Φ_2 of G_2, α_1 runs through the root system Φ_1 of G_1 and we conclude that

$$\omega\Bigg(\bigcap_{\alpha_1 \in \Phi_1} \ker \alpha_1\Bigg) \subset \bigcap_{\alpha_2 \in \Phi_2} \ker \alpha_2\ ,$$

or, by Lemma 1.1, $\omega((q_1)_*(\pi_1(T)_p^\wedge) \subset (q_2)_*(\pi_1(T)_p^\wedge)$. Since $(q_1)_*$ and $(q_2)_*$ are monomorphisms, ω restricts to a monomorphism $\omega_T \colon \pi_1(T)_p^\wedge \to \pi_1(T)_p^\wedge$ such that $\omega \circ (q_1)_* = (q_2)_* \circ \omega_T$. This monomorphism will make also the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_2(BT)_p^\wedge \arrow[r, "(Bq_1)_*"] \arrow[d, "\omega_T"'] & \pi_2(BG_1)_p^\wedge \arrow[r, "\partial_1"] \arrow[d, "f_*"'] & (K_1)_p \arrow[r] & 1 \\
0 \arrow[r] & \pi_2(BT)_p^\wedge \arrow[r, "(Bq_2)_*"] & \pi_2(BG_2)_p^\wedge \arrow[r, "\partial_2"] & (K_2)_p \arrow[r] & 1
\end{tikzcd}$$

commute; here the exact rows are the short exact sequences for the principal K_i-bundles $q_i \colon H \times T \longrightarrow G_i$, $i = 1, 2$. Hence the only obstruction, $\partial_2 \circ f_* \circ (Bq_1)_*$, to lifting $f \circ Bq_1$ vanishes and there exist $f_H \times f_T \in \varepsilon_{\mathbb{Q}}(BH_p^\wedge \times BT_p^\wedge)$ and $\alpha \in \mathrm{Hom}\,((K_1)_p, (K_2)_p)$ such that $f \circ Bq_1 \simeq Bq_2 \circ (f_H \times f_T)$ and $\alpha \circ \partial_1 = \partial_2 \circ f_*$.

We may view f_H as the 2-connected cover $f\langle 2 \rangle$ of f (which is a self map of $(BG_i)_p^\wedge\langle 2 \rangle = BH_p^\wedge$, $i = 1, 2$). Thus f_H is uniquely determined by f and is a homotopy equivalence if f is one. Also f_T is uniquely determined by f as $\pi_2(f_T) = \omega_T$ is a restriction of $\pi_2(f)$ and, since α is injective (see the proof of (3)), ω_T is an isomorphism if $\pi_2(f)$ is one.

(3) Define i_l , $l = 1, 2$, by the commutative diagram

$$\begin{tikzcd} (K_l)_p \arrow[r, tail] \arrow[d, "i_l"'] & K_l \arrow[d, "j_l"] \\ (T_H \times T)_{p^\infty} \arrow[r, tail, "i"] & H \times T \end{tikzcd}$$

with horizontal inclusions.

Let, as above, $f_H \times f_T$ be the lift of the rational equivalence $f\colon (BG_1)^\wedge_p \longrightarrow (BG_2)^\wedge_p$. Write $f_T = (B\varphi_T)^\wedge_p$ and let $(B\varphi_H)^\wedge_p$ be the restriction of f_H to $(BT_H)^\wedge_p$, $\varphi_T \in \mathrm{Epi}(T_{p^\infty})$, $\varphi_H \in \mathrm{Epi}((T_H)_{p^\infty})$. Then the outer and lower square of the diagram

$$\begin{tikzcd} B(K_1)_p \arrow[rr, "B\alpha"] \arrow[d, "Bi_1"'] & & B(K_2)_p \arrow[d, "Bi_2"] \\ B((T_H \times T)_{p^\infty})^\wedge_p \arrow[rr, "(B\varphi_H)^\wedge_p \times (B\varphi_T)^\wedge_p"] \arrow[d, "Bi"'] & & B((T_H \times T)_{p^\infty})^\wedge_p \arrow[d, "Bi"] \\ B(H \times T)^\wedge_p \arrow[rr, "f_H \times f_T"] & & B(H \times T)^\wedge_p \end{tikzcd}$$

are homotopy commutative meaning that $B(i \circ i_2 \circ \alpha) \simeq B(i \circ (\varphi_H \times \varphi_T) \circ i_1)$. By the theorem of Dwyer & Zabrodsky [D-Z], the homomorphisms $i \circ i_2 \circ \alpha$ and $i \circ (\varphi_H \times \varphi_T) \circ i_1$ are conjugate and even identical because the image $i \circ i_2 \circ \alpha$ is central in $H \times T$. This is equivalent to saying that α fits into the commutative diagram of (3). Note that α must be injective since $\eta_p(f_H) = \varphi_H | Z_H$ is an isomorphism.

Assume, conversely, that the diagram in (3) can be completed commutatively by some α. The above diagram of classifying spaces will then be homotopy commutative and so the " Zabrodsky spaces will then be homotopy commutative and so the " Zabrodsky Lemma" ([Mi], Proposition 9.5; [Z], Lemma 3.1), or Proposition 1.1 in [N$_1$], assures that $f_H \times f_T$ covers some $f\colon (BG_1)^\wedge_p \longrightarrow (BG_2)^\wedge_p$. $\square$

Thus we may say that

$$\varepsilon_{\mathbb{Q}}\left((BG_1)^\wedge_p\,,\,(BG_2)^\wedge_p\right) \subset \varepsilon_{\mathbb{Q}}(BH^\wedge_p) \times \mathrm{Epi}(T_{p^\infty})$$

is the subset consisting of those pairs (f_H, φ) for which $\eta_p(f_H)((K_1)_p) \subset (K_2)_p$ and

$$\begin{tikzcd} (K_1)_p \arrow[r, "\varphi_1"] \arrow[d, tail, "\eta_p(f_H)|(K_1)_p"'] & T_{p^\infty} \arrow[d, two heads, "\varphi"] \\ (K_2)_p \arrow[r, "\varphi_2"'] & T_{p^\infty} \end{tikzcd}$$

commutes. Moreover, a pair (f_H, φ), for which this diagram commutes, represents a homotopy equivalence $(BG_1)^\wedge_p \to (BG_2)^\wedge_p$ if and only if $f_H \in \mathrm{Aut}(BH^\wedge_p)$, $\varphi \in \mathrm{Aut}(T_{p^\infty})$, and $\eta_p(f_H)((K_1)_p) = (K_2)_p$. In case $G_1 = G_2 =: G$, $\varepsilon_{\mathbb{Q}}(BG^\wedge_p)$ $(\mathrm{Aut}(BG^\wedge_p))$ is even a submonoid (subgroup) of $\varepsilon_{\mathbb{Q}}(BH^\wedge_p) \times \varepsilon_{\mathbb{Q}}(BT^\wedge_p)$ $(\mathrm{Aut}(BH^\wedge_p) \times \mathrm{Aut}(BT^\wedge_p))$.

We single out the semisimple case

Corollary 2.4. *Let H be a simply connected compact Lie group, $K_1, K_2 \subset Z_H$ central subgroups, and p any prime.*

1. $\varepsilon_{\mathbb{Q}}(B(H/K_1)^\wedge_p\,, B(H/K_2)^\wedge_p) = \{f \in \varepsilon_{\mathbb{Q}}(BH^\wedge_p) \mid \eta_p(f)(K_1)_p \subset (K_2)_p\}$.
2. $\varepsilon_{\mathbb{Q}}(B(H/K_1)^\wedge_p)\,, B(H/K_2)^\wedge_p) = \emptyset$ *if* $|(K_1)_p| > |(K_2)_p|$.
3. *Let* $PH = H/Z_H$. *Then* $\varepsilon_{\mathbb{Q}}(BH^\wedge_p\,, BPH^\wedge_p) = \varepsilon_{\mathbb{Q}}(BH^\wedge_p) = \varepsilon_{\mathbb{Q}}(BPH^\wedge_p)$ *but* $\varepsilon_{\mathbb{Q}}(BPH^\wedge_p\,, BH^\wedge_p) = \emptyset$ *if* $(Z_H)_p \neq 0$.

In the general case we have,

$$\varepsilon_{\mathbb{Q}}\left((BG_1)^\wedge_p\,, (BG_2)^\wedge_p\right) \subset \varepsilon_{\mathbb{Q}}\left(B(H/K_1)^\wedge_p\,, B(H/K_2)^\wedge_p\right) \times \varepsilon_{\mathbb{Q}}(BT^\wedge_p)$$

so again $\varepsilon_{\mathbb{Q}}((BG_1)^\wedge_p, (BG_2)^\wedge_p) = \emptyset$ if $|(K_1)_p| > |(K_2)_p|$. If φ_1 and φ_2 are trivial, i.e. $G_i = (H/K_i) \times T$, $i = 1, 2$, the above inclusion is an identity generalizing (1) in Theorem 2.3.

Example 2.5. i) Let k be a natural number and let $\Gamma_k \subset U(n)$, $n = p^j m\,, (p, m) = 1$, be the subgroup of all diagonal matrices θE where $\theta^k = 1$. Then [Ba]

$$U(n)/\Gamma_k = SU(n) \times S^1/(Z, \varphi_k)\,,\ \varphi_k(\omega) = \omega^k\ ,$$

where ω generates $Z = Z_{SU(n)}$ as in Example 2.2.

Commutativity of

$$\begin{array}{ccc} Z_p & \xrightarrow{\varphi_k} & (S^1)_{p^\infty} \\ {\scriptstyle\psi^\lambda}\downarrow & & \downarrow{\scriptstyle\psi^\kappa} \\ Z_p & \xrightarrow{\varphi_l} & (S^1)_{p^\infty} \end{array}$$

where $\lambda \in (\mathbb{Z}^\wedge_p)^*$ and $0 \neq \kappa \in \mathbb{Z}^\wedge_p$ is equivalent to $l\lambda \equiv k\kappa \bmod p^j$. Consequently, for $n > 2$,

$$\varepsilon_{\mathbb{Q}}\left(B(U(n)/\Gamma_k)^\wedge_p\,, B(U(n)/\Gamma_l)^\wedge_p\right) \cong \{\,(\lambda, \kappa) \in (\mathbb{Z}^\wedge_p)^* \times \mathbb{Z}^\wedge_p \mid l\lambda \equiv k\kappa \bmod p^j\,\}$$

as sets and as monoids if $k = l$. In particular,

$$\varepsilon_{\mathbb{Q}}\left(BU(n)^\wedge_p\right) \cong \{\,(\lambda, \kappa) \in (\mathbb{Z}^\wedge_p)^* \times (\mathbb{Z}^\wedge_p)^* \mid \lambda \equiv \kappa \bmod p^j\,\}, \quad n > 2.$$

If $(k,n)=1$, $U(n)/\Gamma_k = FU_k(n)$ is the funny unitary group of [N-S].

ii) Let $H = \mathrm{Spin}\,(2n+1)\times \mathrm{Sp}\,(n)$, $n>2$. Example 2.2 shows that η_2 is trivial on $\varepsilon_{\mathbb{Q}}(BH_2^\wedge) = \mathrm{Aut}(BH_2^\wedge) = (\mathbb{Z}_2^\wedge)^* \times (\mathbb{Z}_2^\wedge)^*$. Hence

$$\varepsilon_{\mathbb{Q}}\left(B(H/K_1)_p^\wedge\,,\,B(H/K_2)_p^\wedge\right) = \begin{cases}\varepsilon(BH_2^\wedge) & \text{if } K_1 = K_2\\ \emptyset & \text{if } K_1 \neq K_2\end{cases}$$

for any pair of central subgroups $K_1, K_2 \subset Z_H = \mathbb{Z}/2\times\mathbb{Z}/2$; cf. ([J-M-$\mathrm{O}_2$], Example 2.1).

We now turn to Theorem 0.3, i.e. the integral version of Theorem 2.3.

Note that, as in the p-complete case, there exists a homomorphism of monoids

$$\eta\colon \varepsilon_{\mathbb{Q}}(BH) \longrightarrow \mathrm{Aut}(Z_H)$$

given by $\eta(f) = \varphi|Z_H$ where $\varphi\in\mathrm{Aut}(T_H)$ gives a lift to BT_H of f.

Proof of Theorem 0.3. First note that the product maps

$$\begin{array}{lcl}\varepsilon_{\mathbb{Q}}(BH)\times\varepsilon_{\mathbb{Q}}(BT) & \longrightarrow & \varepsilon_{\mathbb{Q}}(BH\times BT)\\ \mathrm{Aut}(BH)\times\mathrm{Aut}(BT) & \longrightarrow & \mathrm{Aut}(BH\times BT)\end{array}$$

are monoid isomorphisms: Again, it suffices to prove surjectivity. For $g\in\varepsilon_{\mathbb{Q}}(BH\times BT)$, let $f_H\in\varepsilon_{\mathbb{Q}}(BH)$ be the map induced by g on the 3-connected cover $(BH\times BT)\langle 3\rangle \simeq BH$ and let $\varphi\in\mathrm{Epi}(T)$ be the epimorphism with $\pi_2(g) = \pi_2(B\varphi)$ on $\pi_2(BH\times BT) = \pi_2(BT)$. Then $g^\wedge \simeq (f_H\times B\varphi)^\wedge$ by point (1) of Theorem 2.3, so, since completion

$$[BH\times BT, BH\times BT] \longrightarrow [(BH\times BT)^\wedge, (BH\times BT)^\wedge]$$

is injective [J-M-O_1, Theorem 3.1], $g\simeq f_H\times B\varphi$. If $g\in\mathrm{Aut}(BH\times BT)$ then, of course, $f_H\in\mathrm{Aut}(BH)$ and $\varphi\in\mathrm{Aut}(T)$.

Suppose that $f\colon BG_1\to BG_2$ is a rational equivalence. There exist by point (2) of Theorem 2.3 maps $f_H^\wedge\colon BH^\wedge\to BH^\wedge$ and $f_T^\wedge\colon BT^\wedge\to BT^\wedge$ making the completed diagram

$$\begin{tikzcd}(BH\times BT)^\wedge \arrow[r, "f_H^\wedge\times f_T^\wedge"] \arrow[d, "(Bq_1)^\wedge"'] & (BH\times BT)^\wedge \arrow[d, "(Bq_2)^\wedge"] \\ (BG_1)^\wedge \arrow[r, "f^\wedge"'] & (BG_2)^\wedge\end{tikzcd}$$

homotopy commutative. Since the projection maps Bq_1 and Bq_2 are rational equivalences, there also exists a homotopy equivalence $f(0)$ making the rationalized diagram

$$\begin{array}{ccc}
(BH \times BT)_{(0)} & \xrightarrow{f(0)} & (BH \times BT)_{(0)} \\
{\scriptstyle (Bq_1)_{(0)}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle (Bq_2)_{(0)}} \\
(BG_1)_{(0)} & \underset{f_{(0)}}{\xrightarrow{\simeq}} & (BG_2)_{(0)}
\end{array}$$

homotopy commute. Applying the rationalization functor to the first and the formal completion functor to the second of the above diagrams, and remembering that $((Bq_1)^\wedge)_{(0)}$ and $((Bq_2)^\wedge)_{(0)}$ are homotopy equivalences, we see that the rationalization of $f_H^\wedge \times f_T^\wedge$ agrees with the formal completion of $f(0)$ as self maps of $((BH \times BT)^\wedge)_{(0)}$. By Sullivan's Arithmetic Square [B-K], $f_H^\wedge \times f_T^\wedge$ and $f(0)$ come from a self map of $BH \times BT$. This self map is a rational equivalence (its rationalization $f(0)$ is a homotopy equivalence) and therefore of the form $f_H \times B\varphi$ for some $f_H \in \varepsilon_{\mathbb{Q}}(BH)$ and some $\varphi \in \mathrm{Epi}(T)$. Since

$$(Bq_2 \circ (f_H \times B\varphi))^\wedge \simeq (Bq_2)^\wedge \circ (f_H^\wedge \times f_T^\wedge) \simeq f^\wedge \circ (Bq_1)^\wedge \simeq (f \circ Bq_1)^\wedge$$

and completion is injective, $Bq_2 \circ (f_H \times B\varphi) \simeq f \circ Bq_1$. By point (3) of Theorem 2.3, $\eta(f_H)(K_1) \subset K_2$ and $\varphi_2 \circ (\eta(f_H)|K_1) = \varphi \circ (\varphi_1|K_1)$.

The uniqueness clauses of Theorem 2.3 together with injectivity of completion show that f_H and φ are uniquely determined by f.

Conversely, suppose $f_H \in \varepsilon_{\mathbb{Q}}(BH)$ and $B\varphi \in \varepsilon_{\mathbb{Q}}(BT)$ are rational equivalences such that $\eta(f_H)(K_1) \subset K_2$ and

$$\begin{array}{ccc}
K_1 & \xrightarrow{\varphi_1} & T \\
{\scriptstyle \eta(f_H)|K_1}\downarrow & & \downarrow{\scriptstyle \varphi} \\
K_2 & \xrightarrow[\varphi_2]{} & T
\end{array}$$

commutative. Then there exists by point (3) of Theorem 2.3 a map $f^\wedge$ such that

$$\begin{array}{ccc}
(BH \times BT)^\wedge & \xrightarrow{(f_H)^\wedge \times (B\varphi)^\wedge} & (BH \times BT)^\wedge \\
{\scriptstyle (Bq_1)^\wedge}\downarrow & & \downarrow{\scriptstyle (Bq_2)^\wedge} \\
(BG_1)^\wedge & \xrightarrow[f^\wedge]{} & (BG_2)^\wedge
\end{array}$$

commutes. Since Bq_1 and Bq_2 are rational equivalences, there also exists a homotopy equivalence $f_{(0)}\colon (BG_1)_{(0)} \to (BG_2)_{(0)}$ such that the corresponding diagram in the rational category commutes. Exploiting, as above, that Bq_1 and Bq_2 are rational equivalences, we see that $f^\wedge$ and $f_{(0)}$ come from a map $f\colon BG_1 \to BG_2$ which must necessarily be a rational equivalence.

It is easy to see (using Theorem 2.3) that if $f \in \varepsilon_{\mathbb{Q}}(BG_1, BG_2)$, $f_H \in \varepsilon_{\mathbb{Q}}(BH)$, and $\varphi \in \mathrm{Epi}(T)$ satisfy $Bq_2 \circ (f_H \times B\varphi) \simeq f \circ Bq_1$, then f is a homotopy equivalence if and only if f_H is a homotopy equivalence, φ is an automorphism, and $\eta(f_H)(K_1) = K_2$. $\square$

In particular, $\alpha[n_1, \ldots, n_t; R] \in \varepsilon_{\mathbb{Q}}(BH) \times \varepsilon_{\mathbb{Q}}(BT)$ given, with notation as in Section 0, by

$$\alpha[n_1, \ldots, n_t; R] = \left(\prod_{i=1}^{t} \psi^{n_i},\ R \right),$$

defines a rational equivalence $BG \to BF_RG[n_1, \ldots, n_t]$.

Example 2.6. i) With $\Gamma_k \subset U(n)$ as in Example 2.5,

$$\begin{aligned}\varepsilon_{\mathbb{Q}}\left(B(U(n)/\Gamma_k)\,,\, B(U(n)/\Gamma_l)\right) \cong &\\ \{\, (\lambda, \kappa) \in \mathbb{Z} \times \mathbb{Z} \mid (\lambda, n\,!) = 1\ ,\ \lambda l \equiv \kappa k \bmod n \,\}&\end{aligned}$$

for $n > 2$. In particular, when $(k, n) = 1$,

$$\begin{aligned}\varepsilon_{\mathbb{Q}}\left(BFU_k(n)\right) \cong \varepsilon_{\mathbb{Q}}\left(BU(n)\right) \cong &\\ \{(\lambda, \kappa) \in \mathbb{Z} \times \mathbb{Z} \mid (\lambda, n!) = 1,\ \kappa \equiv \lambda \bmod n\}, &\quad n > 2,\end{aligned}$$

as monoids.

ii) For $H = \mathrm{Spin}\,(2n+1) \times \mathrm{Sp}\,(n)\,,\ n > 2\,$,

$$\varepsilon_{\mathbb{Q}}(BH)\ = \{\, (\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z} \mid (\lambda, n\,!) = 1 = (\mu, n\,!) \,\}$$

by ([Mø], Theorem 3.1) and [J-M-O$_1$], and

$$\varepsilon_{\mathbb{Q}}\left(B(H/K_1)\,,\, B(H/K_2)\right) = \begin{cases} \varepsilon_{\mathbb{Q}}(BH) & \text{if } K_1 = K_2 \\ \emptyset & \text{if } K_1 \neq K_2 \end{cases}$$

for any pair of central subgroups $K_1, K_2 \subset Z_H$, cfr. ([J-M-O$_2$], Example 2.1).

The next proposition shows that Theorem 2.3 covers all existing rational equivalences between classifying spaces of compact connected Lie groups.

Proposition 2.7. *Two compact connected Lie groups, G_1 and G_2, are locally isomorphic if any of the following three conditions holds*

1. $\varepsilon_{\mathbb{Q}}(BG_1\,,\, BG_2) \neq \emptyset$
2. $\varepsilon_{\mathbb{Q}}((BG_1)^{\wedge}_2\,,\, (BG_2)^{\wedge}_2) \neq \emptyset$

3. *$\varepsilon_{\mathbb{Q}}((BG_1)^\wedge_p\,,\,(BG_2)^\wedge_p) \neq \emptyset$ where p is odd and G_1 and G_2 contain no simple factors of type $C_n,\ n > 2$.*

Proof. (1) First note the special case

$$\varepsilon_{\mathbb{Q}}(BG_1\,,\,BG_2) \neq \emptyset \Rightarrow G_1 \cong G_2$$

when G_1 and G_2 are simply connected simple Lie groups. To see this, choose a homomorphism $G_1 \supset T_1 \xrightarrow{\omega} T_2 \subset G_2$ of maximal tori such that the diagram

$$\begin{array}{ccc} BT_1 & \xrightarrow{B\omega} & BT_2 \\ \downarrow & & \downarrow \\ BG_1 & \longrightarrow & BG_2 \end{array}$$

homotopy commutes with some rational equivalence at the bottom. The induced map $\omega_*\colon \pi_1(T_1)\otimes\mathbb{Q} \longrightarrow \pi_1(T_2)\otimes\mathbb{Q}$ is an admissible isomorphism. Therefore there exist a bijection $\varphi\colon \Phi_1^+ \longrightarrow \Phi_2^+$ of positive roots and a function $\lambda\colon \Phi_1^+ \longrightarrow \mathbb{Q}^*$ such that $\omega_* \circ \sigma_\alpha = \sigma_{\varphi(\alpha)} \circ \omega_*$ and $\varphi(\alpha) \circ \omega_* = \lambda(\alpha)\alpha$ for all positive roots $\alpha \in \Phi_1^+$; see ([A-M], Theorem 2.12), ([Mø], Theorem 1.3) or [J-M-O$_2$, Lemma 2.3]. These two identities imply that φ preserves the product $n_{\alpha\beta}n_{\beta\alpha}$ of Cartan numbers so since φ, at least after composition with an automorphism from the Weyl group of G_2, can be assumed to preserve the simple roots, φ determines an isomorphism of Coxeter graphs. Hence $G_1 \cong G_2$ or $G_1, G_2 \in \{\mathrm{Spin}\,(2n+1), Sp(n)\}$ for some $n > 2$. But even in the latter case $G_1 \cong G_2$ for Ishiguro showed ([I], Theorem 5(a)) that there exist no essential maps $BSpin(2n+1) \leftrightarrow BSp(n),\ n > 2$.

In the general case, view G_i as a finite quotient of $H_i \times T^{n_i}$ where H_i is simply connected and T^{n_i} is an n_i-dimensional torus. If BG_1 and BG_2 are rationally equivalent so are the 2-connected covers $BH_1 \simeq BG_1\langle 2\rangle$ and $BH_2 \simeq BG_2\langle 2\rangle$ and

$$n_1 = \dim_{\mathbb{Q}}(\pi_2(BG_1)\otimes\mathbb{Q}) = \dim_{\mathbb{Q}}(\pi_2(BG_2)\otimes\mathbb{Q}) = n_2\ .$$

To show $H_1 \cong H_2$ we again exploit admissibility. Choose a homomorphism $H_1 \supset T_1 \xrightarrow{\omega} T_2 \subset H_2$ of maximal tori such that $B\omega\colon BT_1 \longrightarrow BT_2$ covers some rational equivalence $f\colon BG_1 \longrightarrow BG_2$. The induced map $\omega_*\colon \pi_1(T_1)\otimes\mathbb{Q} \longrightarrow \pi_1(T_2)\otimes\mathbb{Q}$ is equivariant w.r.t. some automorphism of the Weyl groups ([A-M], Theorem 2.12) and $\pi_1(T_i)\otimes\mathbb{Q}$ splits into a direct sum of distinct irreducible representations of the Weyl group, ([Ba], Proposition 5, p. 82) or ([J-M-O$_2$], Lemma 2.3), corresponding to the simple factors of H_i. Thus, by Schur's lemma, each simple factor $S_1 \triangleleft H_1$

corresponds to a unique simple factor $S_2 \lhd H_2$ such that ω_* maps the direct summand of $\pi_1(T_1) \otimes \mathbb{Q}$ corresponding to S_1 isomorphically to the direct summand of $\pi_1(T_2) \otimes \mathbb{Q}$ corresponding to S_2. The composition

$$BS_1 \longrightarrow BH_1 \xrightarrow{f} BH_2 \longrightarrow BS_2$$

of the rational equivalence f with the obvious inclusion and projection, is then again a rational equivalence and so $S_1 \cong S_2$ by the first part of this proof. This shows $H_1 \cong H_2$.

(2) is proved similarly using ([I], Theorem 5(b)), which says that all maps $BSpin(2n+1)^\wedge_2 \leftrightarrow BSp(n)^\wedge_2$ are nonessential, to handle the special case of simply connected simple groups. These kind of difficulties do not occur in (3) where admissibility suffices. $\square$

When the prime p is odd, any simple factor of type $C_n,\ n > 2$, can always be replaced by a simple factor of type B_n without changing the homotopy type of the p-completed classifying space.

Finally, a simple test for a map to be a rational equivalence.

Proposition 2.8. *Let G_1 and G_2 be two compact connected Lie groups.*

1. *A map $f\colon BG_1 \to BG_2$ is a rational equivalence if and only if the induced map*

$$H^*(f;\mathbb{Q})\colon H^*(BG_1;\mathbb{Q}) \longleftarrow H^*(BG_2;\mathbb{Q})$$

is an isomorphism.

2. *A map $f\colon (BG_1)^\wedge_p \to (BG_2)^\wedge_p$ is a rational equivalence if and only if the induced map*

$$H^*(f;\mathbb{Z}^\wedge_p) \otimes 1_{\mathbb{Q}}\colon H^*((BG_1)^\wedge_p;\mathbb{Z}^\wedge_p) \otimes \mathbb{Q} \longleftarrow H^*((BG_2)^\wedge_p;\mathbb{Z}^\wedge_p) \otimes \mathbb{Q}$$

is an isomorphism.

Proof of (2). If $H^*(f;\mathbb{Z}^\wedge_p) \otimes \mathbb{Q}$ is an isomorphism of graded algebras, the induced map of indecomposables is an isomorphism of modules over $\mathbb{Q}_p$ naturally identifiable to the dual of the induced homomorphism

$$\pi_*(f) \otimes 1_{\mathbb{Q}}\colon \pi_*((BG_1)^\wedge_p) \otimes \mathbb{Q} \longrightarrow \pi_*((BG_2)^\wedge_p) \otimes \mathbb{Q}$$

between homotopy groups. The rationalization $f_{(0)}\colon ((BG_1)^\wedge_p)_{(0)} \to ((BG_2)^\wedge_p)_{(0)}$ is thus a homotopy equivalence.

Conversely, if f is a rational equivalence, its fibre, F, is a connected torsion space and hence $H^*(F;\mathbb{Z}^\wedge_p)\otimes\mathbb{Q} \cong \mathbb{Q}_p$. The Serre spectral sequence

$$\big(H^*((BG_2)^\wedge_p;\mathbb{Z}^\wedge_p) \otimes \mathbb{Q}\big)\otimes_{\mathbb{Q}_p}\big(H^*(F;\mathbb{Z}^\wedge_p) \otimes \mathbb{Q}\big) \Longrightarrow H^*((BG_1)^\wedge_p;\mathbb{Z}^\wedge_p)\otimes\mathbb{Q}$$

now implies that $H^*(f;\mathbb{Z}^\wedge_p) \otimes \mathbb{Q}$ is an isomorphism. $\square$

3. Local isomorphism systems

This section contains the proofs of Proposition 0.2, Proposition 0.4 and Proposition 0.5.

As in Section 0, $G_1 = H \times T/(K_1, \varphi_1)$ and $G_2 = H \times T/(K_2, \varphi_2)$ are two Lie groups locally isomorphic to some fixed connected compact Lie group $G = H \times T/(K, \varphi)$. Here, H is a simply connected compact Lie group and if $H = \prod_{i=1}^t H_i$, where H_i is simple, the derived group

$$F_R G[n_1, \ldots, n_t] = \\ H \times T \Big/ \left(\left(\prod_{i=1}^t \psi^{n_i} \right)(K),\ R \circ \varphi \circ \left(\prod_{i=1}^t \psi^{n_i} | Z_H \right)^{-1} \right)$$

is defined whenever $R \in \mathrm{Epi}(T)$ is injective on $\varphi(K) \subset T$ and $n_i \in \mathbb{N}$ is prime to the Weyl group order of H_i.

The first proposition is a slight improvement of Proposition 0.2. Let $\det R$ denote the determinant of $\eta(R) \in \mathrm{Mono}(\pi_1(T))$, i.e. $|\det(R)| = |\mathrm{coker}\,(\eta(R))|$.

Proposition 3.1. *Let G_1 and G_2 be two locally isomorphic compact connected Lie groups. Then $G_1 \geq G_2 \geq G_1$ if and only if $F_R G_1 \cong G_2$ for some $R \in \mathrm{Epi}(T)$ with determinant prime to $|K_1|$.*

Proof. We must show $F_R G_1 \geq G_1$. Note that the map $R|\,\mathrm{im}\ \varphi_1 : \mathrm{im}\ \varphi_1 \to \mathrm{im}\,(R\varphi_1)$ is an isomorphism. The extension problem

$$\begin{array}{ccc} \mathrm{im}\,(R\varphi_1) & \xrightarrow[\cong]{(R|\,\mathrm{im}\ \varphi_1)^{-1}} & \mathrm{im}\ \varphi_1 \\ \downarrow & & \downarrow \\ T & \xrightarrow{?} & T \end{array}$$

is equivalent to the lifting problem

$$\begin{array}{ccc} \pi_1(T/\,\mathrm{im}\,(R\varphi_1)) & \xrightarrow{?} & \pi_1(T/\,\mathrm{im}\ \varphi_1) \\ \partial \downarrow & & \downarrow \partial \\ \mathrm{im}\,(R\varphi_1) & \xrightarrow[\cong]{(R|\,\mathrm{im}\ \varphi_1)^{-1}} & \mathrm{im}\ \varphi_1 \end{array}$$

whose solution Lemma 3.2 assures; here, ∂ is the homomorphism classifying the principal bundle $T \longrightarrow T/\,\mathrm{im}\,(R\varphi_1)$ or $T \longrightarrow T/\,\mathrm{im}\ \varphi_1$. Thus there exists an epimorphism $S \in \mathrm{Epi}(T)$ making the diagram

$$\begin{tikzcd} K_1 \arrow[r, "R\varphi_1"] \arrow[d, equal] & T \arrow[d, "S"] \\ K_1 \arrow[r, "\varphi_1"] & T \end{tikzcd}$$

commute meaning (Theorem 0.1) that $F_R G_1 \geq G_1$ (and that $F_S F_R G_1 \cong G_1$).

Conversely, assume $G_1 \geq G_2 \geq G_1$ and let $G_1 \to G_2$ be an epimorphism determined by $\alpha_1 \in \mathrm{Out}(H)$ and $\beta_1 \in \mathrm{Epi}(T)$ such that $\alpha_1(K_1) \subset K_2$ and $\beta_1\varphi_1 = \varphi_2\alpha_1|K_1$. Since also $G_2 \geq G_1$, $|K_1| = |K_2|$ and $\beta_1 \mid \mathrm{im}\ \varphi_1 : \mathrm{im}\ \varphi_1 \to \mathrm{im}\ \varphi_2$ is an isomorphism. Choose, using Lemma 3.2, some $R \in \mathrm{Epi}(T)$ such that

$$\begin{tikzcd} K_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha_1"', "\cong"] & \mathrm{im}\ \varphi_1 \arrow[r, hook] \arrow[d, "\beta_1"', "\cong"] & T \arrow[d, "R"] \\ K_2 \arrow[r, "\varphi_2"'] & \mathrm{im}\ \varphi_2 \arrow[r, hook] & T \end{tikzcd}$$

commutes and $(|K_1|, \det R) = 1$. Then $R\varphi_1|K_1 = \varphi_2\alpha_1|K_1$ so, by Theorem 0.1, $(\alpha_1, 1) \in \mathrm{Aut}(H) \times \mathrm{Aut}(T)$ covers a Lie group isomorphism $F_R G_1 \xrightarrow{\cong} G_2$. $\square$

Lemma 3.2. *Let R be a commutative ring, M a finitely generated R-module, F a finitely generated free R-module and $\partial_1 : F \twoheadrightarrow M$, $\partial_2 : F \twoheadrightarrow M$ epimorphisms.*

i) *If R is a local principal ideal domain, then there exists an isomorphism $f : F \longrightarrow F$ such that $\partial_2 f = \partial_1$.*

ii) *If $R = \mathbb{Z}$, M is a finite abelian group and n is a natural number such that $nM = 0$ then there exists a monomorphism $f : F \longrightarrow F$ such that $\partial_2 f = \partial_1$ and $(n, \det f) = 1$.*

Proof. i) Let k denote the residue field of R. Suppose first that $\partial_i \otimes_R k : F \otimes_R k \longrightarrow M \otimes_R k$ is an isomorphism, $i = 1, 2$. Let $f : F \longrightarrow F$ be any R-homomorphism such that $\partial_2 f = \partial_1$. Nakayama's lemma ([Ma], 1. M) applied to the kernel and cokernel of f shows that f is an isomorphism. In the general case, choose a minimal basis ([Ma], 1. N) of M, i.e. a free finitely generated R-module F_i' and an epimorphism $\partial_i' : F_i' \twoheadrightarrow M$ inducing an isomorphism $\partial_i' \otimes_R k$ of vector spaces. By projectivity, there exists a homomorphism η_i such that the diagram

$$\begin{tikzcd} & F_i \arrow[dl, "\eta_i"'] \arrow[d, "\partial_i"] \\ F_i' \arrow[r, "\partial_i'"'] & M \end{tikzcd}$$

commutes; η_i is an epimorphism by Nakayama's lemma. Thus $F \cong F_i' \oplus F_i''$ is a direct sum of two finitely generated free modules, $(\partial_i|F_i') \otimes_R k$ is an isomorphism, $\partial_i|F_i'' = 0$, and $\dim_k(F_i'' \otimes_R k) = \dim_k(F \otimes_R k) - \dim_R(M \otimes_R k)$, $i = 1, 2$. Define $f := f' \oplus f''$ where $f' : F_1' \longrightarrow F_2'$ is any homomorphism such that $\partial_2' f = \partial_1'$ and $f'' : F_1'' \xrightarrow{\cong} F_2''$ is any isomorphism.

ii) M is a module over the Artinian ring $\mathbb{Z}/n\mathbb{Z}$ and has as such a projective cover $\pi : P(M) \twoheadrightarrow M$. For any other epimorphism $p : P \twoheadrightarrow M$ from a projective module P onto M there exists a module K and an isomorphism $P(A) \oplus K \cong P$ such that $p|P(A) = \pi$ and $p(K) = 0$ ([Bass], Lemma 2.3). In particular, we have a commutative diagram

$$\begin{tikzcd} & P(A) \oplus K_1 \arrow[d, "\cong"] & P(A) \oplus K_2 \arrow[d, "\cong"'] & \\ F \arrow[r, "\mathrm{can}"] \arrow[dr, "\partial_1"'] & F/nF \arrow[r, dashed, "\overline{f}"] \arrow[d, "\overline{\partial_1}"] & F/nF \arrow[d, "\overline{\partial_2}"'] & F \arrow[l, "\mathrm{can}"'] \arrow[dl, "\partial_2"] \\ & A \arrow[r, equal] & A & \end{tikzcd}$$

where $\overline{\partial_1}|P(A) = \pi = \overline{\partial_2}|P(A)$ and $\partial_1 K_1 = 0 = \partial_2 K_2$. Since F/nF is a finite abelian group, $K_1 \cong K_2$; let $f' : K_1 \to K_2$ be any isomorphism. The map $\overline{f} := 1_{P(A)} \oplus f'$ is an isomorphism of F/nF with $\overline{\partial_1} = \overline{\partial_2} \circ f$. Take $f : F \to F$ to be any homomorphism covering f. Since $f \otimes 1_{\mathbb{Z}/n\mathbb{Z}}$ is an isomorphism of F/nF, the determinant of f is invertible in the ring $\mathbb{Z}/n\mathbb{Z}$. $\square$

Proof of Proposition 0.4. (1) Suppose $BG_1 \geq BG_2$. Then there exists an $f \in \varepsilon_{\mathbb{Q}}(BH)$ and an $R \in \mathrm{Epi}(T)$ making the diagram

$$\begin{tikzcd} Z_H \arrow[d, "\eta(f)"'] & K_1 \arrow[l, hook'] \arrow[r, "\varphi_1"] \arrow[d] & T \arrow[d, "R"] \\ Z_H & K_2 \arrow[l, hook'] \arrow[r, "\varphi_2"'] & T \end{tikzcd}$$

commute. The rational equivalence f has, cf. the remarks preceding Example 2.2, the form $f = B\alpha \circ \prod \psi^{n_i}$ for some (outer) automorphism α of H and some natural numbers n_i prime to the Weyl group order of H_i. Rearranging the above commutative diagram we obtain another commutative diagram

$$\begin{tikzcd}
H \arrow[d, "\alpha"'] & \left(\prod \psi^{n_i}\right)(K_1) \arrow[l, hook'] \arrow[d] \arrow[r, "\varphi_1 \circ (\prod \psi^{n_i}|Z_H)^{-1}"] & T \arrow[d, "R"] \\
H & K_2 \arrow[l, hook'] \arrow[r, "\varphi_2"'] & T
\end{tikzcd}$$

showing that $G_1[n_1, \ldots, n_t] \geq G_2$.
(2) If BG_1 and BG_2 are homotopy equivalent, then, because $\mathrm{Aut}(BH) = \mathrm{Out}(H)$ e. g. by ([Mø], Corollary 3.2), the first of the above diagrams commutes with $\eta(f) = \eta(B\alpha) = \alpha$ for some $\alpha \in \mathrm{Out}(H)$ and some $R \in \mathrm{Aut}(T)$ and thus G_1 and G_2 are isomorphic ([Ba], Corollary 6) Lie groups. $\square$

We shall later need a slightly stronger version of point (1) in Proposition 0.4. Let W_H denote the Weyl group of H.

Proposition 3.3. *Let G_1 and G_2 be two locally isomorphic connected compact Lie groups. Then $BG_1 \geq BG_2$ if and only if $G_1[n_1 \ldots, n_t] \geq G_2$ for some natural numbers n_i with $(\prod n_i, |W_H|) = 1$.*

Since the result of the process $G_1 \to G_1[n_1, \ldots, n_t]$ only depends on n_i modulo the Weyl group order of H_i, this follows immediately from Proposition 0.4 and

Lemma 3.4. *Let $w_1, \ldots, w_t$ and $m_1, \ldots, m_t$ be natural numbers with $(m_i, w_i) = 1$ for $1 \leq i \leq t$. Then there exist natural numbers (primes) n_i such that $n_i \equiv m_i \bmod w_i$ for all $1 \leq i \leq t$ and $(\prod n_i, \prod w_i) = 1$.*

Proof. By Dirichlet's Theorem there are infinitely many prime numbers congruent to m_i modulo w_i. Choose one such prime which is larger than $\prod w_i$ and call it n_i. Do this for each i. The result follows. $\square$

The proof of Proposition 3.3 shows that any rational equivalence $f : BG_1 \to BG_2$ has a factorization of the form

$$BG_1 \xrightarrow{(\prod \psi^{m_i}, 1_T)} BG_1[n_1, \ldots, n_t] \xrightarrow{B\gamma} BG_2$$

where $m_i \in \mathbb{N}$ is prime to, and congruent to n_i modulo the order of, the Weyl group of H_i, $(\prod n_i, |W_H|) = 1$, and $\gamma = (\alpha, R)$ is a Lie group epimorphism of $G_1[n_1, \ldots, n_t]$ onto G_2.

Note also, in connection with the proof of point (2) of Proposition 0.4, that Theorem 2.3 contains as a special case the isomorphism

$$\mathrm{Aut}(BG) \cong \mathrm{Out}(G)$$

from [J-M-O$_2$, Corollary 3.2].

Combining (the proofs of) Proposition 3.1 and Proposition 3.3 results in

Proposition 3.5. *The following three conditions are equivalent:*

1. $BG_1 \geq BG_2 \geq BG_1$
2. $F_R G_1[n_1, \ldots, n_t] \cong G_2$ *for some* $R \in \mathrm{Epi}(T)$ *with* $(\det R,\, |K_1|) = 1$ *and some* $n_i \in \mathbb{N}$ *with* $(\prod n_i,\, |W_H|) = 1$
3. $G_1[n_1, \ldots, n_t] \geq G_2 \geq G_1[n_1, \ldots, n_t]$ *for some* $n_i \in \mathbb{N}$ *with* $(\prod n_i,\, |W_H|) = 1$

Example 3.6. Let $Z = \{(\omega_1 E,\, \omega_2 E) \mid \omega_1^5 = 1 = \omega_2^5\}$ denote the center of $H = SU(5) \times SU(5)$.

i) There are no covering homomorphisms between

$$G_1 = H \times S^1/(Z,\, \varphi_1) \quad \text{and} \quad G_2 = G_1[1,\, 13] = H \times S^1/(Z,\, \varphi_2),$$

where $\varphi_1(\omega_1 E,\, \omega_2 E) = \omega_1\omega_2$ and $\varphi_2(\omega_1 E,\, \omega_2 E) = \omega_1\omega_2^2$, even though $BG_1 \geq BG_2 \geq BG_1$; in particular $F_R G_1 \neq G_2$ for all $R \in \mathrm{Epi}(T)$, $5 \nmid \det R$, so condition (2) in Proposition 3.5 can not be sharpened to $F_R G_1 = G_2$.

ii) Let

$$G_1 = H \times S^1 \times S^1/(Z,\, \varphi_1) \text{ and } G_2 = F_R G_1 = H \times S^1 \times S^1/(Z,\, \varphi_2)$$

where $\varphi_1(\omega_1 E, \omega_2 E) = (\omega_1, \omega_2)$, $R(t_1, t_2) = (t_1 t_2, t_2)$ and $\varphi_2(\omega_1 E, \omega_2 E) = R\varphi_1(\omega_1 E, \omega_2 E) = (\omega_1, \omega_2^2)$. Then $BG_1 \geq BG_2 \geq BG_1$ because $G_1 \geq G_2 \geq G_1$ but $G_1[n_1,\, n_2] \neq G_2$ for all choices of $n_1,\, n_2 \in \mathbb{N}$ with $(n_1 n_2,\, 5!) = 1$, so condition (2) in Proposition 3.5 can not be sharpened to $G_1[n_1,\, n_2] = G_2$.

iii) For

$$G_1 = H/\Delta_1 \quad \text{and} \quad G_2 = G_1[1,\, 7] = H/\Delta_2,$$

where $\Delta_i = \{(\omega E,\, \omega^i E)\}$, $i = 1, 2$, the groups $\mathrm{Out}(G_1) \cong \mathrm{Aut}(BG_1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\mathrm{Out}(G_2) \cong \mathrm{Aut}(BG_2) \cong \mathbb{Z}/2$ are not isomorphic. No covering homomorphism exists between G_1 and G_2 as no Lie group automorphism of H takes Δ_1 to Δ_2.

The structure of the local isomorphism system of a connected compact Lie group G is conveniently represented by an oriented graph: Define $G_1 \sim G_2$ if $G_2 \in [G_1, G_1]$ and represent the partially ordered set of these equivalence classes by an oriented graph in the usual fashion. Indicate the number of elements in each equivalence class.

The oriented graph representing the local isomorphism system of $G = SU(5) \times SU(5)$ is

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[d] \arrow[dr] & \\ 1 \arrow[dr] & 1 \arrow[d] & 1 \arrow[dl] \\ & 1 & \end{tikzcd}$$

where G is at the top and the projective group G/Z_G is at the bottom. In the middle row are the quotients of G by the special subgroups Δ_0, Δ_1, and Δ_2 (in the notation of Example 3.6).

Now define the homotopy local isomorphism system of G to be the set of isomorphism classes of connected compact Lie groups locally isomorphic to G equipped with the relation $G_1 \gtrsim G_2$ if $BG_1 \geq BG_2$. Also this ordered set may be represented by an ordered graph in the same way as was done for the local isomorphism system. This time we group together all groups equivalent under the equivalence relation $G_1 \approx G_2$ if $G_2 \in \langle G_1, G_1 \rangle$. For $G = SU(5) \times SU(5)$ the result is

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ 1 \arrow[dr] & & 2 \arrow[dl] \\ & 1 & \end{tikzcd}$$

because $(\psi^1 \times \psi^7)(\Delta_1) = \Delta_2$ and no other relations are introduced.

4. The genus set of BG

For a compact connected Lie group $G = H \times T/(K, \varphi)$, the set $\langle G, G \rangle$ consists (Proposition 3.5) of the derived groups $F_R G[n_1, \ldots, n_t]$ where $(\prod n_i, |W_H|) = 1 = (\det R, |K|)$. If the prime p divides $|K|$, p does not divide neither $\prod n_i$ nor $\det R$, so $\prod \psi^{n_i}$ is a homotopy equivalence of $BH_p^\wedge$ and $R|T_{p^\infty}$ an automorphism of T_{p^∞}. It follows (Theorem 2.3) that

$\beta[n_1,\dots,n_t;R]_p \in \mathrm{Aut}(BH_p^\wedge) \times \mathrm{Aut}(T_{p^\infty})$ given by

$$\beta[n_1,\dots,n_t;R]_p = \begin{cases} (\prod \psi^{n_i},\ R) & \text{if } p \mid |K| \\ (1,\ 1) & \text{if } p \nmid |K| \end{cases}$$

defines a homotopy equivalence $BG^\wedge \xrightarrow{\beta[n_1,\dots,n_t;R]} BF_RG[n_1,\dots,n_t]^\wedge$ such that the diagram of completed classifying spaces

$$\begin{tikzcd} BH^\wedge \times BT^\wedge \arrow[r, "{\beta[n_1,\dots,n_t;R]}", "\cong"'] \arrow[d, "Bq"'] & BH^\wedge \times BT^\wedge \arrow[d, "Bq"] \\ BG^\wedge \arrow[r, "\cong", "{\beta[n_1,\dots,n_t;R]}"'] & BF_RG[n_1,\dots,n_t]^\wedge \end{tikzcd}$$

commutes where q indiscriminately denotes the projection of $H \times T$ onto G or $F_RG[n_1,\dots,n_t]$. Thus BG and $BF_RG[n_1,\dots,n_t]$ are spaces of the same genus. Moreover, $BG \geq BF_RG[n_1,\dots,n_t]$. This is the point of departure for the proof of Theorem 0.8. But first a p-adic version of Proposition 3.5.

For any prime p, the Lie group

$$G^{(p)} := H \times T \big/ (K_p, \varphi|K_p)$$

was defined in Section 0 as the quotient of $H \times T$ by the graph of the restriction $\varphi|K_p$ of φ to the p-primary subgroup of K. Note the principal bundle

$$K\big/K_p \longrightarrow G^{(p)} \longrightarrow G$$

implying that $(BG^{(p)})_p^\wedge \simeq BG_p^\wedge$.

Proposition 4.1. *Let G_1 and G_2 be two locally isomorphic compact connected Lie groups and p a prime. Then the following four conditions are equivalent:*

1. $(BG_1)_p^\wedge \simeq (BG_2)_p^\wedge$
2. $F_RG_1[n_1,\dots,n_t]^{(p)} \cong G_2^{(p)}$ *for some* $R \in \mathrm{Epi}(T)$ *and some* $n_i \in \mathbb{N}$ *such that* $p \nmid \det R$ *and* $(\prod n_i,\ |W_H|) = 1$
3. $BG_1^{(p)} \geq BG_2^{(p)} \geq BG_1^{(p)}$
4. $G_1[n_1,\dots,n_t]^{(p)} \geq G_2^{(p)} \geq G_1[n_1,\dots,n_t]^{(p)}$ *for some natural numbers* n_i *with* $(\prod n_i,\ |W_H|) = 1$

Proof. If $(BG_1)^\wedge_p \simeq (BG_2)^\wedge_p$ then there exist $f \in \varepsilon_{\mathbb{Q}}(BH^\wedge_p)$ and $\beta \in \mathrm{Aut}(T_{p^\infty})$ such that

$$\begin{tikzcd}
(Z_H)_p \arrow[d, "\eta_p(f)"'] & (K_1)_p \arrow[l, hook'] \arrow[r, "\varphi_1"] \arrow[d, "\cong"'] & T \arrow[d, "\cong"', "\beta"] \\
(Z_H)_p & (K_2)_p \arrow[l, hook'] \arrow[r, "\varphi_2"'] & T
\end{tikzcd}$$

commutes. By the remarks preceding Example 2.2, the homotopy equivalence f has the form

$$f = g \circ (\psi^{u_1} \times \cdots \times \psi^{u_t})$$

where $u_i \in (\mathbb{Z}^\wedge_p)^*$ is a p-adic unit and g is a composition of automorphisms of H and possibly some exceptional isogenies. However, η_p is trivial on any exceptional isogeny, so $\eta_p(f) = \alpha \circ \prod \psi^{n_i}|Z_H$ for some $\alpha \in \mathrm{Out}(H)$ and some natural numbers $n_i \in \mathbb{N}$ with $n_i \equiv u_i \bmod |Z_H|$ and, by Lemma 3.4, $(\prod n_i,\, |W_H|) = 1$. According to Lemma 3.2, the derived commutative diagram

$$\begin{tikzcd}
H \arrow[d, "\alpha"'] & \left(\prod \psi^{n_i}\right)(K_1)_p \arrow[l, hook'] \arrow[r, "\varphi_1 \circ (\prod \psi^{n_i}|Z_H)^{-1}"] \arrow[d, "\cong"'] & \mathrm{im}\ \varphi_1 \arrow[r, hook] \arrow[d, "\beta|\,\mathrm{im}\ \varphi_1"', "\cong"] & T \arrow[d, dashed, "R"] \\
H & (K_2)_p \arrow[l, hook'] \arrow[r, "\varphi_2"'] & \mathrm{im}\ \varphi_2 \arrow[r, hook] & T
\end{tikzcd}$$

can be completed by some $R \in \mathrm{Epi}(T)$ whose determinant is not divisible by p. Hence $(\alpha,\, 1_T)$ covers a Lie group isomorphism $F_R G_1[n_1, \ldots, n_t]^{(p)} \cong G_2^{(p)}$.

The implications $(2) \Rightarrow (3) \Rightarrow (4)$ follow from Proposition 3.5.

Now assume (4). Since $BG_1[n_1, \ldots, n_t]^{(p)} \geq BG_2^{(p)}$ and $|(K_1)_p| = |(K_2)_p|$ there exist $\alpha_1 \in \mathrm{Out}(H)$ and $\beta_1 \in \mathrm{Epi}(T)$ such that the diagram

$$\begin{tikzcd}
H \arrow[d, "\alpha_1"'] & \left(\prod \psi^{n_i}\right)(K_1)_p \arrow[l, hook'] \arrow[r, "\varphi_1 \circ (\prod \psi^{n_i}|Z_H)^{-1}"] \arrow[d] & T \arrow[d, "\beta_1"] \\
H & (K_2)_p \arrow[l, hook'] \arrow[r, "\varphi_2"'] & T
\end{tikzcd}$$

commutes where $\alpha_1|(K_1)_p : \left(\prod \psi^{n_i}\right)(K_1)_p \longrightarrow (K_2)_p$ and $\beta_1|\,\mathrm{im}\ \varphi_1 : \mathrm{im}\ \varphi_1 \longrightarrow \mathrm{im}\ \varphi_2$ are isomorphisms. By Lemma 3.2 there exists $\gamma \in \mathrm{Aut}(T_{p^\infty})$ such that $\gamma \circ \varphi_1 = \varphi_2 \circ \alpha_1|(K_1)_p$ meaning that

$$(BG_1^{(p)})^\wedge_p \simeq (BG_1[n_1, \ldots, n_t]^{(p)})^\wedge_p \simeq (BG_2^{(p)})^\wedge_p$$

where $(BG_i^{(p)})_p^\wedge \simeq (BG_i)^\wedge$ for $i = 1, 2$ by the fibration mentioned above. $\square$

The above proposition implies (Corollary 0.7) that if $G_1 \geq G_2 \geq G_1$, then BG_1 and BG_2 are of the same genus. The converse is not true as shown by

Example 4.2. i) Let p and q be distinct primes and let $G_i = SU(pq) \times SU(pq)/K_i$ where $K_1 = \mathbb{Z}/pq \times 1$ and $K_2 = \mathbb{Z}/p \times \mathbb{Z}/q$. Then $G_1^{(p)} \cong G_2^{(p)}$ and $G_1^{(q)} \cong G_2^{(q)}$ so BG_1 and BG_2 are of the same genus but neither $G_1 \geq G_2$ nor $G_2 \geq G_1$ and neither $BG_1 \geq BG_2$ nor $BG_2 \geq BG_1$.
ii) Let p be a prime, $Z = \{\, \omega^i E \,\} \cong \mathbb{Z}/p$, $\omega = e^{2\pi i/p}$, the center of $SU(p)$, and $\varphi_k : Z \longrightarrow S^1$ the homomorphism given by $\varphi_k(\omega E) = \omega^k$ where $k \in \mathbb{Z}$. Then

$$SU(p) \times S^1/(Z, \varphi_k) \geq SU(p) \times S^1/(Z, \varphi_l) \geq SU(p) \times S^1/(Z, \varphi_k)$$

whenever $p \nmid k$ and $p \nmid l$ but

$$SU(p) \times S^1/(Z, \varphi_k) \cong SU(p) \times S^1/(Z, \varphi_l)$$

only if $k \equiv \pm l \mod p$ by Theorem 0.1. Hence condition (2) in Proposition 4.1 can not in general be sharpened to $G_1^{(p)} \cong G_2^{(p)}$.

More preparation is necessary before the proof of Theorem 0.8. Let B be a connected nilpotent space of finite type. A homotopy self-equivalence V of the formal completion $(B^\wedge)_{(0)}$ is called π_*-continuous if $\pi_*(V)$ is $\mathbb{Z}^\wedge \otimes \mathbb{Q}$-linear on $\pi_*((B^\wedge)_{(0)}) \cong \pi_*(B) \otimes \mathbb{Z}^\wedge \otimes \mathbb{Q}$. $\mathrm{CAut}((B^\wedge)_{(0)})$ denotes the subgroup of all π_*-continuous elements of $\mathrm{Aut}((B^\wedge)_{(0)})$ [W, Definition 3.3]. For any $V \in \mathrm{CAut}((B^\wedge)_{(0)})$ the homotopy inverse limit B_V of the diagram

$$B^\wedge \longrightarrow (B^\wedge) \xleftarrow[\cong]{V} (B^\wedge) \longleftarrow B_{(0)}$$

is a nilpotent space of finite type of the genus of B. In fact, Wilkerson showed [W, Theorem 3.8] that the assignment $V \to B_V$ induces a bijection

$$\mathrm{Aut}(B_{(0)}) \backslash \mathrm{CAut}((B^\wedge)_{(0)}) / \mathrm{Aut}(B^\wedge) \xrightarrow{\cong} G_0^\wedge(B)$$

of pointed sets.

Consider the subsets $G_0^\wedge(B)^+$ and $G_0^\wedge(B)^-$ of $G_0^\wedge(B)$ defined by

$$G_0^\wedge(B)^+ = \{X \in G_0^\wedge(B) \mid X \geq B\}, \quad G_0^\wedge(B)^- = \{X \in G_0^\wedge \mid B \geq X\}$$

where $X \geq B$ ($B \geq X$) if there exists a rational equivalence $X \to B$ ($B \to X$). Furthermore, let $\overline{\mathrm{Aut}}(B), \overline{\varepsilon}_{\mathbb{Q}}(B), \overline{\mathrm{Aut}}(B^\wedge), \overline{\varepsilon}_{\mathbb{Q}}(B^\wedge)$ denote the images in $\mathrm{CAut}((B^\wedge)_{(0)})$ of, respectively, $\mathrm{Aut}(B), \varepsilon_{\mathbb{Q}}(B), \mathrm{Aut}(B^\wedge), \varepsilon_{\mathbb{Q}}(B^\wedge)$.

Lemma 4.3. (Cf. [Mø, Proposition 3.9]) $X \in G_0^\wedge(B)^-$ *(*$X \in G_0^\wedge(B)^+$*) if and only if* $X \simeq B_{V^{-1}}$ *(*$X \simeq B_V$*) for some* $V \in \overline{\varepsilon}_{\mathbb{Q}}(B^\wedge)$*.*

Proof. Suppose $X \in G_0^\wedge(B)^-$. Let $\alpha : B \to X$ be a rational equivalence and $\beta : B^\wedge \to X^\wedge$ a homotopy equivalence. The commutative diagram

$$\begin{tikzcd}
B_{(0)} \arrow[r] \arrow[d, "\alpha_{(0)}"', "\simeq"] & (B^\wedge)_{(0)} \arrow[r, "\beta_{(0)}^{-1}\circ(\alpha^\wedge)_{(0)}", "\simeq"'] \arrow[d, "(\alpha^\wedge)_{(0)}"', "\simeq"] & (B^\wedge)_{(0)} \arrow[d, "\simeq"', "\beta_{(0)}"] & B^\wedge \arrow[l] \arrow[d, "\simeq"', "\beta"] \\
X_{(0)} \arrow[r] & (X^\wedge)_{(0)} \arrow[r, equal] & (X^\wedge)_{(0)} & X^\wedge \arrow[l]
\end{tikzcd}$$

shows that $X \simeq B_{V^{-1}}$ with $V = \beta_{(0)}^{-1}\circ(\alpha^\wedge)_{(0)} = (\beta^{-1}\circ\alpha^\wedge)_{(0)} \in \overline{\varepsilon}_{\mathbb{Q}}(B^\wedge)$.

Conversely, suppose $\beta \in \varepsilon_{\mathbb{Q}}(B^\wedge)$ and put $V = \beta_{(0)} \in \overline{\varepsilon}_{\mathbb{Q}}(B^\wedge)$. Then

$$\begin{tikzcd}
B_{(0)} \arrow[r] \arrow[d, equal] & (B^\wedge)_{(0)} \arrow[r, equal] \arrow[d, equal] & (B^\wedge)_{(0)} \arrow[d, "\simeq"', "V"] & B^\wedge \arrow[l] \arrow[d, "\beta"] \\
B_{(0)} \arrow[r] & (B^\wedge)_{(0)} \arrow[r, "\simeq", "V"'] & (B^\wedge)_{(0)} & B^\wedge \arrow[l]
\end{tikzcd}$$

commutes and induces a rational equivalence from B to $B_{V^{-1}}$.

Similar arguments apply to $G_0^\wedge(B)^+$. □

In the case of $B = BG$ and $X = BF_RG[n_1, \ldots, n_t]$, we have (see Section 2) a rational equivalence

$$\alpha[n_1, \ldots, n_t; R] \in \varepsilon_{\mathbb{Q}}(BG, BF_RG[n_1, \ldots, n_t]) \subset \varepsilon_{\mathbb{Q}}(BH) \times \varepsilon_{\mathbb{Q}}(BT)$$

as well as a homotopy equivalence

$$\begin{aligned} \beta[n_1, \ldots, n_t; R] &\in \varepsilon_{\mathbb{Q}}(BG^\wedge, BF_RG[n_1, \ldots, n_t]^\wedge) \\ &\subset \varepsilon_{\mathbb{Q}}(BH^\wedge) \times \varepsilon_{\mathbb{Q}}(BT^\wedge). \end{aligned}$$

The commutative diagram

$$\begin{tikzcd}
(BH\times BT)_{(0)} \arrow[r] \arrow[d, equal] & (BH^\wedge\times BT^\wedge)_{(0)} \arrow[d, equal] & (BH^\wedge\times BT^\wedge)_{(0)} \arrow[l, "\beta_{(0)}"', "\simeq"] \arrow[r, "Bq", "\simeq"'] \arrow[d, "\simeq"', "\beta_{(0)}"] & (BG^\wedge)_{(0)} \arrow[d, "\simeq"', "\beta_{(0)}"] & BG^\wedge \arrow[l] \arrow[d, "\simeq"', "\beta"] \\
(BH\times BT)_{(0)} \arrow[r] & (BH^\wedge\times BT^\wedge)_{(0)} \arrow[r, equal] & (BH^\wedge\times BT^\wedge)_{(0)} \arrow[r, "\simeq", "Bq"'] & (BF_RG[n_i]^\wedge)_{(0)} & BF_RG[n_i]^\wedge \arrow[l]
\end{tikzcd}$$

shows that $BF_RG[n_1, \ldots, n_t]$ as an element of

$$\operatorname{Aut}((BH\times BT)_{(0)})\backslash \operatorname{CAut}((BH^\wedge\times BT^\wedge)_{(0)})/\operatorname{Aut}(BG^\wedge) \cong G_0^\wedge(BG)$$

corresponds to the equivalence class containing $\beta := \beta[n_1, \dots, n_t; R]$. Another representative for the same equivalence class is $V[n_1, \dots, n_t; R]^{-1}$ where

$$V[n_1, \dots, n_t; R] := \beta[n_1, \dots, n_t; R]^{-1} \circ \alpha[n_1, \dots, n_t; R]$$

given, as an element of $\varepsilon_{\mathbb{Q}}(BH^\wedge) \times \varepsilon_{\mathbb{Q}}(BT^\wedge)$, by

$$V[n_1, \dots, n_t; R]_p = \begin{cases} (1,\, 1) & \text{if } p \mid |K| \\ (\prod \psi^{n_i},\, R) & \text{if } p \nmid |K| \end{cases}$$

is (Theorem 2.3) a rational self-equivalence of $BG^\wedge$.

Lemma 4.4. *$G_0^\wedge(B)^- = \{B\}$ if and only if $\overline{\varepsilon}_{\mathbb{Q}}(B^\wedge) = \overline{\mathrm{Aut}}(B^\wedge) \cdot \overline{\varepsilon}_{\mathbb{Q}}(B)$ and $G_0^\wedge(B)^+ = \{B\}$ if and only if $\overline{\varepsilon}_{\mathbb{Q}}(B^\wedge) = \overline{\varepsilon}_{\mathbb{Q}}(B) \cdot \overline{\mathrm{Aut}}(B^\wedge)$.*

Proof. Suppose $G_0^\wedge(B)^- = \{B\}$ and let $V \in \varepsilon_{\mathbb{Q}}(B^\wedge)$. Since V^{-1} classifies (Lemma 4.3) B, $V^{-1} = RU$ in $\mathrm{CAut}((B^\wedge)_{(0)})$ for some $R \in \mathrm{Aut}(B_{(0)})$ and some $U \in \mathrm{Aut}(B^\wedge)$. Note that $R^{-1} = UV \in \overline{\mathrm{Aut}}(B_{(0)}) \cap \overline{\varepsilon}_{\mathbb{Q}}(B^\wedge) = \overline{\varepsilon}_{\mathbb{Q}}(B)$ so $V = U^{-1}R^{-1} \in \overline{\mathrm{Aut}}(B^\wedge) \cdot \overline{\varepsilon}_{\mathbb{Q}}(B)$.

The converse is clear by Wilkerson's double coset formula for the genus set. $\square$

In the special case where $B = BT$ is the classifying space of a torus, $G_0^\wedge(BT) = \{BT\}$, and it follows from Lemma 4.4 that in $\varepsilon_{\mathbb{Q}}(BT^\wedge)$

$$\mathrm{Aut}(BT^\wedge) \cdot \varepsilon_{\mathbb{Q}}(BT) = \varepsilon_{\mathbb{Q}}(BT^\wedge) = \varepsilon_{\mathbb{Q}}(BT) \cdot \mathrm{Aut}(BT^\wedge)$$

as the homomorphisms

$$\varepsilon_{\mathbb{Q}}(BT) \xrightarrow{\text{completion}} \varepsilon_{\mathbb{Q}}(BT^\wedge) \xrightarrow{\text{rationalization}} \mathrm{CAut}((BT^\wedge)_{(0)})$$

are injective.

The condition in Lemma 4.4 is easily checked for simple or simply connected compact Lie groups.

Lemma 4.5. *Let G be a simple Lie group with Weyl group W_G. Then any rational self-equivalence $g \in \varepsilon_{\mathbb{Q}}(BG^\wedge)$ has the form $g = f \circ \psi^n$ for some homotopy self-equivalence $f \in \mathrm{Aut}(BG^\wedge)$ and some natural number n prime to $|W_G|$.*

Proof. As $BG^\wedge = \prod_p BG_p^\wedge$, $g = \prod_p g_p$ is a product of rational equivalences $g_p \in \varepsilon_{\mathbb{Q}}(BG_p^\wedge)$. According to [J-M-O1,J-M-O2], $g_p = f_p \circ \psi^{n_p}$ where $f_p \in \mathrm{Aut}(BG_p^\wedge)$ and n_p is a power of p such that $n_p = 1$ if p divides $|W_G|$.

Also $\left(\prod_p f_p^{-1}\right) \circ g = \prod_p \psi^{n_p}$ is a rational equivalence of $BG^\wedge$, so the induced map

$$\pi_4\left(\left(\prod_p \psi^{n_p}\right)_{(0)}\right) : \pi_4((BG^\wedge)_{(0)}) \longrightarrow \pi_4((BG^\wedge)_{(0)})$$

is an isomorphism. This isomorphism can be identified to multiplication in the module $\pi_4((BG^\wedge)_{(0)}) \cong \mathbb{Z}^\wedge \otimes \mathbb{Q}$ by the adic integer (n_p^2). In particular, the equation $(n_p^2)x = 1$ has a solution. Recall that the elements of the ring $\mathbb{Z}^\wedge \otimes \mathbb{Q}$ are sequences $x = (x_p)$ where x_p is a p-adic number which actually is a p-adic integer for almost all p. Hence $n_p = 1$ for $p \geq P$ for some prime P. Let $n = \prod_{p<P} n_p$. Then n is prime to $|W_G|$ and $(n_p) = (u_p)n$ where (u_p) is a unit in $\mathbb{Z}^\wedge$. Now

$$g = \prod_p f_p \circ \prod_p \psi^{n_p} = \left(\prod_p f \circ \psi^{u_p}\right) \circ \psi^n$$

where $f := \prod_p (f_p \circ \psi^{u_p}) \in \operatorname{Aut}(BG^\wedge)$. $\square$

Lemma 4.6. *Let $H = \prod_{i=1}^t H_i$ be a simply connected compact Lie group written as the product of its simple factors H_i. Then any rational self-equivalence $g \in \varepsilon_{\mathbb{Q}}(BH^\wedge)$ has the form $g = f \circ \prod_{i=1}^t \psi^{n_i}$ for some homotopy self-equivalence $f \in \operatorname{Aut}(BH^\wedge)$ and some natural numbers n_i prime to the Weyl group order of H_i.*

Proof. We may assume that g has the form $g = g_1 \times \cdots \times g_t$ where $g_i \in \varepsilon_{\mathbb{Q}}(BH_i^\wedge)$: The remarks preceding Example 2.2 imply that g after composition with a homotopy equivalence of the form $\prod_p \sigma_p \in \prod_p \operatorname{Aut}(BH_p^\wedge)$, where σ_p permutes identical factors in the product $BH_p^\wedge = \prod_{i=1}^t (BH_i)_p^\wedge$, becomes such a product map. By Lemma 4.5, $g_i = f_i \circ \psi^{n_i}$ with $f_i \in \operatorname{Aut}(BH_i^\wedge)$ and n_i a natural number prime to the Weyl group order of H_i. Now

$$g = \prod_{i=1}^t f_i \circ \prod_{i=1}^t \psi^{n_i}$$

where $f := \prod_{i=1}^t f_i \in \operatorname{Aut}(BH^\wedge)$. $\square$

Alternatively, any rational equivalence $g \in \varepsilon_{\mathbb{Q}}(BH^\wedge)$ has the form $g = (\prod \psi^{n_i}) \circ f'$ with $f' \in \operatorname{Aut}(BH^\wedge)$.

An obvious consequence of Lemma 4.3–4.6 is

Proposition 4.7. *$G_0^\wedge(BG)^+ = \{BG\} = G_0^\wedge(BG)^-$ whenever G is simple or a simply connected compact Lie group.*

We may now consider the special case, Corollary 0.9, of Theorem 0.8 where G is simple, cf. [Mø, Example 3.11].

Proof of Corollary 0.9. Assume that $f : BG \longrightarrow X$ is essential. As completion induces, cf. [J-M-O$_1$, Theorem 3.1], an injection $[BG, X] \rightarrowtail [BG^\wedge, X^\wedge]$, there exists a prime p such that $f_p^\wedge$ is essential and hence [J-M-O$_1$] a rational equivalence. Then also the original map f is a rational equivalence (Proposition 2.8) and the result follows from Proposition 4.7. $\square$

Finally we deal with Theorem 0.8. Using the notation introduced in this section,

$$\begin{aligned} G_0^\wedge(BG)^- &= \Big\{BF_RG[n_1, \dots, n_t] \mid (\det R, |K|) = 1 = \Big(\prod n_i, |W_H|\Big)\Big\} \\ &= G_0^\wedge(BG)^+ \end{aligned}$$

is an alternative formulation of that theorem.

Proof of Theorem 0.8. Let $X \in G_0^\wedge(BG)^-$. By Lemma 4.3, X is classified by $V^{-1} \in \mathrm{CAut}((BH^\wedge \times BT^\wedge)_{(0)})$ for some $V \in \varepsilon_{\mathbb{Q}}(BG^\wedge) \subset \varepsilon(BH^\wedge) \times \varepsilon_{\mathbb{Q}}(BT^\wedge)$. Any rational equivalence of $BH^\wedge$ has (Lemma 4.6) the form $f \circ \prod \psi^{n_i}$ for some $f \in \mathrm{Aut}(BH^\wedge)$ and some natural numbers n_i prime to the order of the Weyl group of the ith simple factor H_i of H and any rational equivalence of $BT^\wedge$ has the form $\beta \circ S$ for some $\beta \in \mathrm{Aut}(BT^\wedge)$ and some $S \in \varepsilon_{\mathbb{Q}}(BT)$. Thus

$$V = \Big(f \circ \prod \psi^{n_i},\ \beta \circ S\Big)$$

where the diagrams

$$\begin{tikzcd} (Z_H)_p \arrow[d, "\eta_p(f_p)\circ\prod\psi^{n_i}"'] & K_p \arrow[l, hook'] \arrow[r, "\varphi"] \arrow[d, "\cong"'] & \varphi(K_p) \arrow[r, hook] \arrow[d, "\cong"] & T_{p^\infty} \arrow[d, "\beta_p\circ S"] \\ (Z_H)_p & K_p \arrow[l, hook'] \arrow[r, "\varphi"] & \varphi(K_p) \arrow[r, hook] & T_{p^\infty} \end{tikzcd}$$

commute and have isomorphisms as indicated. These diagrams remain commutative if n_i is replaced by any natural number with the same residue modulo the order of the Weyl group of H_i. Note that such a replacement does not change the equivalence class of V^{-1}, which is also the equivalence class of $(f^{-1},\ \beta^{-1})$, in the double coset formula for $G_0^\wedge(BG)$. According to Lemma

3.4, we may therefore assume that the product $\prod n_i$ is prime to the order of the Weyl group W_H of H. The rearranged commutative diagrams

$$\begin{tikzcd}
\left(\prod \psi^{n_i}\right) K_p \arrow[rr, "\varphi\circ(\psi^{n_i}|Z_H)^{-1}"] \arrow[d, "\eta_p(f_p)"', "\cong"] & & \varphi(K_p) \arrow[r, hook] \arrow[d, "\cong"'] & T_{p^\infty} \arrow[d, "S"] \\
(K_p) \arrow[rr, "\beta_p^{-1}\circ\varphi"'] & & (\beta_p^{-1}\circ\varphi)(K_p) \arrow[r, hook] & T_{p^\infty}
\end{tikzcd}$$

can be pieced together ny taking the direct sum over all primes p for which $K_p \neq 1$ to

$$\begin{tikzcd}
\left(\prod \psi^{n_i}\right) K \arrow[rr, "\varphi\circ(\psi^{n_i}|Z_H)^{-1}"] \arrow[d, "\eta(f)"', "\cong"] & & \varphi(K) \arrow[r, hook] \arrow[d, "\cong"', "S|\varphi(K)"] & T \arrow[d, dashed, "R"] \\
K \arrow[rr, "\beta^{-1}\circ\varphi"'] & & (\beta^{-1}\circ\varphi)(K) \arrow[r, hook] & T
\end{tikzcd}$$

where $\eta(f) = \bigoplus \eta_p(f_p)$, $\beta^{-1} \circ \varphi = \bigoplus \beta_p^{-1} \circ \varphi|K_p$ and $S|\varphi(K) = \bigoplus S|\varphi(K_p)$. Lemma 3.2 shows that this diagram can be completed by some $R \in \mathrm{Epi}(T)$ with $(\det R, |W_H|) = 1$. Then

$$\beta_p R\varphi = \varphi \circ \left(\eta_p(f_p) \circ \prod \psi^{n_i}\right) |K_p$$

so $(f_p \circ \prod \psi^{n_i},\, \beta_p R) \in \varepsilon_{\mathbb{Q}}(BG_p^\wedge)$. Define

$$U_p = \begin{cases} (f_p \circ \prod \psi^{n_i},\, \beta_p R) & \text{if } p \mid |K| \\ (f_p,\, \beta_p) & \text{if } p \nmid |K| \end{cases}$$

and note that $U := \prod U_p$ is a homotopy equivalence of $BG^\wedge$. The space $X \in G_0^\wedge(BG)^-$ is therefore also represented by

$$\left(\prod \psi^{n_i},\, S\right) V^{-1} U = \begin{cases} (\prod \psi^{n_i},\, R) & \text{if } p \mid |K| \\ (1,\, 1) & \text{if } p \nmid |K| \end{cases}$$

showing that $X = BF_R G[n_1, \ldots, n_t]$.

If $X \in G_0^\wedge(BG)^+$ then X is classified by some $V \in \varepsilon_{\mathbb{Q}}(BG^\wedge) \subset \varepsilon_{\mathbb{Q}}(BH^\wedge) \times \varepsilon_{\mathbb{Q}}(BT^\wedge)$. Write V on the form

$$V = \left(\prod \psi^{n_i} \circ f,\; S \circ \beta\right)$$

with $f \in \mathrm{Aut}(BH^\wedge)$, $\beta \in \mathrm{Aut}(BT^\wedge)$, $S \in \varepsilon_{\mathbb{Q}}(BT)$, and proceed as above. $\square$

In other words, $G_0^\wedge(BG)^- = B(\langle G, G\rangle) = G_0^\wedge(BG)^+$ is a finite set consisting of those Lie group classifying spaces BL for which $BG \geq BL \geq BG$. For example,

$$\begin{aligned}
G_0^\wedge(BSO(n))^+ &= \{BSO(n)\}\\
G_0^\wedge(BSU(n)\times BSU(n))^+ &= \{BSU(n)\times BSU(n)\}\\
G_0^\wedge(B(SU(5)\times SU(5)/\Delta_1))^+ &= \{B(SU(5)\times SU(5)/\Delta_1),\\
&\quad\ B(SU(5)\times SU(5)/\Delta_2)\}\\
G_0^\wedge(BU(n))^+ &= \{B(U(n)/\Gamma_k) \mid (k,n)=1\,\}
\end{aligned}$$

in the notation of Example 2.5 and Example 3.6. (The two spaces in the third of the above examples have non-isomorphic groups of homotopy self-equivalences, see also [Mø, Example 3.7].)

The full genus set $G_0^\wedge(BG)$ contains in general (Example 4.2) other Lie group classifying spaces than those in $G_0^\wedge(BG)^\pm$; namely (Proposition 0.6) all homotopy types BL for which $L^{(p)} \in \langle G^{(p)}, G^{(p)}\rangle$ for all primes p. Alternatively, $X \in G_0^\wedge(BG)$ is [N-S, N_2] the classifying space of a compact connected Lie group if X has a maximal torus.

References

[A–M] J.F. Adams, Z. Mahmud, Maps between classifying spaces, Invent. Math. **35** (1976), 1–41

[A-W] J.F. Adams, Z. Wojtkowiak, Maps between p-completed classifying spaces, Proc. Royal Soc. Edinburgh **112a** (1989), 231–235

[Bass] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. **95** (1960), 466–488

[Ba] P.F. Baum, Local isomorphisms of compact connected Lie groups, Pacific J. Math **22** (1967), 197–204

[B-K] A.K. Bousfield, D.M. Kan, Homotopy limits, completions and localizations. Lectures Notes in Math. 304, Springer-Verlag 1987

[Bo] N. Bourbaki, Groupes et algebrés de Lie, (Chapitres 4–6), Hermann: Paris, 1968

[B-tD] T. Bröcker, T. tom Dieck, Representations of compact Lie groups, Graduate Texts in Mathematics 98, Springer: New York, Berlin, Heidelberg, Tokyo, 1985

[D-Z] W. Dwyer, A. Zabrodsky, Maps between classifying spaces, pp. 106–119. In: Algebraic Topology, Barcelona 1986 (J. Aguadé, R. Kane: eds.), Lecture Notes in Mathematics 1298, Springer: Berlin - Heidelberg, 1987

[F] L. Fuchs, Infinite Abelian Groups, Volume I, Academic Press: New York, London, 1970

[G–M] H.H. Glover, G. Mislin, On the genus of generalized flag manifolds, Enseign. Math. **27** (1981), 211–219

[H–R] P. Hilton, J. Roitberg, On principal S^3-bundles over spheres, Ann. of Math. **90**(2) (1969), 91–107

[I] K. Ishiguro, Classifying spaces of compact simple Lie groups and p-tori, pp. 210–226. In: Algebraic Topology. Homotopy and Group Cohomology. Proceedings Barcelona 1990 (J. Aguadé, M. Castellet, F.R. Cohen: eds.), Lecture Notes in Mathematics 1509, Springer: Berlin - Heidelberg, 1992

[I-M-N] K. Ishiguro, J.M. Møller, D. Notbohm, Rational self-equivalences of spaces in the genus of a product of quaternionic projective spaces, J. Math. Soc. Japan **51** (1999), 45–61

[J-M-O$_1$] S. Jackowski, J.McClure, R. Oliver, Homotopy classification of self-maps of BG via G-actions. Part I, Ann. of Math. **135**(2) (1992), 183–226; Part II, Ann. of Math. **135**(2) (1992), 227–270

[J-M-O$_2$] S. Jackowski, J.McClure, R. Oliver, Self-homotopy equivalences of classifying spaces of compact connected Lie groups, Fund. Math. **147** (1995), 99–126

[McG] C.A. McGibbon, Which group structures on S^3 have a maximal torus? In: Geometric Applications of Homotopy Theory I, Proceedings, Evanston 1977 (M.G. Barratt, M.E. Mahowald: eds.), Lecture Notes in Mathematics 657, Springer: Berlin, Heidelberg, New York, 1978, pp. 353–360

[Ma] H. Matsumura, Commutative Algebra, Second Edition, The Benjamin/Cummings Publishing Company: London, 1980

[Mi] H.R. Miller, The Sullivan conjecture on maps from classifying spaces, Ann. of Math. **120**(2) (1984), 39–87

[Mø] J.M. Møller, The normalizer of the Weyl group, Math. Ann. **294** (1991), 59–80

[N$_1$] D. Notbohm, Maps between classifying spaces and applications, Mathematica Gottingensis **20**, 1991

[N$_2$] D. Notbohm, Fake Lie groups with maximal tori. IV, Math. Ann. **294** (1992), 109–116

[N$_3$] D. Notbohm, On the functor 'Classifying Space' for compact Lie groups. Preprint

[N-S] D. Notbohm, L. Smith, Fake Lie groups and maximal tori III, Math. Ann. **290** (1991), 629–642

[O] A. Osse, Sur le classifiant d'un groupe de Lie compact connexe, C.R. Acad. Sci. Paris, Série I **315** (1992), 833–838

[P] S. Papadima, Rigidity Properties of Compact Lie Groups Modulo Maximal Tori, Math. Ann. **275** (1986), 637–652

[R] D. Rector, Loop structures on the homotopy type of S^3, pp. 99–105. In: Symposium on Algebraic Topology, Seattle 1971 (P. Hilton: ed.), Lecture Notes in Mathematics 249, Springer: Berlin, Heidelberg, New York, 1971

[R–S] D. Rector, J. Stasheff, Lie groups from a homotopy point of view, pp.121–131. In: Localization in Group Theory and Homotopy Theory and Related Topics, Seattle 1974 (P. Hilton: ed.), Lecture Notes in Mathematics 418, Springer: Berlin, Heidelberg, New York, 1974

[S] H. Scheerer, Homotopieäquivalente kompakte Liesche Gruppen, Topology **7** (1968), 227–232

[W] C.W. Wilkerson, Applications of minimal simplicial groups, Topology **15** (1976), 111–130

[Z$_1$] A. Zabrodsky, Hopf Spaces, North–Holland Mathematics Studies 22, North–Holland: Amsterdam - New York - Oxford, 1976

[Z$_2$] A. Zabrodsky, Hopf Spaces, On phantom maps and a theorem of H. Miller, Israel J. Math. **58** (1987), 129–143