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The normalizer of the Weyl group

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0 Introduction

Consider a compact connected Lie group G with maximal torus $T \subset G$. The Weyl group $W_G = N(T)/T$ of G is, simply by applying the cohomology functor to the Weyl group operation on T, faithfully represented in $H^1(T; R)$ where R is some suitable commutative ring. The title of this paper refers to the normalizer, denoted $N_R(W_G)$, of W_G considered as a subgroup of the R-module automorphism group of $H^1(T; R)$.

The normalizer $N_{\mathbb{Z}_p^\wedge}(W_G)$ obtained by taking coefficients in the p-adics is of particular interest here because of its close relation to the group of homotopy classes of self homotopy equivalences of the p-completed classifying space $BG_p^\wedge$. The link between these two groups is provided by the Adams-Wojtkowiak theorem [A-W] implying that any self homotopy equivalence of $BG_p^\wedge$ restricts to a self homotopy equivalence of $BT_p^\wedge$ such that the induced map on $H^2(BT_p^\wedge; \mathbb{Z}_p^\wedge) = H^1(T; \mathbb{Z}_p^\wedge)$ normalizes the Weyl group. (Since the restriction to $BT_p^\wedge$ is unique up to left W_G-action, one actually gets an anti-homomorphism, studied in detail in [W2] and [JMO3], from the group of self homotopy equivalences of the p-completed classifying space into the quotient group $N_{\mathbb{Z}_p^\wedge}(W_G)/W_G$.) The main idea of this paper is to apply information about the algebraic structure of the normalizer of the Weyl group to an investigation of self homotopy equivalences and genus sets of Lie group classifying spaces.

The normalizer is described algebraically in Theorem 1.3 as an extension of two groups defined (almost) explicitly in terms of the group of units of R and the group of automorphisms of the Coxeter graph of G. Two consequences of Theorem 1.3 are mentioned in Corollary 1.6 and Corollary 1.7. The first of these is the main input for the treatment of self homotopy equivalences of BG and the second one will be needed in the discussion of genus sets. Corollary 1.6 shows that the normalizer behaves very rigidly under formation of product Lie groups. Indeed, the normalizer associated to a simply connected compact Lie group is generated by the normalizers associated to each of the simple factors together with permutations of identical factors (where $\mathrm{Spin}(2n+1)$ and $\mathrm{Sp}(n)$, $n > 2$, are considered identical if 2 is invertible in R). Corollary 1.7 concerns the action, for a simply connected simple G, of $N_R(W_G)$ on the cohomology algebra $H^*(BG; k)$ with coefficients in the field of fractions, k, of R. This action is obtained by identifying the cohomology of BG with the ring of invariants $H^*(BT; k)^{W_G}$. It is shown that $N_R(W_G)$ acts on $H^4(BG; k) = k$ as multiplication by a square unit from R (except that in some special cases multiplication by 2 or 3 may also occur).

The next section, concerned with the cardinality of genus sets of Lie group classifying spaces, is motivated by Rector's demonstration [R] of the existence of uncountably many distinct homotopy types of the same genus as $BSU(2)$. As shown in Theorem 2.3, the same extreme non-rigidity conclusion holds if $SU(2)$ is replaced by any other compact connected Lie group which is not a torus. Genus sets of localized classifying spaces are also considered. The proofs of these results use the Arithmetic Square.

It is not true in general, of course, that any self homotopy equivalence of a Cartesian product of spaces is homotopic to a product of self homotopy equivalences of each of the spaces. Nevertheless, the above is true for products of classifying spaces of simply connected compact Lie groups whose Dynkin diagrams have no identical components. This is a consequence of Corollary 3.2 expressing the group of self-homotopy equivalences of BG, for a simply connected G, explicitly by means of the self homotopy equivalences of the simple factors in G. Actually, a little more general statement about rational self equivalences of BG can be found in Theorem 3.1. The paper closes with a short discussion of rational equivalences between members of the genus of BG.

1 The normalizer of the Weyl group

Let G be a simply connected compact Lie group with maximal torus $T \subsetneq G$ and Weyl group $W = W_G = N(T)/T$.

The Weyl group acts by conjugation on T and hence also on $\pi_1(T, 1)$ and $\mathrm{Hom}(\pi_1(T), R) = H^1(T; R)$ for any commutative ring R. The object of this section is to describe the normalizer $N_R(W_G) \subset \mathrm{Aut}_R H^1(T; R)$ of W_G. The ring R will always be assumed to be an integral domain of characteristic zero; typically $R = \mathbb{Z}_{(p)}$, $\mathbb{Q}$, $\mathbb{Z}_p^\wedge$ or $\mathbb{Q}_p$, the p-local integers, the rationals, the p-adic integers or the p-adic numbers.

In terms of Lie theory, the action of W on $H^1(T; R)$ may be described as follows; see [B-tD, H].

Let LT be the Lie algebra of T. The integral lattice I is defined to be the kernel of the universal covering map $\exp: LT \to T$; thus $I = \pi_1(T)$ and $H^1(T; R) = \mathrm{Hom}_{\mathbb{Z}}(I, R)$. The real roots of G are certain $\mathbb{R}$-linear forms on LT; the root system $\Phi \subset LT^*$ is the set of all roots. Actually, the roots are integral, i.e.

$$\Phi \subset \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{R}) = LT^* .$$

Each root $\alpha \in \Phi$ has an inverse root $\alpha^* \in I$; the inverse root system

$$\Phi^* \subset I \subset I \otimes_{\mathbb{Z}} \mathbb{R} = LT$$

is the set of all inverse roots. Define

$$\sigma_\alpha \colon \operatorname{Hom}_{\mathbb{Z}}(I, R) \to \operatorname{Hom}_{\mathbb{Z}}(I, R)$$

to be the R-automorphism given by

$$\sigma_\alpha(\beta) = \beta - \beta(\alpha^*)\alpha, \quad \beta \in \operatorname{Hom}_{\mathbb{Z}}(I, R) .$$

Here we view α as an element of $\operatorname{Hom}_{\mathbb{Z}}(I, \mathbb{Z}) \subset \operatorname{Hom}_{\mathbb{Z}}(I, R)$. The map σ_α is an idempotent, W is isomorphic to the subgroup of $\operatorname{Aut}_R \operatorname{Hom}_{\mathbb{Z}}(I, R)$ generated by $\{\sigma_\alpha \mid \alpha \in \Phi\}$, and the isomorphism $\operatorname{Hom}(I, R) \cong H^1(T; R)$ is an isomorphism of W-modules.

The integers $n_{\alpha\beta} = \beta(\alpha^*)$, $\alpha, \beta \in \Phi$, are the Cartan numbers of G.

Choose a set of positive roots $\Phi^+ \subset \Phi$ and a set of simple roots, i.e. a basis for LT^*, $\Delta \subset \Phi^+$. The Dynkin diagram of G is a graph with Δ as its set of vertices and $n_{\alpha\beta} n_{\beta\alpha}$ edges connecting the simple roots α and β. If two simple roots are connected by more than one edge, the Dynkin diagram has an arrow pointing to the shorter (w.r.t. a W-invariant metric on LT^*) root.

We define the R-Dynkin diagram of G to be diagram obtained from the Dynkin diagram by removing all arrows on r-fold edges whenever r is invertible in R, $r = 2, 3$. In particular, the $\mathbb{Z}$-Dynkin diagram is the Dynkin diagram and the $\mathbb{Q}$-Dynkin diagram is the Coxeter graph. Let $\Gamma_R(G)$ denote the group of all automorphisms of the R-Dynkin diagram. Then $\Gamma_{\mathbb{Z}}(G) \subset \Gamma_R(G) \subset \Gamma_{\mathbb{Q}}(G)$. For an irreducible root system, $\Gamma_{\mathbb{Q}}(G)$ has order $\leqq 2$ except that $\Gamma_{\mathbb{Z}}(D_4) = \Gamma_{\mathbb{Q}}(D_4) = \Sigma_3$ has order 6. The root system B_2 has Γ_R equal to $\mathbb{Z}/2$ if $2 \in R^*$ and to $\{1\}$ if $2 \notin R^*$.

Viewing $W \subset \operatorname{Aut}_R H^1(T; R)$ as a group of R-automorphisms of $H^1(T; R)$ we consider in the next two theorems the centralizer

$$Z_R(W_G) = \{A \in \operatorname{Aut}_R H^1(T; R) \mid \forall w \in W \colon Aw = wA\}$$

and the normalizer

$$N_R(W_G) = \{A \in \operatorname{Aut}_R H^1(T; R) \mid AW = WA\}$$

of W_G.

A version of the first result can also be found on p. 651 of [N-S] and for $R = \mathbb{Q}$ or $\mathbb{R}$ Papadima [P] proved the existence of a short exact sequence similar to the one in Theorem 1.3.

Theorem 1.1. *The centralizer $Z_R(W_G)$ is isomorphic to the abelian group $R^* \times \cdots \times R^*$ where the number of factors equals the number of connected components of the Dynkin diagram of G.*

The description of the normalizer is less elegant. Let $\operatorname{Perm}(\Phi^+)$ be the group of permutations of the positive roots. Consider the subgroup $P_R(G)$ of those $\varphi \in \operatorname{Perm}(\Phi^+)$ for which there exist a transformation $A \in N_R(W_G)$ and a function $\lambda \colon \Phi^+ \to R^*$ such that $A(\alpha) = \lambda(\alpha)\varphi(\alpha)$ for all positive roots $\alpha \in \Phi^+$. There is a homomorphism

$$W \to P_R(G) \colon w \to w^+$$

where

$$w^+(\alpha) = \begin{cases} w\alpha & \text{if } w\alpha \in \Phi^+ \\ -w\alpha & \text{if } w\alpha \notin \Phi^+ \end{cases}$$

for any positive root α. Let W^+ denote the image of this homomorphism.

Proposition 1.2. (a) *The isotropy subgroup*

$$P_R(G)_\Delta := \{\varphi \in P_R(G) \mid \varphi\Delta = \Delta\}$$

is isomorphic to $\Gamma_R(G)$.

(b) $P_R(G) = (W^+)\cdot\Gamma_R(G)$.

W^+ is a normal subgroup of $P_R(G)$ which may have a non-trivial intersection with $\Gamma_R(G)$; thus $P_R(G)$ is in general not equal to (but rather a quotient of) the semi-direct product of these two subgroups.

Theorem 1.3. *There is a short exact sequence*

$$0 \to Z_R(W_G) \to N_R(W_G) \to P_R(G) \to 1$$

of groups.

Proof of Theorem 1.1. Let $A \in Z_R(W_G)$. For any root α, $-A\alpha = A\sigma_\alpha\alpha = \sigma_\alpha A\alpha$, so $A\alpha = \lambda(\alpha)\alpha$ for some $\lambda(\alpha) \in R^*$. The statement

$$\forall\, \alpha, \beta \in \Phi\colon A\sigma_\alpha(\beta) = \sigma_\alpha A(\beta)$$

is then equivalent to

$$\forall\, \alpha, \beta \in \Phi\colon \lambda(\alpha) n_{\alpha\beta} = \lambda(\beta) n_{\alpha\beta}\ .$$

In particular, if $\alpha, \beta \in \Delta$ are simple roots connected by an edge in the Coxeter graph, $n_{\alpha\beta} \neq 0$ and $\lambda(\alpha) = \lambda(\beta)$. Thus $\lambda\colon \Delta \to R^*$ is constant on each connected component of the Coxeter graph.

Conversely, suppose that $\lambda\colon \Delta \to R^*$ is constant on each component of the Coxeter graph. Let A be the R-automorphism of $H^1(T; R)$ that satisfies $A\alpha = \lambda(\alpha)\alpha$, $\alpha \in \Delta$. Then $A \in Z_R(W_G)$. □

Proof of Theorem 1.3. Let $k \supset R$ be the field of fractions of R. Equip LT with a W-invariant euclidean metric $\langle\,,\rangle$ such that $\langle I, I\rangle \subset \mathbb{Z}$; the Killing form multiplied by the factor -2 will do [B-tD, p. 214]. Extension of scalars gives us a non-degenerate W-invariant k-valued bilinearform on $I \otimes k$ and thereby also on the dual vector space $\mathrm{Hom}_k(I \otimes k, k) = \mathrm{Hom}_{\mathbb{Z}}(I, k)$. The symmetry σ_α, $\alpha \in \Phi$, then becomes reflection w.r.t. $\alpha^\perp$, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha$$

for all $x \in \mathrm{Hom}_{\mathbb{Z}}(I, k)$.

Suppose $A \in \mathrm{Aut}_R\mathrm{Hom}(I, R) \subset \mathrm{Aut}_k\mathrm{Hom}(I, k)$ normalizes W. In particular, $w := A\sigma_\alpha A^{-1} \in W$ for any given positive root $\alpha \in \Phi^+$. Note that w sends $A\alpha$ to $-A\alpha$ and fixes pointwise the hyperplane $A(\alpha^\perp)$. I claim that $w = \sigma_\beta$ for some (positive) root β.

First note that we must have $A(\alpha^\perp) = \beta^\perp$ for some (positive) root β. If not, the difference

$$A(\alpha^\perp) - \bigcup_{\beta \in \Phi} \beta^\perp \subset \mathrm{Hom}(I, k)$$

would be non-empty. Choose an x in it. Since $x \in A(\alpha^\perp)$, $wx = x$. However, W acts freely [B-tD, Theorem V.4.1] on $\mathrm{Hom}(I, \mathbb{Q}) - \cup\, \beta^\perp$ and thus, by extension of scalars, also freely on $\mathrm{Hom}(I, k) - \cup\, \beta^\perp$. We arrive at the contradiction $w = 1$.

The orthogonal transformation w carries β into some vector on the line $k\beta$. Write $A\alpha = \lambda\beta + u$ for some $\lambda \in k$ and $u \in \beta^{\perp}$. Since

$$- \lambda\beta - u = - A\alpha = wA\alpha = \lambda w\beta + u$$

the uniqueness of such decompositions shows that $u = 0$. Hence $A\alpha = \lambda\beta$ and, because $w\beta = - \beta$ and w fixes $\beta^{\perp}$ pointwise, $\sigma_\beta = w = A\sigma_\alpha A^{-1}$.

We conclude that for any $A \in N_R(W_G)$ and any positive root $\alpha \in \Phi^+$ there is a unique positive root $\varphi(\alpha) \in \Phi^+$ and scalar $\lambda(\alpha) \in R^*$ such that $A\alpha = \lambda(\alpha)\varphi(\alpha)$ and $A\sigma_\alpha = \sigma_{\varphi(\alpha)}A$.

The map $N_R(W_G) \to P_R(G)$, taking A to the associated permutation $\varphi \in \mathrm{Perm}(\Phi^+)$, is easily seen to be a homomorphism. By the very definition of $P_R(G)$, it's surjective. The kernel, consisting of those $A \in N_R(W_G)$ for which $A\sigma_\alpha = \sigma_\alpha A$, $\alpha \in \Phi^+$, is the centralizer $Z_R(W_G)$. □

Proof of Proposition 1.2. (a) There is an obvious homomorphism

$$P_R(G)_\Delta \to \Gamma_R(G) \colon \varphi \to \varphi \mid \Delta \,.$$

Let's first verify that indeed $\varphi \mid \Delta \in \Gamma_R(G)$, i.e. that the permutation $\varphi \mid \Delta$ of the vertices of the Dynkin diagram is an automorphism of the R-Dynkin diagram. Each $\varphi \in P_R(G)_\Delta$ comes from some $A \in N_R(W_G)$ with $A\alpha = \lambda(\alpha)\varphi(\alpha)$ and $A\sigma_\alpha = \sigma_{\varphi(\alpha)}A$. The statement

$$\forall \alpha, \beta \in \Delta \colon A\sigma_\alpha(\beta) = \sigma_{\varphi(\alpha)}A(\beta)$$

is equivalent to

$$\forall \alpha, \beta \in \Delta \colon \lambda(\alpha)n_{\alpha\beta} = \lambda(\beta)n_{\varphi(\alpha)\varphi(\beta)} \,.$$

Consequently,

$$\begin{aligned} \lambda(\alpha)\lambda(\beta)n_{\alpha\beta}n_{\beta\alpha} &= (\lambda(\alpha)n_{\alpha\beta})(\lambda(\beta)n_{\beta\alpha}) \\ &= (\lambda(\beta)n_{\varphi(\alpha)\varphi(\beta)})(\lambda(\alpha)n_{\varphi(\beta)\varphi(\alpha)}) \\ &= \lambda(\alpha)\lambda(\beta)n_{\varphi(\alpha)\varphi(\beta)}n_{\varphi(\beta)\varphi(\alpha)} \end{aligned}$$

so $n_{\alpha\beta}n_{\beta\alpha} = n_{\varphi(\alpha)\varphi(\beta)}n_{\varphi(\beta)\varphi(\alpha)}$ since R is an integral domain. This means that $\varphi \mid \Delta \in \Gamma_{\mathbb{Q}}(G)$ is an automorphism of the Coxeter graph. Suppose that the graph contains an r-fold edge, $r = 2, 3$, between two simple roots α and β such that $|\alpha| > |\beta|$ but $|\varphi(\alpha)| < |\varphi(\beta)|$. This means that φ reverses the direction of an arrow:

$$\alpha \circ\!\!\Rightarrow\!\!\circ\, \beta \qquad \varphi(\alpha) \circ\!\!\Leftarrow\!\!\circ\, \varphi(\beta) \qquad (r = 2)\,.$$

In this situation [B-tD, Proposition V.5.9], $n_{\alpha\beta} = - 1$ and $n_{\varphi(\alpha)\varphi(\beta)} = - r$. From the equation $\lambda(\alpha) = - \lambda(\alpha)n_{\alpha\beta} = - \lambda(\beta)n_{\varphi(\alpha)\varphi(\beta)} = r\lambda(\beta)$ we see that r must be a unit in R. This means that $\varphi \mid \Delta$ is an automorphism of the R-Dynkin diagram.

Let $A \in N_R(W_G)$ with $A\alpha = \lambda(\alpha)\varphi(\alpha)$, $\alpha \in \Phi^+$. Suppose $\varphi(\alpha) = \alpha$ for all simple roots, i.e. $A\alpha = \lambda(\alpha)\alpha$ for $\alpha \in \Delta$. Since Δ is a basis for $\mathrm{Hom}(I, k)$, it follows that $A\alpha = \lambda(\alpha)\alpha$ for all positive roots $\alpha \in \Phi^+$, i.e. that $\varphi \in \mathrm{Perm}(\Phi^+)$ is the identity transformation. This shows that the homomorphism $\varphi \to \varphi \mid \Delta$ is injective.

In order to prove surjectivity we note that the Dynkin diagram of G consists of finitely many irreducible components. For a simple Lie group $\Gamma_R = \Gamma_{\mathbb{Z}}$ except when the group is B_2, F_4 or G_2 and 2 or 3 is invertible in R. For a general simply connected compact Lie group, $\Gamma_R(G)$ decomposes into the (semi-direct) product of the automorphism groups corresponding to the simple factors of G with the group of permutations of identical components of the R-Dynkin diagram. Furthermore, when $2 \in R^*$ any pair of irreducible components of type B_n and C_n may be interchanged. Hence it suffices to prove surjectivity in the following three cases

(i) $R = \mathbb{Z}$.

(ii) $2 \in R^*$ and $G = \mathrm{Sp}(2)$, F_4, $\mathrm{Spin}(2n+1) \times \mathrm{Sp}(n)$, $n > 2$.

(iii) $3 \in R^*$ and $G = G_2$.

In case (i), any automorphism of the Dynkin diagram is known to be induced by an (outer) Lie group automorphism $f: (G, T) \to (G, T)$; i.e. any permutation in $\Gamma_{\mathbb{Z}}(G)$ comes from an $H^1(f; \mathbb{Z}) \in N_{\mathbb{Z}}(W_G)$. In cases (ii) and (iii), Adams and Mahmud demonstrate that the (exotic) permutations of the roots extend to automorphisms of $\mathrm{Hom}(I, R) = H^1(T; R)$. These automorphisms are admissible, i.e. they normalize W. This finishes the proof of (a).

(b) Let $\varphi \in P_R(G)$ come from $A \in N_R(W_G)$: $A\alpha = \lambda(\alpha)\varphi(\alpha)$ for $\alpha \in \Phi^+$. Choose $w \in W$ such that $w^{-1}\Delta = \varphi\Delta$. The product $wA \in N_R(W_G)$ and, for $\alpha \in \Delta$, $wA\alpha = \lambda(\alpha)w\varphi(\alpha)$ where $w\varphi(\alpha) \in \Delta \subset \Phi^+$ meaning that $w^+\varphi\alpha = w\varphi\alpha$. Hence $w^+\varphi \in \Gamma_R(G)_\Delta$. $\square$

We next list some consequences of Theorem 1.3.

Let G be a simply connected compact simple Lie group and $G^n = G \times \cdots \times G$ the n-fold Cartesian product. Choose $T^n = T \times \cdots \times T$ as a maximal torus for G^n where T is a maximal torus for G. Since $H^1(T^n) = H^1(T) \times \cdots \times H^1(T)$ and $W_{G^n} = W_G \times \cdots \times W_G$, there is an obvious homomorphism

$$N_R(W_G) \wr \Sigma_n = N_R(W_G)^n \rtimes \Sigma_n \to N_R(W_{G^n}) \,.$$

The semi-direct, or wreath, product is w.r.t. to the usual right action

$$(A_1, \ldots, A_n) \cdot \sigma = (A_{\sigma^{-1}(1)}, \ldots, A_{\sigma^{-1}(n)})$$

of the symmetric group Σ_n on $N_R(W_G)^n$.

Corollary 1.4. $N_R(W_{G^n}) \cong N_R(W_G) \wr \Sigma_n$ *for any simply connected compact simple Lie group G and any $n \geqq 1$.*

Proof. Since $\Phi^+(G) \cup \cdots \cup \Phi^+(G)$ may serve as the set of positive roots for G^n we also have a homomorphism

$$P_R(G)^n \wr \Sigma_n \to P_R(G^n)$$

on the quotients. This map is clearly a monomorphism and it's also an epimorphism because

$$\begin{aligned} P_R(G^n) &= (W^+)^n \cdot \Gamma_R(G^n) \\ &= (W^+ \times \cdots \times W^+) \cdot (\Gamma_R(G) \times \cdots \times \Gamma_R(G)) \cdot \Sigma_n \\ &= (W^+ \cdot \Gamma_R(G) \times \cdots \times W^+ \cdot \Gamma_R(G)) \cdot \Sigma_n \\ &= (P_R(G) \times \cdots \times P_R(G)) \cdot \Sigma_n \end{aligned}$$

where we've used $\Gamma_R(G^n) \cong \Gamma_R(G) \wr \Sigma_n$. These homomorphisms extend to a map

$$\begin{tikzcd} 0 \arrow[r] & Z_R(W_G)^n \arrow[r] \arrow[d, "\cong"'] & N_R(W_G) \wr \Sigma_n \arrow[r] \arrow[d] & P_R(G) \wr \Sigma_n \arrow[r] \arrow[d, "\cong"'] & 1 \\ 0 \arrow[r] & Z_R(W_{G^n}) \arrow[r] & N_R(W_{G^n}) \arrow[r] & P_R(G^n) \arrow[r] & 1 \end{tikzcd}$$

of short exact sequences. The middle vertical arrow is an isomorphism since the two outer ones are. □

A similar argument shows

Corollary 1.5. *Let G and H be two simply connected compact Lie groups such that no component of the R-Dynkin of one of the groups is isomorphic to any component of the R-Dynkin diagram of the other group. Then $N_R(W_G) \times N_R(W_H) \cong N_R(W_G \times W_H)$.*

When $G = \mathrm{Spin}(2n+1)$ and $H = \mathrm{Sp}(n)$, $n > 2$, and $2 \in R^*$, the R-Dynkin diagrams, B_n and C_n, of G and H are isomorphic and Corollary 1.5 doesn't apply. But since [A-M, Example 2.7] $H^1(T(\mathrm{Spin}(2n+1)); R)$ and $H^1(T(\mathrm{Sp}(n)); R)$ are isomorphic as $W_{\mathrm{Spin}(2n+1)} = W_{\mathrm{Sp}(n)}$-modules we get immediately

$$N_R(W_{\mathrm{Spin}(2n+1) \times \mathrm{Sp}(n)}) \cong N_R(W_{\mathrm{Sp}(n) \times \mathrm{Sp}(n)}) .$$

For a general simply connected compact Lie group G, write G as a product

$$G = \prod_{i=1}^{s} (B_i^{b_i} \times C_i^{c_i}) \times \prod_{i=1}^{t} H_i^{h_i}$$

of distinct simple Lie groups such that (B_i, C_i) has type (B_{n_i}, C_{n_i}) for some $n_i > 2$, $1 \leqq i \leqq s$, and the groups H_i, $1 \leqq i \leqq t$, have Dynkin diagrams that are not of type B_n or C_n, $n > 2$. Combining Corollary 1.4 and 1.5 we get

Corollary 1.6. *$N_R(W_G)$ is isomorphic to*

$$\bigoplus_{i=1}^{s} N_R(W_{B_i}) \wr \Sigma_{b_i} \oplus N_R(W_{C_i}) \wr \Sigma_{c_i} \oplus \bigoplus_{i=1}^{t} N_R(W_{H_i}) \wr \Sigma_{h_i}$$

if $2 \notin R^*$ and to

$$\bigoplus_{i=1}^{s} N_R(W_{B_i}) \wr \Sigma_{b_i + c_i} \oplus \bigoplus_{i=1}^{t} N_R(W_{H_i}) \wr \Sigma_{h_i}$$

if $2 \in R^*$.

Now recall Borel's theorem $H^*(BG; k) \cong H^*(BT; k)^{W_G}$. By restricting the action of $N_R(W_G)$ on the polynomial algebra $H^*(BT; k)$ to the ring of W_G-invariants we obtain a homomorphism

$$N_R(W_G) \to \mathrm{Aut}_k H^*(BG; k)$$

whose image we list below for the simple Lie groups. We write $A = \{\psi^r \mid r \in R^*\}$ for the (central) subgroup of unstable Adams operations and $\langle \varepsilon \rangle$ for the cyclic group generated by an (exotic) automorphism $\varepsilon \in \mathrm{Aut}_k H^*(BG; k)$.

Also, let $H^4(N_R(W_G); k)$ be the image of the homomorphism

$$N_R(W_G) \to \mathrm{Aut}_k H^*(BG; k) \to \mathrm{Aut}_k H^4(BG; k) = k - \{0\} .$$

Thus $H^4(N_R(W_G); k)$ is the group of degrees in $H^4(BG; k)$ of maps induced by the elements of $N_R(W_G)$. Write $S(R^*)$ for the multiplicative group of square units in R. The last statement of the following corollary is not surprising in view of the fact that any outer automorphism induces the identity on $H^4(BG; \mathbb{Q})$.

Corollary 1.7. *For the simply connected compact simple Lie groups G the image of $N_R(W_G)$ in* $\operatorname{Aut} H^*(BG; k)$ *is*

(i) *a group containing A with index $\leqq 6$ if $G = \mathrm{Spin}(8)$.*
(ii) *$A \cdot \langle\varepsilon\rangle$, where $\varepsilon^2 = \psi^1$, if $G = \mathrm{Spin}(4n)$, $n > 2$.*
(iii) *$A \cdot \langle\varepsilon\rangle$, where $\varepsilon^2 = \psi^2$, if $G = \mathrm{Sp}(2)$ or F_4 and $2 \in R^*$.*
(iv) *$A \cdot \langle\varepsilon\rangle$, where $\varepsilon^2 = \psi^3$, If $G = G_2$ and $3 \in R^*$.*
(v) *A in all other cases.*

Furthermore

$$H^4(N_R(W_G); k) = \begin{cases} S(R^*) & \textit{in case (i), (ii) and (v)} \\ S(R^*) \cdot \langle 2 \rangle & \textit{in case (iii)} \\ S(R^*) \cdot \langle 3 \rangle & \textit{in case (iv)}. \end{cases}$$

Proof. I. $\Gamma_R(G) \cong \{1\}$: In this case (A_1; B_2 and $2 \notin R^*$; B_l, $l > 2$; C_l, $l > 2$; E_7; E_8; F_4 and $2 \notin R^*$; G_2 and $3 \notin R^*$) $P_R(G) = W^+$ so $N_R(W_G) = R^* \cdot W$. (The product is direct if and only if $-1 \notin W$.)

II. $\Gamma_R(G) \cong \mathbb{Z}/2$:

(a) A_l, $l \geqq 2$; D_l, $l \geqq 5$ odd; E_6. The nontrivial element of $\Gamma_R(G)$ is $(w_0)^+$ where $w_0 \in W$ is the element of maximal length [B, Planche I, IV, V]. Hence $P_R(G) = W^+$ and $N_R(W_G) = R^* \cdot W$. (The outer automorphism of G corresponds to $-w_0$ inducing ψ^{-1} on $H^*(BG; k)$.)

(b) D_l, $l \geqq 6$ even. $LT \cong \mathbb{R}^l$ with coordinates $(\theta_1, \ldots, \theta_l)$ and integral lattice $I = \{(\theta_1, \ldots, \theta_l) \in \mathbb{Z}^l \mid \sum \theta_i \in 2\mathbb{Z}\}$. The integral forms

$$\alpha_1 = \theta_1 - \theta_2, \ldots, \alpha_{l-1} = \theta_{l-1} - \theta_l, \quad \alpha_l = \theta_{l-1} + \theta_l$$

form a system of simple roots. One easily checks that there is a unique isomorphism ε of $\operatorname{Hom}(I, \mathbb{Z})$ satisfying $\varepsilon\theta_i = \theta_i$ for $i < l$ and $\varepsilon\theta_l = -\theta_l$. Moreover, $\varepsilon \in N_R(W_G)$, $\varepsilon^2 = 1$, and, since ε projects to a generator of $\Gamma_R(G)$, $N_R(W_G) = R^* \cdot W \cdot \langle\varepsilon\rangle$. Note also that ε preserves the invariant form $\sum \theta_i^2$.

(c) B_2 and $2 \in R^*$. $LT \cong \mathbb{R}^2$ with coordinates (θ_1, θ_2) and integral lattice $I = \mathbb{Z}^2$. The simple roots

$$\alpha_1 = \theta_1 - \theta_2, \quad \alpha_2 = 2\theta_2$$

form a basis for $\operatorname{Hom}(I, R)$. Let ε be the R-automorphism with $\varepsilon\alpha_1 = \alpha_2$, $\varepsilon\alpha_2 = 2\alpha_1$. Then $\varepsilon \in N_R(W_G)$, $\varepsilon^2 = 2$, and, since ε projects to the generator of $\Gamma_R(G)$, $N_R(W_G) = R^* \cdot W \cdot \langle\varepsilon\rangle$. The invariant form $\theta_1^2 + \theta_2^2$ is taken to twice itself by ε.

(d) F_4 and $2 \in R^*$. The simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ form a $\mathbb{Z}$-basis for $\operatorname{Hom}(I, \mathbb{Z})$ because the Cartan matrix has determinant 1. Let ε denote the R-automorphism of $\operatorname{Hom}(I, R)$ taking $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to $\alpha_4, \alpha_3, 2\alpha_2, 2\alpha_1$, respectively. Then $\varepsilon \in N_R(W_G)$, $\varepsilon^2 = 2$, and, since ε projects to the generator of $\Gamma_R(G)$, $N_R(W_G) = R^* \cdot W \cdot \langle\varepsilon\rangle$. According to [A-M, Table 2.14] the map induced by ε has degree 2 on $H^4(BF_4; k)$.

(e) G_2 and $3 \in R^*$: Similar to (d).

III. $\Gamma_R(D_4) \cong \Sigma_3$: The root system is described in II(b) with $l = 4$. When $l = 4$ there is another abelian group isomorphism δ of $\operatorname{Hom}(I, \mathbb{Z})$ that permutes α_1 and

α_3 and leaves α_2 and α_4 fixed. Moreover, $\delta \in N_R(W_G)$, $\delta^2 = 1$, and, since the subgroup $\langle \delta, \varepsilon \rangle \subset N_R(W_G)$ generated by δ and ε maps isomorphically onto $\Gamma_R(D_4)$, $N_R(W_G) = R^* \cdot W \cdot \langle \delta, \varepsilon \rangle$. From the formulas

$$\delta\theta_1 = \tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3 - \theta_4) \quad \delta\theta_2 = \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 + \theta_4)$$

$$\delta\theta_3 = \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 + \theta_4) \quad \delta\theta_4 = \tfrac{1}{2}(-\theta_1 + \theta_2 + \theta_3 + \theta_4)$$

we see that (also) δ preserves the invariant form $\sum \theta_i^2$. □

See [W3, Theorems B and C] for related results.

2 Genus sets of classifying spaces

Let X be a connected nilpotent space of finite type and L a set of primes. Denote by $G(X_L)$ the genus set of all homotopy types Y of $\mathbb{Z}_L$-nilpotent spaces of finite type over $\mathbb{Z}_L$ such that $Y_{(p)} \simeq X_{(p)}$ for all $p \in L$.

We investigate here the genus set of a classifying space BG localized, first, at a finite set of primes and, next, away from a finite set of primes.

Theorem 2.1. *Let G be a compact connected Lie group whose universal covering group is non-contractible and not homotopy equivalent to $SU(2)$. Then*

$$\operatorname{card} G(BG_L) = \aleph_0$$

for all finite sets L of at least two primes.

The compact connected Lie groups not covered by Theorem 2.1 are tori, $T = T^d$, of dimension $d \geqq 0$ and $SU(2) \times T$, $U(2) \times T$ and $SO(3) \times T$ (where $U(2)_L = SU(2)_L \times T_L^1$ and $SO(3)_L = SU(2)_L$ if $2 \notin L$).

Proposition 2.2. *Let G be any of the Lie groups $U(2) \times T$, $\dim T \geqq 0$, or $SU(2) \times T$, $SO(3) \times T$, $\dim T \geqq 1$. Then $G(BG_L) \neq *$ for all (finite) sets L of at least two primes.*

Rector [R] shows the genus set $G(BSU(2))$ is uncountably large. The next theorem shows that in fact $G(BG)$ is uncountably large for all compact connected Lie groups but tori. This even remains true after inverting finitely many primes.

Theorem 2.3. *Let G be any non-abelian compact connected Lie group and L any cofinite set of primes. Then the genus set $G(BG_L)$ is uncountably infinite.*

Now follow the proofs. The proof of Theorem 2.2 consists of Proposition 2.4 – Lemma 2.8 and the proof of Theorem 2.4 makes use of Lemma 2.9 – Lemma 2.12.

We begin with a couple of remarks of a general nature. Note that when $M \supset L$ are sets of primes the natural map $G(X_M) \to G(X_L)$ is surjective. Note also that according to Glover and Mislin [G-M] there is, for finite L, a bijection between $G(X_L)$ and

$$\operatorname{Aut}(X_{(0)}) \backslash \operatorname{Aut}(X_{(0)})^L \Big/ \prod_{p \in L} \operatorname{Aut}(X_{(p)}) .$$

Here, $\operatorname{Aut}(Z)$ denotes the group of homotopy classes of self homotopy equivalences of the space Z and $\operatorname{Aut}(X_{(0)})^L = \prod_{p \in L} \operatorname{Aut}(X_{(0)})$ is the group of functions of L into $\operatorname{Aut}(X_{(0)})$.

Proposition 2.4. *Let L be a finite set of at least two primes. Then $G(X_L)$ is trivial if and only if the obvious map*

$$\mathrm{Aut}(X_{(p)}) \times \mathrm{Aut}(X_{L-\{p\}}) \to \mathrm{Aut}(X_{(0)})$$

is surjective for all $p \in L$.

The "obvious map" mentioned in Proposition 2.4 is the one that takes $f \in \mathrm{Aut}(X_{(p)})$ and $g \in \mathrm{Aut}(X_{L-\{p\}})$ to $f_{(0)} \circ g_{(0)}$. We omit the proof as only the special case where $L = \{p, q\}$ contains just two distinct primes $p \neq q$ will be needed. Note that in this case

$$G(X_{\{p,q\}}) = \mathrm{Aut}(X_{(p)}) \backslash \mathrm{Aut}(X_{(0)}) / \mathrm{Aut}(X_{(q)}) .$$

Specializing to the case $X = BG$, the above formula for $G(BG_{\{p,q\}})$ indicates that a key point in the proof of Theorem 2.1 is to show that the image of the anti-homomorphism

$$H^*(-; \mathbb{Q}): \mathrm{Aut}(BG_{(p)}) \to \mathrm{Aut}\, H^*(BG; \mathbb{Q}) \cong \mathrm{Aut}(BG_{(0)})$$

is rather small. This follows from the theorem of Adams and Wojtkowiak [A-W] which says that for any given p-complete homotopy equivalence $f: BG_p^\wedge \to BG_p^\wedge$ there exists a homotopy equivalence k, unique up to left W_G-action, such that the diagram

$$\begin{array}{ccc} BT_p^\wedge & \xrightarrow{k} & BT_p^\wedge \\ {\scriptstyle B_i}\downarrow & & \downarrow{\scriptstyle B_i} \\ BG_p^\wedge & \xrightarrow{f} & BG_p^\wedge \end{array}$$

commutes up to homotopy. Here, G can be any compact connected Lie group with maximal torus $i: T \subsetneq G$. Moreover, k is admissible in the sense that

$$H^2(k; \mathbb{Z}_p^\wedge) \in N_{\mathbb{Z}_p^\wedge}(G) \subset \mathrm{Aut}_{\mathbb{Z}_p^\wedge} H^2(BT; \mathbb{Z}_p^\wedge) .$$

Therefore we have

Lemma 2.5. *The image of the map*

$$\mathrm{Aut}(BG_p^\wedge) \to \mathrm{Aut}_{\mathbb{Q}_p} H^*(BG; \mathbb{Q}_p)$$

is contained in the image of the map

$$N_{\mathbb{Z}_p^\wedge}(W_G) \to \mathrm{Aut}_{\mathbb{Q}_p} H^*(BG; \mathbb{Q}_p) .$$

The existence of the first map in Lemma 2.5 relies on the natural isomorphism $H^*(BG_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \cong H^*(BG; \mathbb{Q}_p)$. Wojtkowiak [W2, Theorems 1.3–1.4] shows that, when $(p, |W|) = 1$, the first map is injective and

$$\mathrm{Aut}(BG_p^\wedge) \cong N_{\mathbb{Z}_p^\wedge}(W_G)/W_G$$

so that the two maps have the same image in this case.

Lemma 2.6. *Let G be any non-trivial simply connected compact Lie group but $SU(2)$. Then*

$$\mathrm{card}\, G(BG_{\{p,q\}}) = \aleph_0$$

for any choice of two distinct primes $p \neq q$.

Proof. First assume that G is simple, $G \neq 1$, $G \neq SU(2)$. For a prime p, let $\bar{A}_p \subset \operatorname{Aut} H^*(BG; \mathbb{Q})$ be the central subgroup of unstable Adams operations determined by p-adic units; i.e. $\bar{A}_p$ contains $\psi^u, u \in (\mathbb{Z}_p^\wedge)^*$, provided $u^k \in \mathbb{Z}_{(p)}$ whenever $H^{2k}(BG; \mathbb{Q}) \neq 0$. Lemma 2.5 and Corollary 1.7 show that $H^*(f; \mathbb{Q})$ is contained in a finite extension E_p of $\bar{A}_p$ for all $f \in \operatorname{Aut}(BG_{(p)})$. Using the abbreviation $A = \operatorname{Aut} H^*(BG; \mathbb{Q})$ we have surjections

$$\operatorname{Aut}(BG_{(p)}) \backslash A / \operatorname{Aut}(BG_{(q)}) \twoheadrightarrow E_p \backslash A / E_q \twoheadleftarrow \bar{A}_p \backslash A / \bar{A}_q$$

where the fibres of the right map contain at most $|E_p \colon \bar{A}_p| \cdot |E_q \colon \bar{A}_q|$ points. The right set is countably infinite because rank $G > 1$; the two other sets are then also countably infinite.

For general compact simply connected G, write

$$G = \prod_{i=1}^{s} (B^{b_i} \times C^{c_i}) \times \prod_{i=1}^{t} H_i^{h_i}$$

as in Corollary 1.6. Lemma 2.5 and Corollary 1.6 show that the images of $\operatorname{Aut}(BG_{(p)})$ and $\operatorname{Aut}(BG_{(q)})$ in $\operatorname{Aut} H^*(BG; \mathbb{Q})$ are contained in the subgroup

$$\bigoplus_{i=1}^{s} \operatorname{Aut} H^*(BB_i; \mathbb{Q}) \wr \Sigma_{b_i + c_i} \oplus \bigoplus_{i=1}^{t} \operatorname{Aut} H^*(BH_i; \mathbb{Q}) \wr \Sigma_{h_i} .$$

Note that the corresponding set of double cosets is countably infinite if at least one of the numbers $b_i + c_i$, h_i or $s + t$ is > 1; i.e. whenever G is non-simple. □

This finishes the proof of Theorem 2.1 for simply connected Lie groups. We now deal with the non-simply connected ones.

Lemma 2.7. *Let G be a compact connected Lie group with universal covering group $\bar{G}$. Then*

$$\operatorname{card} G(BG_L) \geqq \operatorname{card} G(B\bar{G}_L)$$

for any finite set L of at least two primes.

Lemma 2.7 follows immediately from the more general statement below by noting that $B\bar{G}$ is homotopy equivalent to the 2-connective cover $BG\langle 2 \rangle$ of BG.

Lemma 2.8. *Let X be a simply connected H_0-space of finite type with n-connective cover $X\langle n \rangle$, $n > 1$, and L any finite set of at least two primes. The n-connective cover map*

$$-\langle n \rangle \colon G(X_L) \to G(X\langle n \rangle_L)$$

is surjective.

Proof. Naturality (of the Postnikov approximations to the path space fibration over X) determines a homomorphism

$$\operatorname{Aut}(X) \to \operatorname{Aut}(X\langle n \rangle), \quad n > 1 .$$

Similar homomorphisms exist for the various localizations of X. The rational version, in particular, is surjective because $X_{(0)}$ is a product of Eilenberg-MacLane spaces. The induced map will then take

$$G(X_L) = \operatorname{Aut}(X_{(0)}) \backslash \operatorname{Aut}(X_{(0)})^L \Big/ \prod_{p \in L} \operatorname{Aut}(X_{(p)})$$

onto

$$G(X\langle n\rangle_L) = \mathrm{Aut}(X\langle n\rangle_{(0)})\backslash \mathrm{Aut}(X\langle n\rangle_{(0)})^L \Big/ \prod_{p\in L} \mathrm{Aut}(X\langle n\rangle_{(p)}) . \quad \square$$

This finishes the proof of Theorem 2.1.

Proof of Proposition 2.2 for $G = U(2)\times T$. We may choose coordinates $(\theta_1, \theta_2; \eta_1, \ldots, \eta_d)$, $d = \dim T \geqq 0$, for the Lie algebra of the standard maximal torus $T(G) = S^1 \times S^1 \times T$ in G such that the integral lattice $I = \mathbb{Z}^2 \times \mathbb{Z}^d$ and the real roots are the integral forms $\theta_1 - \theta_2$ and $\theta_2 - \theta_1$. For any $A \in N_R(W_G)$, $A\tau = \tau A$ where $\tau \in W_G \cong \mathbb{Z}/2$ is the non-trivial element. The computation

$$-A(\theta_1 - \theta_2) = A\tau(\theta_1 - \theta_2) = \tau A(\theta_1 - \theta_2)$$

shows that $A(\theta_1 - \theta_2)$ is a (-1)-eigenvector for τ, i.e. $A(\theta_1 - \theta_2) = \lambda(\theta_1 - \theta_2)$ for some $\lambda \in R^*$. The theorem by Adams and Wojtkowiak [A-W] then implies that $(\theta_1 - \theta_2)^2 \in H^4(BT(G); \mathbb{Q})^{W_G} = H^4(BG; \mathbb{Q})$ is an eigenvector for $H^4(f; \mathbb{Q})$ for all $f \in \mathrm{Aut}(BG_{(p)})$. Since this property is not shared by all members of $\mathrm{Aut}(BG_{(0)})$,

$$G(BG_{\{p,q\}}) = \mathrm{Aut}(BG_{(p)})\backslash \mathrm{Aut}(BG_{(0)})/\mathrm{Aut}(BG_{(q)}) \neq *$$

for any choice of primes $p \neq q$. $\square$

The proof of Theorem 2.3 makes use of Lemma 2.9–2.12 below. First we need a small observation concerning Legendre symbols $\left(\frac{a}{p}\right)$, p prime, $a \in \mathbb{Z} - p\mathbb{Z}$.

Lemma 2.9. *Let E be a finite set of primes and s a prime not in E. Then there exist infinitely many odd primes p such that $\left(\frac{-1}{p}\right) = \left(\frac{s}{p}\right) = -1$ and $\left(\frac{e}{p}\right) = +1$ for all $e \in E$.*

Proof. Put $\varepsilon := \prod_{e\in E} e$. Let p be any of the infinitely many (Dirichlet) primes of the form $p = 4m\varepsilon - 1$. For any odd prime $e \in E$, $p \equiv -1 (\mathrm{mod}\, e)$ and by the quadratic reciprocity law [S, Theorem I.6]

$$\left(\frac{-1}{e}\right)\left(\frac{e}{p}\right) = \left(\frac{p}{e}\right)\left(\frac{e}{p}\right) = (-1)^{\frac{e-1}{2}} = \left(\frac{-1}{e}\right)$$

so $\left(\frac{e}{p}\right) = +1$. If $2 \in E$, $p \equiv -1 (\mathrm{mod}\, 8)$ so $\left(\frac{2}{p}\right) = +1$ by [S, Theorem I.5]. Moreover, $\left(\frac{-1}{p}\right) = -1$ as $p \equiv -1 (\mathrm{mod}\, 4)$.

Let now $s \notin E$ be a prime. If $s = 2$, then $2 \notin E$ so $p \equiv 3 (\mathrm{mod}\, 8)$, i.e. $\left(\frac{2}{p}\right) = -1$, for any of the infinitely many primes of the form $p = 4(2m+1)\varepsilon - 1 = m8\varepsilon + (4\varepsilon - 1)$. If $s > 2$, choose natural numbers a and b such that $\left(\frac{a}{s}\right) = -\left(\frac{-1}{s}\right)$ and $4b\varepsilon - 1 \equiv a (\mathrm{mod}\, s)$. Then $p \equiv a (\mathrm{mod}\, s)$ for any of the infinitely many primes of the form $p = 4(b + ms)\varepsilon - 1 = m4\varepsilon s + (4b\varepsilon - 1)$ so by the quadratic reciprocity law

$$\left(\frac{s}{p}\right) = \left(\frac{-1}{s}\right)\left(\frac{p}{s}\right) = \left(\frac{-1}{s}\right)\left(\frac{a}{s}\right) = -1 .$$

This finishes the proof. $\square$

For the rest of this section, the letter L will stand for some cofinite set of primes and E for the complementary finite set of primes. Π denotes the set of all primes. For any given partition $\Pi = E \cup L$, the set

$$S = \left\{ p \in L, p > 2 \,\middle|\, \left(\frac{-1}{p}\right) = -1 \quad \text{and} \quad \forall e \in E\colon \left(\frac{e}{p}\right) = +1 \right\}$$

is infinite. Therefore there are uncountably many sequences $(a_p)_{p \in \Pi}$ satisfying

$$a_p = \begin{cases} +1 & p \notin S \quad \text{or} \quad p = s \\ \pm 1 & p \in S \quad \text{but} \quad p \neq s\,. \end{cases}$$

Here, s is some fixed prime in S. The idea of the proof is to associate to each such sequence a space in the appropriate genus set. For technical reasons it is convenient to consider the localization genus set $G(BG_L)$ as a subset of the completion genus set $G_0^\wedge(BG_L)$. The completion genus set $G_0^\wedge(X_L)$ consists of all homotopy types of $\mathbb{Z}_L$-nilpotent spaces Y of finite type over $\mathbb{Z}_L$ such that $Y_p^\wedge \simeq X_p^\wedge$, $p \in L$, and $Y_{(0)} \simeq X_{(0)}$.

Lemma 2.10. *Let G be any non-trivial simply connected compact simple Lie group. Then the genus set $G(BG_L)$ is uncountably infinite.*

Proof. Wilkerson's double coset formula [Wi]

$$G_0^\wedge(BG_L) = \operatorname{Aut}(BG_{(0)}) \backslash C\operatorname{Aut}((BG_L^\wedge)_{(0)}) / \operatorname{Aut}(BG_L^\wedge)$$

expresses the completion genus set as a quotient of the subgroup $C\operatorname{Aut}((BG_L^\wedge)_{(0)})$ consisting of those elements of $\operatorname{Aut}(BG_L^\wedge)_{(0)})$ that induce $\mathbb{Q} \otimes \mathbb{Z}_L^\wedge$-linear automorphisms in homotopy. In particular, the self-homotopy equivalence $\prod_{p \in L} a_p \times 1$ of the formal completion

$$K(\mathbb{Q} \otimes \mathbb{Z}_L^\wedge, 4) \times (BG\langle 4 \rangle_L^\wedge)_{(0)}$$

of BG_L represents a space $B(a_p)$ in the completion genus $G_0^\wedge(BG_L)$. Moreover, since each a_p is a rational number, $B(a_p)$ is in fact contained in the localization genus $G(BG_L)$. It suffices to show

$$B(a_p) = B(b_p) \Rightarrow (a_p) = (b_p)$$

for any two sequences (a_p) and (b_p) that meet the requirements.

Assume $B(a_p) = B(b_p)$. Then

$$R \circ \left(\prod a_p \times 1 \right) = \left(\prod b_p \times 1 \right) \circ N$$

for some $R \in \operatorname{Aut}(BG_{(0)})$ and some $N \in \operatorname{Aut}(BG_L^\wedge)$. In

$$H^4(BG_{(0)}; \mathbb{Q}) \otimes \mathbb{Z}_L^\wedge = H^4(BG_L^\wedge; \mathbb{Z}_L^\wedge) \otimes \mathbb{Q} = H^4(BG; \mathbb{Q} \otimes \mathbb{Z}_L^\wedge)$$

we get

$$(*) \qquad \forall p \in L\colon \rho a_p = v_p b_p$$

where ρ is the degree of $H^4(R; \mathbb{Q}) \otimes \mathbb{Z}_p^\wedge$ and v_p the degree of $H^4(N; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. We may assume that 2 and 3 are in E. Then 2 and 3 will be square units in $\mathbb{Z}_p^\wedge$ for all $p \in S$; by the computation of $H^4(N_R(W_G); \mathbb{Q}_p)$ in Corollary 1.7 the same is true for v_p.

Formula $(*)$ implies $\rho \in \mathbb{Z}_L^*$, i.e. $\rho = \pm e_1^{i_1} \cdots e_k^{i_k}$ for some exponents $i_j \in \mathbb{Z}$ and some primes $e_j \in E$, $1 \leq j \leq k$. All primes in E are, by construction, quadratic residues mod p for any prime $p \in S$. In particular, $|\rho|$ is a quadratic residue mod s. As also $\rho = v_s$ is a quadratic residue mod s whereas -1 is not, we must have $\rho = e_1^{i_1} \dots e_k^{i_k} > 0$. Hence ρ is a square unit in $\mathbb{Z}_p^\wedge$ for all $p \in S$. We observed above that also v_p is a square unit in $\mathbb{Z}_p^\wedge$ for all $p \in S$ and that -1 is not. Therefore $(*)$ now says $a_p = -1 \Leftrightarrow b_p = -1$, $p \in S$. If $p \notin S$, $a_p = 1 = b_p$ by definition. Hence $(a_p)_{p \in \Pi} = (b_p)_{p \in \Pi}$. □

Lemma 2.11. *Let G be any non-trivial simply connected compact Lie group. Then the genus set $G(BG_L)$ is uncountably infinite.*

Proof. A detailed proof is given only in the case where $G = H \times \cdots \times H = H^n$, $n > 1$, for some simple Lie group $H \neq 1$. I'll show that the map

$$- \times BH_L^{n-1} \colon G(BH_L) \to G(BH_L \times BH_L^{n-1})$$

has an uncountable image.

Write $\Pi = E \cup L$ where 2 and 3 are in E. It's enough to show

$$B(a_p) \times BH_L^{n-1} \simeq B(b_p) \times BH_L^{n-1} \Rightarrow B(a_p) \simeq B(b_p)$$

for any two sequences $(a_p)_{p \in L}$ and $(b_p)_{p \in L}$ associated to the given partition. So assume the left statement is true.

The space $B(a_p) \times BH_L^{n-1} \in G_0^\wedge(BG_L)$ is represented by the self map $(\prod_{p \in L} \operatorname{diag}(a_p, 1, \dots, 1)) \times 1$ of the formal completion

$$(BG_L^\wedge)_{(0)} = K(\mathbb{Q}^n \otimes \mathbb{Z}_L^\wedge, 4) \times (BG\langle 4 \rangle_L^\wedge)_{(0)} .$$

By the double coset formula for the completion genus there exist $R \in \operatorname{Aut}(BG_{(0)})$ and $N_p \in \operatorname{Aut}(BG_p^\wedge)$ such that for all $p \in L$

$$R(\operatorname{diag}(a_p, 1, \dots, 1) \times 1) = (\operatorname{diag}(b_p, 1, \dots, 1) \times 1) N_p$$

in $C\operatorname{Aut}((BG_p^\wedge)_{(0)})$. The induced $\mathbb{Q}_p$-linear maps on

$$H^4(BG; \mathbb{Q}_p) = H^4(BG_{(0)}; \mathbb{Q}_p) = H^4(BG_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$$

satisfy

$$(*) \qquad \forall p \in L \colon \operatorname{diag}(a_p, 1, \dots, 1)\rho = v_p \operatorname{diag}(b_p, 1, \dots, 1)$$

where $\rho = H^4(R; \mathbb{Q}) \otimes \mathbb{Z}_p^\wedge$ and $v_p = H^4(N_p; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$.

The matrix v_p acting on $H^4(BT; \mathbb{Q}_p)^{W_G} = H^4(BG_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is induced from some element of

$$N_{\mathbb{Z}_p^\wedge}(W_G) = N_{\mathbb{Z}_p^\wedge}(W_H) \wr \Sigma_n .$$

Therefore $v_p = \operatorname{diag}(v_p^1, \dots, v_p^n)\sigma_p$ for some units $v_p^i \in \mathbb{Z}_p^\wedge$ and some permutation matrix σ_p. Since we're assuming that 2 and 3 are in E, the p-adic numbers v_p^i are squares by Corollary 1.7.

We rewrite

$$(*) \qquad \forall p \in L \colon \operatorname{diag}(a_p, 1, \dots, 1)\rho = \operatorname{diag}(v_p^1, \dots, v_p^n)\sigma_p \operatorname{diag}(b_p, 1, \dots, 1)$$

and from this we see that $\rho\sigma_p$ must be a diagonal matrix and that the permutation matrix σ_p is independent of p; $\sigma_p = \sigma$, say, for all $p \in L$. If $\rho\sigma = \operatorname{diag}(r_1, \ldots, r_n)$, we get

$$(*) \qquad \forall p \in L\colon \operatorname{diag}(r_1 a_p, r_2, \ldots, r_n) = \operatorname{diag}(v_p^1, \ldots, v_p^j b_p, \ldots, v_n)$$

where $j = \sigma(1)$. As in the proof of Lemma 2.10 we conclude $a_p = 1 = b_p$ if $j \neq 1$ and $a_p = b_p$ if $j = 1$, $p \in L$. $\square$

Lemma 2.12. *Let $G \neq 1$ be a compact simply connected Lie group and let T be a d-dimensional torus, $d \geqq 0$. Then the image of the map*

$$- \times BT_L\colon G(BG_L) \to G(BG_L \times BT_L)$$

is uncountably infinite.

Proof. We shall restrict ourselves to the case where G is a simply connected compact simple Lie group. As above we may assume $L = \Pi - E$ for some finite set of primes, E, containing 2, 3 and all prime divisors of $|W_G|$. Let (a_p) and (b_p) be two sequences as defined above. We'll show

$$B(a_p)_L \times BT_L \simeq B(b_p)_L \times BT_L \Rightarrow B(a_p)_L \simeq B(b_p)_L\,.$$

Assume the left statement is true. The formal completion of $BG_L \times BT_L$ is

$$K(\mathbb{Q} \otimes \mathbb{Z}_L^\wedge, 4) \times (BG\langle 4\rangle_L^\wedge)_{(0)} \times (BT_L^\wedge)_{(0)}\,.$$

Wilkerson's double coset formula for $G_0^\wedge(BG_L \times BT_L)$ shows that in $C\operatorname{Aut}(((BG \times BT)_L^\wedge)_{(0)})$,

$$R\left(\prod_{p \in L} a_p \times 1 \times 1\right) = \left(\prod_{p \in L} b_p \times 1 \times 1\right) N$$

for some $R \in \operatorname{Aut}(BG_{(0)} \times BT_{(0)})$ and some $N \in \operatorname{Aut}(BG_L^\wedge \times BT_L^\wedge)$. In $C\operatorname{Aut}((BG_L^\wedge)_{(0)})$ we get

$$R_{11}\left(\prod_{p \in L} a_p \times 1\right) = \left(\prod_{p \in L} b_p \times 1\right) N_{11}$$

where R_{11} is the composite

$$BG_{(0)} \to BG_{(0)} \times BT_{(0)} \xrightarrow{R} BG_{(0)} \times BT_{(0)} \to BG_{(0)}$$

and N_{11} is defined similarly. Both R_{11} and N_{11} are homotopy equivalences since they induce isomorphisms in homotopy. Therefore the spaces $B(a_p)_L$ and $B(b_p)_L$ are homotopy equivalent according to the double coset formula. $\square$

In order to finish the proof of Theorem 2.3, let now G be any non-abelian compact connected Lie group. G is the base space of a finite covering map $H \times T \to G$ for some non-trivial compact simply connected Lie group H. Hence

$$BH_L \times BT_L = BG_L\,,$$

provided $|\pi_1(G)\colon \pi_1(T)| \in \mathbb{Z}_L^*$, so $G(BG_L) = G(BH_L \times BT_L)$ is uncountably infinite by Lemma 2.12.

Remark 2.13. (1) Two (nilpotent and finite type) connected spaces are said to be clones [M-M2] of each other if they have homotopically identical p-localizations at all primes p as well as identical nth stage Postnikov approximations for all $n \geqq 0$. In other words, the clones of X are the homotopy types in the intersection $G(X) \cap SNT(X)$ where $SNT(X)$ denotes the set of all homotopy types with the same n-type for all n as X. $SNT(X)$ and the completion genus $G_0^\wedge$ are related by the short exact sequence

$$* \to SNT(X) \to G_0^\wedge(X) \to \varprojlim G_0^\wedge(X^{(n)}) \to *$$

of sets. When L is a finite set of primes, and G is any Lie group, Zabrodsky [Z] shows that $G_0^\wedge(BG_L^{(n)}) = G(BG_L^{(n)}) = *$ for all Postnikov approximations $BG_L^{(n)}$ to BG_L. Thus $SNT(BG_L) = G_0^\wedge(BG_L)$ and

$$G(BG_L) \subset SNT(BG_L)$$

is the set of clones of BG_L.

When it comes to the integral space BG it is known that $SNT(BG)$ is uncountably large [M-M1, Theorem 4] for all non-abelian compact connected Lie groups $G \neq \mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SO}(3), \mathrm{PSU}(3)$. However, the cardinality of the set of clones of BG is not known.

(2) The exact size of the genus sets mentioned in Proposition 2.2 are unknown. It follows from [M-M1, Theorem 3] that $SNT(BSU(2)_L)$, and hence also $G(BSU(2)_L)$, is trivial for all finite sets of primes L.

3 Rational self equivalences of BG

A map $\alpha\colon Y \to Z$ between nilpotent spaces is called a rational equivalence if its rationalization $\alpha_{(0)}\colon Y_{(0)} \to Z_{(0)}$ is a homotopy equivalence. Let $\varepsilon_0(Y, Z) \subset [Y, Z]$ denote the set of homotopy classes of rational equivalences from Y into Z and $\varepsilon_0(Y) = \varepsilon_0(Y, Y)$ the monoid of rational self equivalences of Y. The group of invertible elements of $\varepsilon_0(Y)$ is $\mathrm{Aut}(Y)$, the group of homotopy classes of homotopy self equivalences of Y. We are here interested in the case where Y is (the localization of) a Lie group classifying space or of the same genus as such a space.

Let L be a nonempty set of primes and let $G = G_1^{n_1} \times \cdots \times G_t^{n_t}$ be a simply connected compact Lie group written as a product of its simple factors such that $G_i \neq G_j$ for $i \neq j$. We adopt the convention not to distinguish between $\mathrm{Spin}(2n+1)$ and $\mathrm{Sp}(n)$, $n > 2$, in case $2 \notin L$.

Theorem 3.1. *The obvious map*

$$\bigoplus_{i=1}^{t} \varepsilon_0((BG_i)_L) \wr \Sigma_{n_i} \to \varepsilon_0(BG_L)$$

is a monoid isomorphism.

Corollary 3.2. $\bigoplus_{i=1}^{t} \mathrm{Aut}((BG_i)_L) \wr \Sigma_{n_i} \cong \mathrm{Aut}(BG_L)$.

A special (and motivating) case of this corollary,

$$\mathrm{Aut}(BSU(2)^n) \cong \Sigma_n$$

can be found in a paper [I] by Iwase.

The proof of Theorem 3.1, relying on the simple structure of $N_R(W_G)$ as well as results from [JMO3], consists of the following three lemmas.

Lemma 3.3. *Let $G = G(1) \times G(2)$ be the product of the two simply connected compact Lie groups $G(1)$ and $G(2)$. Assume that the $\mathbb{Q}$-Dynkin diagrams of $G(1)$ and $G(2)$ have no isomorphic components. Then the product map*

$$\varepsilon_0(BG(1)_L) \times \varepsilon_0(BG(2)_L) \to \varepsilon_0(BG_L)$$

is a monoid isomorphism.

Lemma 3.4. *Let G be any simply connected compact simple Lie group. Then the obvious map*

$$\varepsilon_0(BG_L) \wr \Sigma_n \to \varepsilon_0(BG_L^n)$$

is a monoid isomorphism for any integer $n \geqq 1$.

Lemma 3.5. *Let $G = \mathrm{Spin}(2n+1)^b \times \mathrm{Sp}(n)^c$, $n > 2$, and let L be a set of primes containing 2. Then the product map*

$$\varepsilon_0(B\,\mathrm{Spin}(2n+1)_L^b) \times \varepsilon_0(B\,\mathrm{Sp}(n)_L^c) \to \varepsilon_0(BG_L)$$

is a monoid isomorphism.

Proof of Lemma 3.3. Choose maximal tori $T(i) \subset G(i)$, 1, 2, and let $T = T(1) \times T(2)$ serve as maximal torus for $G(1) \times G(2) = G$.

Let $f\colon BG_L \to BG_L$ be a rational equivalence. Choose [A-W, Theorem 1.1; JMO3, Proposition 1.2] for each $p \in L$ a rational equivalence k_p such that the diagram

$$(*) \qquad \begin{array}{ccc} BT(1)_p^\wedge \times BT(2)_p^\wedge & \xrightarrow{k_p} & BT(1)_p^\wedge \times BT(2)_p^\wedge \\ \downarrow & & \downarrow \\ BG(1)_p^\wedge \times BG(2)_p^\wedge & \xrightarrow{f_p^\wedge} & BG(1)_p^\wedge \times BG(2)_p^\wedge \end{array}$$

commutes up to homotopy. By Corollary 1.5, the induced map

$$H^2(k_p; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \in N_{\mathbb{Q}_p}(W_G) = N_{\mathbb{Q}_p}(W_{G(1)}) \times N_{\mathbb{Q}_p}(W_{G(2)})$$

is a product homomorphism. Then also $k_p = k(1)_p \times k(2)_p$ is the product of two maps $k(i)_p \in \varepsilon_0(BT(i)_p^\wedge)$, $i = 1, 2$. Define $f(i)$ to be the composite

$$BG(i)_L \to BG_L \xrightarrow{f} BG_L \to BG(i)_L$$

of f and the appropriate inclusion and projection. Since the lift $k(i)_p$ of $f(i)_p^\wedge$ is a rational equivalence, so is $f(i)$, cfr. [JMO3, Proposition 1.2(iii)]. Note that $(*)$ stays homotopy commutative if f is replaced by $f(1) \times f(2)$ and that consequently [JMO3, Corollary 1.10], $f_L^\wedge = f(1)_L^\wedge \times f(2)_L^\wedge$ and $f = f(1) \times f(2)$ as $\varepsilon_0(BG_L) \to \varepsilon_0(BG_L^\wedge)$ is injective [W4, Lemma 2; JMO1, Theorem 3.1]. This proves surjectivity.

Injectivity follows from the general statement that the product map

$$[X_1, X_1] \times [X_2, X_2] \to [X_1 \times X_2, X_1 \times X_2]$$

is an injection for any two spaces X_1 and X_2. □

The proofs of the remaining two lemmas are similar to the one above. In each case it suffices, for a general rational equivalence $f\colon BG_L \to BG_L$, to show that the restrictions $k_p\colon BT_p^\wedge \to BT_p^\wedge$ of $f_p^\wedge$, $p \in L$, are product maps (composed with a permutation map).

Proof of Lemma 3.4. Choose a maximal torus $T \subset G$ and let the Cartesian product T^n serve as maximal torus for G^n.

For each $p \in L$,

$$H^2(k_p; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \in N_{\mathbb{Q}_p}(W_{G^n}) = N_{\mathbb{Q}_p}(W_G) \wr \Sigma_n$$

so the rational equivalence k_p has the form

$$k_p = (k(1)_p \times \cdots \times k(n)_p) \circ \sigma_p$$

for some $k(i)_p \in \varepsilon_0(BT_p^\wedge)$ and some permutation $\sigma_p \in \Sigma_n \subset \mathrm{Aut}((BT_p^\wedge)^n)$, $1 \leqq i \leqq n$, $p \in L$. The permutation σ_p is independent of p because Σ_n is faithfully represented in $H^4(BG_L^n; \mathbb{Q}_p) = H^4(BG_L; \mathbb{Q}_p)^n$ by $H^4(f_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} = H^4(f; \mathbb{Q}) \otimes \mathbb{Z}_p^\wedge = H^4(f; \mathbb{Q}_p)$. Thus $k_p = (k(1)_p \times \cdots \times k(n)_p) \circ \sigma$ where σ is the common value of the σ_p, $p \in L$. $\square$

Proof of Lemma 3.5. Write $G = G(1) \times G(2)$ where $G(1) = \mathrm{Spin}(2n+1)^b$, $G(2) = \mathrm{Sp}(n)^c$ with maximal tori $T(1) \subset G(1)$, $T(2) \subset G(2)$ of the form $T(1) \cong T(0)^b$, $T(2) \cong T(0)^c$ for $T(0) = \mathbb{R}^n/\mathbb{Z}^n$.

For each $p \in L$,

$$H^2(k_p; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \in N_{\mathbb{Q}_p}(W_G) \cong N_{\mathbb{Q}_p}(W_{\mathrm{Spin}(2n+1)}) \wr \Sigma_{b+c}$$

so $k_p = (k(1)_p \times k(2)_p) \circ \sigma_p$ where $k(1)_p$, $k(2)_p$ are product of b, c rational self equivalences of $BT(0)_p^\wedge$ and $\sigma_p \in \Sigma_{b+c} \subset \mathrm{Aut}(BT(1)_p^\wedge \times BT(2)_p^\wedge) = \mathrm{Aut}((BT(0)_p^\wedge)^{b+c})$ is a permutation map. At the prime $2 \in L$,

$$H^2(k_2; \mathbb{Z}_2^\wedge) \in N_{\mathbb{Z}_2^\wedge}(W_{G(1)\times G(2)}) = N_{\mathbb{Z}_2^\wedge}(W_{G(1)}) \times N_{\mathbb{Z}_2^\wedge}(W_{G(2)})$$

because $f_2^\wedge$ and k_2 are genuine homotopy equivalences [JMO3, Proposition 1.2 and 1.3]. Thus the common value $\sigma_p = \sigma_2 \in \Sigma_b \times \Sigma_c \subset \Sigma_{b+c}$ and k_p is a product of rational self equivalences on $BT(1)_p^\wedge$ and $BT(2)_p^\wedge$. $\square$

For the rest of this section, G can be any compact connected Lie group.

Fix a space $Y \in G_0^\wedge(BG_L)$ of the same completion genus as BG_L where L is any set of at least two primes. We shall investigate $\mathrm{Aut}(Y)$ and the sets of rational equivalences between Y and BG_L. By assumption, there exist homotopy equivalences

$$f\colon Y_{(0)} \to BG_{(0)} \quad \text{and } g\colon Y_L^\wedge \to BG_L^\wedge$$

and Y is homotopy equivalent to the homotopy pull back of the diagram

$$BG_{(0)} \to (BG_L^\wedge)_{(0)} \xleftarrow{\;B\;} (BG_L^\wedge)_{(0)} \leftarrow BG_L^\wedge$$

where $B = f \circ g^{-1}$ and the unmarked arrows represent canonical maps.

Proposition 3.6. $\mathrm{Aut}(Y)$ *is isomorphic to (the image in* $\mathrm{Aut}(BG_{(0)})$ *of) the group of all pairs*

$$(R, U) \in \mathrm{Aut}(BG_{(0)}) \times \mathrm{Aut}(BG_L^\wedge)$$

for which $RB = BU$ *in* $C\,\mathrm{Aut}((BG_L^\wedge)_{(0)})$.

Proof. Use [MM2, Lemma 4] and, for the parenthesis, Lemma 3.8 below. □

The next example shows that the group of self-homotopy equivalences is not a generic invariant. Recall that

$$\mathrm{Aut}(\mathbb{H}P_L^\infty \times \mathbb{H}P_L^\infty) \cong (\mathrm{Aut}(\mathbb{H}P_L^\infty) \times \mathrm{Aut}(\mathbb{H}P_L^\infty)) \rtimes \Sigma_2$$

by Corollary 3.2.

Example 3.7. There exists a space $Y \in G(\mathbb{H}P_L^\infty \times \mathbb{H}P_L^\infty)$ with $\mathrm{Aut}(Y) \cong \mathrm{Aut}(\mathbb{H}P_L^\infty)$.

Proof. The idea is to destroy the symmetry in the product $X := \mathbb{H}P^\infty \times \mathbb{H}P^\infty$.

Pick some prime $s \in L$. Let $B \in \mathrm{GL}(2, \mathbb{Z}_L^\wedge)$ be the matrix whose coordinate $B_p \in \mathrm{GL}(2, \mathbb{Z}_p^\wedge)$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ when $p = s$ and the identity matrix for all other primes $p \in L$. Let Y be the homotopy pull back associated to $B \in \mathrm{GL}(2, \mathbb{Z}_L^\wedge) \subset \mathrm{GL}(2, \mathbb{Q} \otimes \mathbb{Z}_L^\wedge) = C\,\mathrm{Aut}((X_L^\wedge)_{(0)})$ as above. Y is in the localization genus $G(X_L)$.

Let $U_p \in \mathrm{Aut}(X_p^\wedge)$. According to Lemma 2.5 and Corollary 1.4, $U_p^* = H^4(U_p; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ acting on $H^4(X_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} = \mathbb{Q}_p^2$ is either diagonal or anti-diagonal:

$$U_p^* = \begin{pmatrix} \mu_p & 0 \\ 0 & \nu_p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\varepsilon_p}$$

where $\varepsilon_p = 0, 1$ and μ_p, ν_p are p-adic (square) units.

Suppose $R \in \mathrm{GL}(2, \mathbb{Q}) = \mathrm{Aut}(X_{(0)})$ represents an element of $\mathrm{Aut}(Y)$ as in Proposition 3.6. Then $R^* = H^4(R; \mathbb{Q}) \otimes \mathbb{Z}_p^\wedge = U_p^*$ for $s \neq p \in L$ and

$$B_s^* R^* (B_s^{-1})^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} R^* \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

are both either diagonal or anti-diagonal. This forces R to be a multiple of the identity matrix; indeed $R = m \times m$ for some $m \in \mathrm{Aut}(\mathbb{H}P_L^\infty) \subset \mathbb{Z}_L^*$. Conversely, any R of this form is in (the subgroup of $\mathrm{GL}(2, \mathbb{Q})$ corresponding to) $\mathrm{Aut}(Y)$ by Proposition 3.6. □

In particular, when L is the set of all primes, we obtain a space of the same genus as $\mathbb{H}P^\infty \times \mathbb{H}P^\infty$ but with no symmetries besides the identity.

Now pick another member Z of $G_0^\wedge(BG_L)$. Choose homotopy equivalences

$$e: Z_{(0)} \to BG_{(0)} \quad \text{and} \quad h: Z_L^\wedge \to BG_L^\wedge$$

so that Z corresponds to the double coset containing $C := e \circ h^{-1} \in C\,\mathrm{Aut}((BG_L^\wedge)_{(0)})$. Consider the set $\varepsilon_0(Y, Z)$ of homotopy classes of rational equivalences from Y into Z.

Lemma 3.8. *The rationalization map*

$$\varepsilon_0(Y, Z) \to [Y_{(0)}, Z_{(0)}]$$

is injective.

Proof. In the commutative diagram

$$\begin{tikzcd}
\varepsilon_0(Y,Z) \arrow[r] \arrow[d] & {[Y_{(0)}, Z_{(0)}]} \arrow[d] \\
\varepsilon_0(Y_L^\wedge, Z_L^\wedge) \arrow[r] \arrow[d, "h\circ - \circ g^{-1}"'] & {[(Y_L^\wedge)_{(0)}, (Z_L^\wedge)_{(0)}]} \arrow[d, "h_{(0)}\circ - \circ g_{(0)}^{-1}"] \\
\varepsilon_0(BG_L^\wedge, BG_L^\wedge) \arrow[r] & {[(BG_L^\wedge)_{(0)}, (BG_L^\wedge)_{(0)}]}
\end{tikzcd}$$

the vertical arrow from the upper left corner is injective because there are no phantom maps from Y to Z and the bottom horizontal map is injective by [JMO3, Corollary 1.10]. □

In the proposition below, $\varepsilon_0(BG_L^\wedge)$ denotes the monoid of rational self equivalences of $BG_L^\wedge$.

Proposition 3.9. *The set $\varepsilon_0(Y,Z)$ is in one-to-one correspondence with (the image in $\mathrm{Aut}(BG_{(0)})$ of) the set of all pairs*

$$(R, U) \in \mathrm{Aut}(BG_{(0)}) \times \varepsilon_0(BG_L^\wedge)$$

for which the equation $RB = CU$ holds in $C\,\mathrm{Aut}((BG_L^\wedge)_{(0)})$.

Proof. Identify $\varepsilon_0(Y,Z)$ with its image in $\mathrm{Aut}(BG_{(0)})$ under the injection

$$\varepsilon_0(Y,Z) \to [Y_{(0)}, Z_{(0)}] \xrightarrow{e\circ - \circ f^{-1}} [BG_{(0)}, BG_{(0)}]$$

taking $\alpha \in \varepsilon_0(Y,Z)$ to $R := e\circ\alpha_{(0)}\circ f^{-1}$. In $C\,\mathrm{Aut}((BG_L^\wedge)_{(0)})$, $RB = e\circ\alpha\circ f^{-1}\circ f\circ g^{-1} = e\circ\alpha\circ g^{-1} = e\circ h^{-1}\circ h\circ\alpha\circ g^{-1} = CU$ for $U := h\circ\alpha_L^\wedge\circ g^{-1}$.

If, conversely, $RB = CU$ for some $R\in\mathrm{Aut}(BG_{(0)})$ and some $U\in\varepsilon_0(BG_L^\wedge)$ then the universal property of pull backs applied to the Arithmetic Square produces a rational equivalence $\alpha\colon Y\to Z$ with $e\circ\alpha_{(0)}\circ f^{-1} = R$ and $h\circ\alpha_L^\wedge\circ g^{-1} = U$. □

Example 3.10. There are no rational equivalences between $\mathbb{H}P_L^\infty\times\mathbb{H}P_L^\infty$ and the space Y constructed in Example 3.7.

Proof. If there existed a rational equivalence from $\mathbb{H}P_L^\infty\times\mathbb{H}P_L^\infty$ into Y, resp. from Y into $\mathbb{H}P_L^\infty\times\mathbb{H}P_L^\infty$, then we could find an $R\in\mathrm{Aut}(\mathbb{H}P_{(0)}^\infty\times\mathbb{H}P_{(0)}^\infty) = \mathrm{GL}(2,\mathbb{Q})$ such that R^* and

$$B_s^*R^* = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}R^*, \quad \text{resp.} \quad R^*(B_s^{-1})^* = R^*\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix},$$

were both diagonal or anti-diagonal. But this just isn't possible. □

Even in the simplest genus set, $G(\mathbb{H}P^\infty)$, the above phenomenon may occur. This fact was also known to Frank Adams.

Example 3.11. There exists a space $Y\in G(\mathbb{H}P^\infty)$ such that $[\mathbb{H}P^\infty, Y] = * = [Y, \mathbb{H}P^\infty]$.

Proof. Let $Y \in G(\mathbb{H}P^\infty)$ be the homotopy type represented by $B = (b_p) \in (\mathbb{Q} \otimes \mathbb{Z}^\wedge)^* = C\,\mathrm{Aut}(((\mathbb{H}P^\infty)^\wedge)_{(0)})$ where $b_2 = -1$ and $b_p = +1$ for all odd primes. Suppose that there existed a rational equivalence between Y and $\mathbb{H}P^\infty$. Then there exists a non-zero integer r such that $-r$ is a square in $\mathbb{Z}_2^\wedge$ and r is a square in $\mathbb{Z}_p^\wedge$ for $p > 2$. Write $r = 2^i t$ where t is an odd integer. Since $-r = 2^i \cdot (-t)$ is a square in $\mathbb{Z}_2^\wedge$, i is even and $t \equiv -1 \pmod 8$ by [S, Theorem II.4]. On the other hand, write $t = s^j \varepsilon$ where s is a prime divisor of t and $(s, \varepsilon) = 1$. Choose (Lemma 2.9) an odd prime p such that t is a unit in $\mathbb{Z}_p^\wedge$ and $\left(\frac{\varepsilon}{p}\right) = +1 = -\left(\frac{s}{p}\right)$. Then

$$(-1)^j = \left(\frac{s}{p}\right) = \left(\frac{2^{-i}\varepsilon^{-1} r}{p}\right) = +1$$

so j is even. By picking out the prime divisors in t one by one in this manner we conclude that t is a square integer contradicting $t \equiv -1 \pmod 8$.

Let α be any map, say, from $\mathbb{H}P^\infty$ to Y. Then $\alpha_p^\wedge \simeq *$ for the self-map $g_p \circ \alpha_p^\wedge$ on $(\mathbb{H}P^\infty)_p^\wedge$ is by the above (rationally) trivial for all p. Hence $\alpha \simeq *$. $\square$

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