Journal of Pure and Applied Algebra 73 (1991) 23–37
North-Holland

Self-homotopy equivalences of group cohomology spaces

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Communicated by J.D. Stasheff
Received 18 July 1988
Revised 20 February 1989

Abstract

Møller, J.M., Self-homotopy equivalences of group cohomology spaces, Journal of Pure and Applied Algebra 73 (1991) 23–37.

The group of homotopy classes of self-homotopy equivalences of a space with only two nonvanishing homotopy groups is computed by means of a differential in a Lyndon spectral sequence.

1. Introduction

For any (based) connected topological space X, let $(\varepsilon(X,*))$ $\varepsilon(X)$ denote the group of (based) homotopy classes of (based) self-homotopy equivalences of X.

Suppose that π is a group and $\theta \in H^{n+1}(\pi, A)$ an $(n+1)$-dimensional, $n \geq 2$, cohomology class of π with coefficients in the $\mathbb{Z}(\pi)$-module A. Associate to θ the unique homotopy type X_θ with $\pi_1(X_\theta) = \pi$, $\pi_n(X_\theta) = A$ as a π-module, $\pi_i(X_\theta) = 0$ for $i \neq 1, n$, and with k-invariant $\theta \in H^{n+1}(\pi, A)$.

The purpose of this paper is to investigate the groups $\varepsilon(X_\theta, *)$ and $\varepsilon(X_\theta)$. In short, the computation of $\varepsilon(X_\theta, *)$ is reduced, or translated, to the computation of a differential $d_2^{0,n+1}$ in a Lyndon spectral sequence naturally associated with the given data. The group $\varepsilon(X_\theta)$ has a similar, although more complicated and less transparent description, but can also be viewed as a quotient group of $\varepsilon(X_\theta, *)$ with respect to a homomorphic image of π.

These results can also be interpreted as the complete computation of the group of self-homotopy equivalences for any finite-dimensional CW-complex with only one nontrivial homotopy group, apart from possibly the fundamental group, below the dimension of the space.

The special case $\theta = 0$, which is much simpler than the general case, was also treated in [10].

Partial results on the group $\varepsilon(X_\theta, *)$ already occur in the literature [1, 7, 11, 12].

2. Reduction of problem

Throughout this paper, let π be a (not necessarily abelian) group, A a π-module, and $\theta \in H^{n+1}(\pi, A)$, $n \geq 2$, some fixed group cohomology class.

Let $\omega : E\pi \to B\pi$ be the functorial Milnor construction [4] of a universal numerable principal π-bundle. For any group endomorphism $\alpha : \pi \to \pi$, let $E\alpha : E\pi \to E\pi$ and $B\alpha : B\pi \to B\pi$ be the induced maps. Choose base points $e_0 \in E\pi$ and $b_0 = \omega(e_0) \in B\pi$ that are preserved by all these induced maps.

The Eilenberg-Mac Lane complex $K(A, n+1)$ has a topological realization as a topological (left) π-module [8]. We shall write the topological group structure as addition. Consider also the loop space $\Omega K(A, n+1)$ and the path space $PK(A, n+1)$, based at $0 \in K(A, n+1)$, with their inherited topological π-module structures: If $u, v \in PK(A, n+1)$ are paths, then $u + v$ is the path $(u+v)(t) = u(t) + v(t)$, $t \in I = [0, 1]$, and ηu, for $\eta \in \pi$, is the path $(\eta u)(t) = \eta u(t)$. Equip the product $E\pi \times PK(A, n+1)$ with the usual π-action: $(e, u)\eta = (e\eta, \eta^{-1}u)$, $e \in E\pi$, $u \in PK(A, n+1)$, $\eta \in \pi$.

Now realise the cohomology class θ as a π-equivariant map $\theta : E\pi \to K(A, n+1)$ and let

$$X_\theta = \{(e, u)\pi \in E\pi \times_\pi PK(A, n+1) \mid \theta(e) = u(1)\} .$$

Then there is a fibration

$$\Omega K(A, n+1) \to X_\theta \xrightarrow{p} B\pi$$

and X_θ is a concrete manifestation of the homotopy type with θ as its only nonzero k-invariant.

For any map $u : K \to Y$ between topological spaces X and Y and any subspace $A \subset X$, denote by $F_u(X, A; Y)$ the space, equipped with the compactly generated topology associated to the compact-open topology [13], of all maps $v : X \to Y$ with $v|_A = u|_A$. The group $\varepsilon(X, *)$ is the group of invertible elements in the monoid $\pi_0 F_1(X, *; X)$ of homotopy classes of based self-maps on X.

Consider the pull back diagrams

$$\begin{array}{ccc} \mathscr{F}_1(X_\theta, *; X_\theta) & \to & F_1(X_\theta, *; X_\theta) \\ \downarrow & & \downarrow{\scriptstyle p} \\ F_1(B\pi, b_0; B\pi) & \xrightarrow{\bar{p}} & F_p(X_\theta, *; B\pi) \end{array} \qquad \begin{array}{ccc} \mathscr{F}_1(X_\theta; X_\theta) & \to & F_1(X_\theta; X_\theta) \\ \downarrow & & \downarrow{\scriptstyle p} \\ F_1(B\pi; B\pi) & \xrightarrow{\bar{p}} & F_p(X_\theta; B\pi) . \end{array}$$

where $(\underline{p})\ \bar{p}$ is (pre-) post-composition with p and $(\mathscr{F}_1(X_\theta, *; X_\theta))\ \mathscr{F}_1(X_\theta; X_\theta)$ is the space of all (based) fibre maps of X_θ. The two horizontal maps $\bar{p}$ are weak homotopy equivalences, see Corollary 2.3 below, the vertical maps $\underline{p}$ are fibrations, and hence the inclusions

$$\mathscr{F}_1(X_\theta, *; X_\theta) \subset F_1(X_\theta, *; X_\theta)\,, \qquad \mathscr{F}_1(X_\theta; X_\theta) \subset F_1(X_\theta; X_\theta)$$

are weak homotopy equivalences, which means that we can consider $(\varepsilon(X_\theta, *))$ $\varepsilon(X_\theta)$ as the group of (based) fibre homotopy classes of (based) fibre homotopy equivalences of X_θ. Moreover, there is a fibration

$$\mathscr{F}_1(X_\theta; X_\theta) \to B\pi$$

which records where the fibre goes under the free fibre maps of X_θ. The fibre, $\mathscr{F}_*$, of this fibration contains $\mathscr{F}_1(X_\theta, *; X_\theta)$ and from [9, Theorem 3.1] one deduces that the inclusion induces an isomorphism on π_i for $0 \le i < n$. In particular, $\pi = \pi_1(B\pi)$ acts on $\pi_0\mathscr{F}_1(X_\theta, *; X_\theta)$ with set of orbits equal to $\pi_0\mathscr{F}_1(X_\theta; X_\theta)$. Thus to compute $\varepsilon(X_\theta)$, it suffices to compute $\varepsilon(X_\theta, *)$ with π-action. Furthermore, the computation of $\varepsilon(X_\theta, *)$ is facilitated by the fact that the base space $F_1(B\pi, *; B\pi)$ is weakly equivalent, see Lemma 2.2 below, to the discrete monoid $\mathrm{End}(\pi)$ of group endomorphisms of π. However, not all self-maps of $B\pi$ can be covered by a fibre map.

Lemma 2.1. *Let $f: X_\theta \to X_\theta$ be a based fibre map. Then $\pi_n(f) \in \mathrm{End}(A)$ is a $\pi_1(f)$-homomorphism and*

$$\pi_1(f)^*\theta = \pi_n(f)_*\theta$$

in $H^{n+1}(\pi, \pi_1(f)^*A)$. □

Here, and in the following, the group endomorphism $\varphi: A \to A$ is called an α-homomorphism, $\alpha \in \mathrm{End}(\pi)$, if $\varphi(\eta a) = \alpha(\eta)\varphi(a)$ for all $\eta \in \pi$, $a \in A$.

The lemma follows easily from an analysis of the transgression homomorphism $\tau_\theta: E_2^{0n} \to H^{n+1}(\pi, A)$ in the Serre spectral sequence for X_θ with local coefficients in A [6].

Motivated by Lemma 2.1, we introduce E_θ to be the subgroup of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(A)$ consisting of all those pairs (α, φ) for which φ is an α-automorphism and $\varphi_*(\alpha^{-1})^*\theta = \theta$ in $H^{n+1}(\pi, A)$. (In [5, IV], the pair (α^{-1}, φ) is called a change of groups.)

Lemma 2.1 states that $(\pi_1(f), \pi_n(f)) \in E_\theta$ for any based fibre map f. We shall later see that in fact any element of E_θ has such a topological realization; at least if $\dim X_\theta \le n$ this is well known [7].

I finish this chapter by proving two facts that were used above. For the first lemma, let X be a connected CW-complex and $u: X \to B\pi$ a map.

Lemma 2.2. *The map induced by* π_1

$$F_u(X, *; B\pi) \to \mathrm{Hom}(\pi_1(X, *), \pi)$$

is a weak homotopy equivalence.

Proof. (Cf. [2, Lemma 2] and [9, Theorem 6.3].) Let $I = [0, 1]$ denote the unit interval and $\dot{I} = \{0, 1\}$ its boundary. Without loss of generality, X has a CW-decomposition with 0-skeleton $X_0 = \{*\}$.

Elementary obstruction theory implies [13, Theorem V.4.3] that the map $f \to \pi_1(f)$ of the lemma induces a bijection on π_0.

Since $I \times S^i \times X$, $i \geq 1$, is formed from $I \times (S^i \vee X) \cup \dot{I} \times S^i \times X$ by attaching cells of dimensions $\geq i + 2 \geq 3$, any two maps $S^i \times X \to B\pi$ that agree on $S^i \vee X = S^i \times \{*\} \cup \{*\} \times X$ are homotopic (rel. $S^i \vee X$). Thus, by adjointness,

$$\pi_i F_u(X, *; B\pi) = \pi_0 F_{u \circ \mathrm{pr}_2}(S^i \times X, S^i \vee X; B\pi) = 0$$

for $i \geq 1$. $\square$

Corollary 2.3. *The two maps*

$$F_1(B\pi, b_0; B\pi) \xrightarrow{\bar{p}} F_p(X_\theta, *; B\pi), \qquad F_1(B\pi; B\pi) \xrightarrow{\bar{p}} F_p(X_\theta; B\pi)$$

are weak homotopy equivalences.

Proof. In the based case, this follows immediately from the naturality of the equivalence of Lemma 2.2. As to the free case, extend $\bar{p}$ to a map of evaluation fibrations

$$\begin{tikzcd}
F_1(B\pi, b_0; B\pi) \arrow[r, "\bar{p}"] \arrow[d] & F_p(X_\theta, *; B\pi) \arrow[d] \\
F_1(B\pi; B\pi) \arrow[r] \arrow[d] & F_p(X_\theta; B\pi) \arrow[d] \\
B\pi \arrow[r, equal] & B\pi
\end{tikzcd}$$

and look at the induced map of the associated long exact homotopy sequences. $\square$

3. Homological algebra

Recall that $\theta \in H^{n+1}(\pi, A)$, where π is any group and A any π-module. The purpose of this section is to define two group extensions related to θ.

For any group G, let $(B_*(G), \partial)$ denote the normalized bar resolution of G; $B_n(G)$ is the free abelian group generated by all symbols $g[g_1|\cdots|g_n]$, $1 \neq g_i$, $g \in G$.

The complex $\mathrm{Hom}_\pi(B_*(\pi), A)$ can be viewed as an E_θ-module complex with action

$$((\alpha, \varphi)f)(b) = \varphi f(\alpha^{-1}b)\,, \quad (\alpha, \varphi) \in E_\theta\,, \quad b \in B_*(\pi)\,,$$

where $\alpha^{-1}(x[x_1|\cdots|x_n]) = \alpha^{-1}(x)[\alpha^{-1}(x_1)|\cdots|\alpha^{-1}(x_n)]$, x, $x_i \in \pi$, and $f \in \mathrm{Hom}_\pi(B_*(\pi), A)$. Form the bicomplex

$$M^{pq} = \mathrm{Hom}_{E_\theta}(B_p(E_\theta), \mathrm{Hom}_\pi(B_q(\pi), A))$$

with respect to this action. The first filtration on M^{**} determines a first quadrant spectral sequence with

$$E_2^{pq} = H^p(E_\theta, H^q(\pi, A))$$

converging to the cohomology of $\mathrm{Tot}(M)$. In particular, there exists a differential

$$d_2 : H^{n+1}(\pi, A)^{E_\theta} \to H^2(E_\theta, H^n(\pi, A))$$

taking θ to some 2-dimensional cohomology class $d_2(\theta)$. We now describe $d_2(\theta)$ more explicitly.

Lemma 3.1. *Choose for each $(\alpha, \varphi) \in E_\theta$ a cochain $\{\alpha, \varphi\} \in \mathrm{Hom}_\pi(B_n(\pi), A)$ with $\{\alpha, \varphi\}\partial = \theta - \varphi\theta\alpha^{-1}$. Then the 2-cocycle $k_* : B_2(E_\theta) \to H^n(\pi, A)$ that takes $[\alpha, \vartheta|\beta, \psi] \in B_2(E_\theta)$ to (the cohomology class of)*

$$(\alpha, \varphi)\cdot\{\beta, \psi\} - \{\alpha\beta, \varphi\psi\} + \{\alpha, \varphi\}$$

lies (up to sign) in the cohomology class $d_2(\theta)$. □

Above, θ (also) denotes a cocycle representing $\theta \in H^n(\pi, A)$.

Remark 3.2. Form the semi-direct product $\pi \rtimes E_\theta$ with respect to the obvious action

$$E_\theta \hookrightarrow \mathrm{Aut}(\pi) \times \mathrm{Aut}(A) \xrightarrow{\mathrm{pr}_1} \mathrm{Aut}(\pi)$$

of E_θ on π. The short split-exact sequence

$$1 \to \pi \to \pi \rtimes E_\theta \rightleftarrows E_\theta \to 1$$

and the $\pi \rtimes E_\theta$-module structure on A provided by

$$(\eta, (\alpha, \varphi))a = \eta\varphi(a)\,, \quad a \in A\,,$$

generate [5, Theorem XI.10.1] a Lyndon–Hochschild–Serre spectral sequence

$$H^*(E_\theta, H^*(\pi, A)) \Rightarrow H^*(\pi \rtimes E_\theta, A)$$

isomorphic on the $E_{\geq 1}$-level to the first spectral sequence of M^{**}. In particular, $H(\mathrm{Tot}(M)) = H(\pi \times E_\theta, A)$.

For $\eta \in \pi$, let $\bar{\eta}(x) = \eta x \eta^{-1}$ denote conjugation by η. Since $(\bar{\eta}, \eta)$ acts trivially on $H^n(\pi, A)$ [5, Proposition IV.5.6], $\bar{\partial}(\eta) = (\bar{\eta}, \eta)$ is a homomorphism $\pi \to E_\theta$. The image is a normal subgroup so $E_\theta/\pi := \operatorname{coker} \bar{\partial}$ is a group. Let also $Z(\pi)_A := \ker \bar{\partial}$. Then

$$1 \to Z(\pi)_A \to \pi \xrightarrow{\bar{\partial}} E_\theta \to E_\theta/\pi \to 1$$

is an exact sequence of groups.

Next we shall see that cochains of the form $\{\bar{\eta}, \eta\}$, $\eta \in \pi$, can be constructed systematically: Let $s_n(\eta): B_n(\pi) \to B_{n+1}(\pi)$ be the $\mathbb{Z}(\pi)$-homomorphism with value

$$\sum_{i=0}^{n} (-1)^i [x_1|\cdots|x_i|\bar{\eta}^{-1}x_{i+1}|\cdots|\bar{\eta}^{-1}x_n]$$

on the free generator $[x_1|\cdots|x_n] \in B_n(\pi)$. Then we have the following lemma:

Lemma 3.3. $-\theta s_n(\eta)\partial = \theta - \eta\theta\bar{\eta}^{-1}$.

Proof. Let $g(\eta)$ be the π-module complex endomorphism of $B_*(\pi)$ with value $\eta[\bar{\eta}^{-1}x_1|\cdots|\bar{\eta}^{-1}x_n]$ on the free generator $[x_1|\cdots|x_n] \in B_n(\pi)$, cf. [5, Proposition IV.5.6]; $s(\eta)$ is a chain homotopy of the identity map to $g(\eta): \partial s(\eta) + s(\eta)\partial = g(\eta) - 1$. Thus $-\theta s(\eta)\partial = -\theta(\partial s(\eta) + s(\eta)\partial) = \theta - \theta g(\eta) = \theta - \eta\theta\bar{\eta}^{-1}$. □

In the following, we always use $\{\bar{\eta}, \eta\} = -\theta s_n(\eta)$. If $\eta \in Z(\pi)_A$, then $\{\bar{\eta}, \eta\}$ is a cocycle; denote its cohomology class by $Z_\theta(\eta)$.

Lemma 3.4. *The function $Z_\theta: Z(\pi)_A \to H^n(\pi, A)$ is a group homomorphism.*

Proof. For given group elements $\zeta, \eta \in \pi$, let $s(\zeta, \eta)$ be he degree-2 π-module complex homomorphism with value

$$\sum_{1\le i<j\le n} (-1)^{i+j+1}[x_1|\cdots|x_{i-1}|\zeta|\bar{\zeta}^{-1}x_i|$$
$$\cdots|\bar{\zeta}^{-1}x_{j-1}|\eta|\bar{\eta}^{-1}\bar{\zeta}^{-1}x_j|\cdots|\bar{\eta}^{-1}\bar{\zeta}^{-1}x_n]$$

on the free generator $[x_1|\cdots|x_n] \in B_n(\pi)$. Then

$$\partial s(\zeta, \eta) - s(\zeta, \eta)\partial = s(\eta)g(\zeta) - s(\zeta\eta) + s(\zeta)$$

and hence the n-cochain $-\theta s(\eta)g(\zeta) + \theta s(\zeta\eta) - \theta s(\zeta)$ is a coboundary. If $\zeta \in Z(\pi)_A$, $\theta s(\eta)g(\zeta) = (\bar{\zeta}, \zeta)\cdot\theta s(\eta) = \theta s(\eta)$ so that if also $\eta \in Z(\pi)_A$, we deduce that $0 = Z_\theta(\eta) - Z_\theta(\zeta\eta) + Z_\theta(\zeta)$. □

Lemma 3.5. *For $\eta \in Z(\pi)_A$ and $(\alpha, \varphi) \in E_\theta$, $(\alpha, \varphi)Z_\theta(\eta) = Z_\theta(\alpha(\eta))$.*

Proof. The cocycle

$$\begin{aligned}
\varphi\{\bar{\eta}, \eta\}\alpha^{-1} &= -\varphi\theta s_n(\eta)\alpha^{-1} = -\varphi\theta\alpha^{-1}s_n(\alpha(\eta)) \\
&= (-\theta + \{\alpha, \varphi\}\partial)s_n(\alpha(\eta)) \\
&= -\theta s_n(\alpha(\eta)) - \{\alpha, \varphi\}s_n(\alpha(\eta))\partial \\
&= \{\overline{\alpha(\eta)}, \alpha(\eta)\} - \{\alpha, \varphi\}s_n(\alpha(\eta))\partial
\end{aligned}$$

is cohomologous to $\{\overline{\alpha(\eta)}, \alpha(\eta)\}$. □

Lemma 3.5 shows that the action of E_θ/π on $H^n(\pi, A)$ descends to one of the quotient $H^n(\pi, A)/Z_\theta$, where Z_θ is short for $Z_\theta(Z(\pi)_A)$, and we shall now define a certain 2-dimensional cohomology class of E_θ/π with coefficients in this E_θ/π-module.

For each orbit $x \in E_\theta/\pi$, choose a representative $(\alpha_x, \varphi_x) \in x \subset E_\theta$. Consider the E_θ/π-module homomorphism

$$k : B_2(E_\theta/\pi) \to H^n(\pi, A)/Z_\theta\,,$$

whose value on the tree generator $[x|y] \in B_2(E_\theta/\pi)$ is

$$k[x|y] = k_*[\alpha_x, \varphi_x|\alpha_y, \varphi_y] - k_*[\alpha_x\alpha_y\alpha_{xy}^{-1}, \varphi_x\varphi_y\varphi_{xy}^{-1}|\alpha_{xy}, \varphi_{xy}]$$

where k_* is the cocycle of Lemma 3.1.

Lemma 3.6. *The cochain k is a cocycle whose cohomology class in $H^2(E_\theta/\pi, H^n(\pi, A)/Z_\theta)$ is independent of the choices made.*

The proof of Lemma 3.6 is deferred until Section 5.

Denote the cohomology classes $d_2(\theta) = [k_*]$ and $[k]$ by $[\theta]_*$ and $[\theta]$, respectively. The main idea of this paper is to show that the group ($\varepsilon(X_\theta, *)$) $\varepsilon(X_\theta)$ of (based) homotopy equivalences of X_θ is the middle term of a group extension in the congruence class corresponding [5, Theorem IV.4.1] to ($[\theta]_*$) $[\theta]$.

4. Based homotopy equivalences

The purpose of this section is to determine the group $\varepsilon(X_\theta, *)$ of based homotopy classes of based homotopy equivalences of X_θ. As shown in Section 2, we may identify $\varepsilon(X_\theta, *)$ with the group of invertible elements in the monoid $\pi_0\mathcal{F}_1(X_\theta, *; X_\theta)$ of based fibre homotopy equivalence classes of based fibre maps of X_θ.

Consider the obvious homomorphism

$$(\pi_1, \pi_n): \varepsilon(X_\theta, *) \to \mathrm{Aut}(\pi) \times \mathrm{Aut}(A)$$

induced by the functors π_1 and π_n. Lemma 2.1 asserts that the image is contained in the subgroup E_θ. We now prove that, in fact, we have the following:

Lemma 4.1. *The image of (π_1, π_n) is E_θ.*

Proof. Suppose $(\alpha, \varphi) \in E_\theta$. Then

$$\theta - \varphi\theta \circ E\alpha^{-1} : (E\pi, e_0) \to (K(A, n+1), 0)$$

is π-homotopic to the constant map. Let

$$\{\alpha, \varphi\} : (E\pi, e_0) \to (PK(A, n+1), 0)$$

be the adjoint of any such π-homotopy such that $\{\alpha, \varphi\}(e)(1) - \theta e - \varphi\theta E\alpha^{-1}(e)$ for all $e \in E\pi$. Then the based fibre map $[\alpha, \varphi]$ that takes $(e, u)\pi \in X_\theta$ to

$$(E\alpha(e), \varphi u + \{\alpha, \varphi\}E\alpha(e))\pi$$

projects onto (α, φ) by (π_1, π_n). $\square$

Lemma 4.2. *The kernel of (π_1, π_n) is $H^n(\pi, A)$.*

Proof. The kernel K consists of based $B\pi$-homotopy classes of based $B\pi$-maps $k: X_\theta \to X_\theta$ with $\pi_n(k) = 1$. Associate to any such k the primary difference $\delta^n(k, 1) \in H^n(X_\theta; A)$ of k and the identity; see [13, VI.6]. In the diagram

$$\begin{array}{ccccc} 0 \to H^n(\pi, A) \to & H^n(X_\theta; A) & \xrightarrow{i^*} \mathrm{Hom}_\pi(A, A) \\ & \uparrow{\scriptstyle \delta^n(-,1)} & \\ & K & \end{array}$$

the row, coming from the Serre spectral sequence, is exact and the vertical arrow is an injective map which, since $i^*\delta^n(k, 1) = \pi_n(k) - 1 = 1 - 1 = 0$, in fact takes K into $H^n(\pi, A)$.

Conversely, let $x: E\pi \to \Omega K(A, n+1)$ be a π-map and $\bar{x}: X_\theta \to X_\theta$ the $B\pi$-map given by

$$\bar{x}((e, u)\pi) = (e, u + x(e))\pi .$$

Then $\delta^n(\bar{x}, 1) = x \in H^n(\pi, A)$. Consequently, $\delta^n(-, 1)$ is a bijection and a homomorphism. □

The results of Lemma 4.1 and Lemma 4.2 are well known at least if $\dim X_\theta \le n$ [7], see also [1, 12]. However, the method used here seems to be more direct than the ones used elsewhere. Another reason for including proofs here is that the applied notation will be used extensively in the following.

The preceding two lemmas show the existence of an extension

$$[\theta]_* : 0 \to H^n(\pi, A) \to \varepsilon(X_\theta, *) \to E_\theta \to 1$$

of $H^n(\pi, A)$ by E_θ with middle term $\varepsilon(X_\theta, *)$. Moreover, if $f: X_\theta \to X_\theta$ is a fibre homotopy equivalence and $k: X_\theta \to X_\theta$ a $B\pi$-map, then

$$\pi_1(f)^* \delta^n(fkf^{-1}, 1) = \delta^n(fk, f) = \pi_n(f)_* \delta^n(k, 1),$$

showing that the associated action of E_θ on $H^n(\pi, A)$ is exactly the one used in Section 3.

The main result of this paper now asserts the following:

Theorem 4.3. *The above extension with middle term $\varepsilon(X_\theta, *)$ belongs to the congruence class $[\theta]_* \in H^2(E_\theta, H^n(\pi, A))$.*

Proof. The idea is to construct a factor set for this topologically defined extension and compare it to the 2-cocycle of Lemma 3.1.

For all $(\alpha, \varphi) \in E_\theta$, choose a π-map, or a π-homotopy,

$$\{\alpha, \varphi\} : E\pi \to PK(A, n+1)$$

as in the proof of Lemma 4.1. The associated fibre homotopy equivalences then satisfy the identity

$$[\alpha, \varphi] \circ [\beta, \psi] = \bar{k}_*[\alpha, \varphi \mid \beta, \psi] \circ [\alpha\beta, \varphi\psi]$$

where

$$\begin{aligned} &k_*[\alpha, \varphi \mid \beta, \psi] \\ &\quad = \varphi\{\beta, \psi\}\alpha^{-1} - \{\alpha\beta, \varphi\psi\} + \{\alpha, \varphi\} : E\pi \to \Omega K(A, n+1) \end{aligned}$$

and $\bar{k}_*[\alpha, \varphi \mid \beta, \psi]$ the associated $B\pi$-map as in the proof of Lemma 4.2.

For $b \in B_n(\pi)$ and $i \in \Gamma_1(I)$ the standard 1-cell of the unit interval I, put

$$\{\alpha, \varphi\}(b) = d^{n+1}(0, \{\alpha, \varphi\})(i \otimes b)$$

where $d^{n+1}(0, \{\alpha, \varphi\}) \in \operatorname{Hom}_\pi(\Gamma_*(I) \otimes B_*(\pi), A)$ is the difference cocycle [13, Theorem VI.5.6] of (the sections associated to) the π-maps 0 and $\{\alpha, \varphi\}$. Then [13, Theorem V.5.6′] $\{\alpha, \varphi\}\partial = \theta - \varphi\theta\alpha^{-1}$. Hence the factor set [5, IV.1] for the extension $[\theta]_*$,

$$[\alpha, \varphi \mid \beta, \psi] \to \delta^n(1, \bar{k}_*[\alpha, \varphi \mid \beta, \psi]),$$

with the primary difference represented by the n-cocycle taking $b \in B_n(\pi)$ to

$$\begin{aligned} &d^{n+1}(0, k_*[\alpha, \varphi \mid \beta, \varphi])(i \otimes b) \\ &\quad = (\varphi\{\beta, \psi\}\alpha^{-1} - \{\alpha\beta, \varphi\psi\} + \{\alpha, \varphi\})(b), \end{aligned}$$

is up to sign a 2-cocycle as described in Lemma 3.1. □

If in particular $\theta = 0$, $d_2(\theta) = 0$, and hence

$$\varepsilon(X_0, *) = H^n(\pi, A) \rtimes E_0$$

in accordance with [10].

It is much easier to compute the higher homotopy groups [9, Theorem 3.1]: $\pi_i(F_1(X_\theta, *; X_\theta), f) = \bar{H}^{n-i}(\pi, \pi_1(f)^*A)$ for $i > 0$.

5. Free homotopy equivalences

Recall from Section 2 that to determine the group $\varepsilon(X_\theta)$ of free homotopy classes of free homotopy equivalence of X_θ it suffices to find the left π-action on $\pi_0\mathcal{F}_1(X_\theta, *; X_\theta)$ arising from the fibration $\mathcal{F}_1(X_\theta, X_\theta) \to B$ whose fibre is $(n-1)$-equivalent to $\mathcal{F}_1(X_\theta, *; X_\theta)$.

We shall need a systematic way of constructing the fibre maps $\{\bar{\eta}, \eta\}$, $\eta \in \pi$. This can be done by using [10, Lemma 2.1] quoted here as Lemma 5.1.

Lemma 5.1. *There exist maps* $\bar{\mu}: E\pi \times E\pi \to E\pi$ *and* $\mu: E\pi \times B\pi \to B\pi$ *such that* $\omega\bar{\mu} = \mu(1 \times \omega)$ *and*

(1) $\bar{\mu}(e_0\eta, e) = (E(\bar{\eta})e)\eta$, $\bar{\mu}(e, e_0\eta) = e\eta$,

(2) $\bar{\mu}(e_1\eta, E(\bar{\eta})^{-1}e) = \bar{\mu}(e_1, e_2)\eta$,

(3) $\bar{\mu}(e_1, e_2\eta) = \bar{\mu}(e_1, e_2)\eta$,

for all $e_1, e_2, e_3 \in E\pi$ *and* $\eta \in \pi$. $\square$

For any $\eta \in \pi$, choose now a path $\sigma(\eta): I \to E\pi$ from $e_0\eta$ to e_0. (Note that $\sigma(\eta)$ is unique up to a homotopy with fixed end-points and that we can assume that $\theta\sigma(\eta)(I) = \{0\}$ for if θ is chosen as a π-fibration, then the π-space $\theta^{-1}(0)$ is connected.) Define

$$\{\bar{\eta}, \eta\}: E\pi \to PK(A, n+1)$$

to be the equivariant function

$$\{\bar{\eta}, \eta\}(e)(t) = \theta s(\eta)(t, e) - \theta s(\eta)(0, e)\,, \quad t \in I\,,\ e \in E\pi\,,$$

where $s(\eta)(t, e) = \bar{\mu}(\sigma(\eta^{-1})(t), e)$. (Note that $s(\eta)$ is a homotopy of $g(\eta)e = (E(\bar{\eta})^{-1}e)\eta^{-1}$ to the identity and that the inverse of the homotopy $\{\bar{\eta}, \eta\}$ via the prism operator [3, 5] induces the chain homotopy $\{\bar{\eta}, \eta\}$ of Section 3.) As in Section 4, let

$$[\bar{\eta}, \eta](e, u)\pi = (E(\bar{\eta})e, \eta u + \{\bar{\eta}, \eta\}E(\bar{\eta})e)\pi\,, \quad (e, u) \in X_\theta\,,$$

be the associated fibre self map of X_θ.

Lemma 5.2. *The action of* $\eta \in \pi$ *on* $\pi_0\mathcal{F}_1(X_\theta, *; X_\theta)$ *is induced by post-composition with* $[\bar{\eta}, \eta]$.

Proof. For $s, t \in I = [0, 1]$, put $\sigma(\eta)_t(s) = \sigma(\eta)(s(t-1)+1)$. From the essential uniqueness of the path $\sigma(\eta)$ and (2) of Lemma 5.1 it follows that the fibre map $[\bar{\eta}, \eta]$ is fibre homotopic to the fibre map

$$(e, u)\pi \to (E(\bar{\eta})e, \eta(u + \theta\bar{\mu}(\sigma(\eta)_0, e) - \theta e))\pi$$

of X_θ.

For $t \in I$ and $(e, u)\pi \in X_\theta$, let now $E_t : X_\theta \to X_\theta$ be the fibre map

$$E_t((e, u)\pi) = (\bar{\mu}(\sigma(\eta)(t), e), u + \theta\bar{\mu}(\sigma(\eta)_t, e) - \theta e)\pi .$$

Then E_1 is the identity, $E_0 \simeq [\bar{\eta}, \eta]$, and $pE_t(*) = \omega\sigma(\eta)(t)$ represents the homotopy class $\eta \in \pi = \pi_1(B\pi)$. In other words, the homotopy

$$I \times \mathcal{F}_1(X_\theta, *; X_\theta) \to \mathcal{F}_1(X_\theta, X_\theta) ,$$

obtained by post-composition with the E_t's, $t \in I$, is a homotopy over η and thus post-composition with $E_0 = [\bar{\eta}, \eta]$ is an η-admissible map; see [13, IV] for the terminology used here. □

The end of the exact homotopy sequence for the fibration $\mathcal{F}_1(X_\theta; X_\theta) \to B\pi$ contains an exact sequence of sets

$$\pi \xrightarrow{\partial} \varepsilon(X_\theta, *) \xrightarrow{i_*} \varepsilon(X_\theta) \to 1$$

where $\partial\eta = [\bar{\eta}, \eta] = \eta \cdot 1$.

Lemma 5.3. *The above sequence is an exact sequence of groups.*

Proof. Clearly, the forgetful map i_* is a group homomorphism. Using Lemma 5.2 and the fact that the group π acts on the set $\varepsilon(X_\theta, *)$ we find that

$$\begin{aligned}\partial(\zeta\eta) &= [\bar{\zeta}\bar{\eta}, \zeta\eta] = (\zeta\eta) \cdot 1 = \zeta \cdot (\eta \cdot 1) = \zeta \cdot [\bar{\eta}, \eta] \\ &= [\bar{\zeta}, \zeta] \circ [\bar{\eta}, \eta] = (\partial\zeta) \circ (\partial\eta)\end{aligned}$$

for all $\zeta, \eta \in \pi$; i.e., also ∂ is a homomorphism. □

Thus $\operatorname{im}\partial$ is a normal subgroup of $\varepsilon(X_\theta, *)$ and the quotient group $\varepsilon(X_\theta, *)/\operatorname{im}\partial$ is isomorphic to $\varepsilon(X_\theta)$. Since $\varepsilon(X_\theta, *)$ is the middle term of the short exact sequence $[\theta]_*$, it follows that the quotient $\varepsilon(X_\theta)$ is the middle term of a short exact sequence of the form

$$[\theta]: \quad 0 \to H^n(\pi, A)/Z_\theta \to \varepsilon(X_\theta) \to E_\theta/\pi \to 1$$

because of the following lemma:

Lemma 5.4. $(\pi_1, \pi_n)\partial = \bar{\partial}$ *and* $H^n(\pi, A) \cap \partial(\pi) = Z_\theta(Z(\pi)_A)$.

Proof. Let $\eta \in \pi$. By constructing, $(\pi_1, \pi_n)\partial(\eta) = (\pi_1, \pi_n)[\bar{\eta}, \eta] = (\bar{\eta}, \eta) = \bar{\partial}(\eta)$, and thus $\partial(\eta) \in H^n(\pi, A) = \ker(\pi_1, \pi_n)$ iff $\eta \in Z(\pi)_A$ and then $\delta^n([\bar{\eta}, \eta], 1) = \{\bar{\eta}, \eta\} = Z_\theta(\eta)$. □

The justification for christening this short exact sequence $[\theta]$ is the following:

Theorem 5.5. *The above group extension belongs to congruence class* $[\theta] \in H^2(E_\theta/\pi, H^n(\pi, A)/Z_\theta)$.

Proof of Lemma 3.6 and Theorem 5.5. Consider the quotient map $[\theta]_* \to [\theta]$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(\pi, A) & \longrightarrow & \varepsilon(X_\theta, *) & \longrightarrow & E_\theta & \rightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H^n(\pi, A)/Z_\theta & \rightarrow & \varepsilon(X_\theta, *)/\pi & \rightarrow & E_\theta/\pi & \rightarrow & 1 \end{array}$$

Choose for each orbit $x \in E_\theta/\pi$ a representative $(\alpha_x, \varphi_x) \in E_\theta$ for x. Define $k[x|y] \in H^n(\pi, A)$, or $k[x|y]: E\pi \to \Omega K(A, n+1)$, to be the solution to the equation

$$[\alpha_x, \varphi_x] \circ [\alpha_y, \varphi_y] = [\bar{\eta}, \eta] \circ \bar{k}[x|y] \circ [\alpha_{xy}, \varphi_{xy}] \tag{1}$$

where $\eta \in \pi$ and $(\bar{\eta}, \eta)(\alpha_{xy}, \varphi_{xy}) = (\alpha_x, \varphi_x)(\alpha_y, \varphi_y)$. Then $k: B_2(E_\theta/\pi) \to H^n(\pi, A)/Z_\theta$ is obviously a factor set [5, IV.4] for the extension $[\theta]$.

To finish the proof, check that (1) is satisfied with

$$k[x|y] = \eta^{-1}(k_*[\alpha_x, \varphi_x|\alpha_y, \varphi_y] - k_*[\alpha_x\alpha_y\alpha_{xy}^{-1}, \varphi_x\varphi_y\varphi_{xy}^{-1}|\alpha_{xy}, \varphi_{xy}])E(\bar{\eta})$$

and note that the self-map $e \to (E(\bar{\eta})e)\eta$ of $E\pi$ can be disregarded since it is π-homotopic to the identity. □

Note finally that the material presented above covers the worst case among all 2-stage Postnikov systems.

To make this remark precise, let X_k be homotopy type with only one nonzero k-invariant $k \in H^{n+1}(\pi, m; A)$ where $n > m > 1$ are integers and π and A are abelian groups. As before, $\mathrm{Aut}(\pi) \times \mathrm{Aut}(A)$ acts on the singular cochain complex $S^*(K(\pi, m), A)$ and on the graded ring $H^*(\pi, m; A)$. Let E_k be the stabilizer of k. The double complex

$$\mathrm{Hom}_{E_k}(B_*(E_k), S^*(K(\pi, m), A)),$$

produces a spectral sequence with

$$E_2^{pq} = H^p(E_k, H^q(\pi, m; A))$$

and a differential

$$d_2^{0,n+1}: H^{n+1}(\pi, m; A)^{E_k} \to H^2(E_k, H^n(\pi, m; A)).$$

The considerations of the first sections of this paper can now be repeated, and this time in a simpler fashion since π is abelian and there is no π-module structure on A, to give the following theorem:

Theorem 5.6. *There exists a short exact sequence*

$$0 \to H^n(\pi, m; A) \to \varepsilon(X_k, *) \to E_k \to 1$$

in the congruence class $d_2^{0,n+1}(k)$; *furthermore*, $\varepsilon(X_k, *) = \varepsilon(X_k)$. □

The last remark of this theorem follows from the stronger statement that the inclusion map

$$F_1(X_k, *; X_k) \subset F_1(X_k; X_k)$$

is $(m-1)$-connected.

The existence of the short exact sequence of Theorem 5.6 was apparently first proved in [11].

Acknowledgment

The recognition of the 2-cocycle k_* from the Lemma 3.1 as a differential image of θ is due to Anders Thorup.

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