EXTENSIONS OF p-COMPACT GROUPS

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Abstract. The classification of short exact sequences of p-compact groups and of rational isomorphisms of not necessarily connected p-compact groups is discussed.

1. Introduction

The concept of a p-compact group was introduced by Dwyer and Wilkerson [8] as a homotopy theoretic version of a compact Lie group. In a subsequent paper [7], they showed that the center of any p-compact group agrees with the centralizer of the identity map. That result is the starting point of this note.

For given p-compact groups, X and Y, let $\mathrm{Ext}(X,Y)$ denote the set of equivalence classes of short exact sequences

$$Y \to G \to X$$

of p-compact groups. Two such extensions of X by Y are declared equivalent if there exists a homomorphism over X and under Y between them.

The discussion of $\mathrm{Ext}(X,Y)$ proceeds along two parallel tracks. One track is concerned with the case where Y is a (completely reducible [12]) connected p-compact group while the other track deals with the case where $Y = Z$ is an abelian p-compact (toral) group. For fixed homotopy actions ρ and ζ of $\pi_0(X)$ on Y and Z let $\mathrm{Ext}_\rho(X,Y) \subseteq \mathrm{Ext}(X,Y)$ and $\mathrm{Ext}_\zeta(X,Z) \subseteq \mathrm{Ext}(X,Z)$ denote the subsets of extensions realizing the actions ρ and ζ, respectively. As is quickly seen, $\mathrm{Ext}_\zeta(X,Z)$ is an abelian group and it turns out [Theorem 3.4] that $\mathrm{Ext}_\rho(X,Y)$ is an affine group with $\mathrm{Ext}_{Z\rho}(X,Z(Y))$ as group of operators. Here Z denotes the conjugation action of the group of self-homotopy equivalences of BY on the classifying space $BZ(Y) = \mathrm{map}(BY,BY)_{B1}$ [7, Theorem 1.3] of the center $Z(Y)$ of Y [13, 7].

The abelian group $\mathrm{Ext}_\zeta(X,Z)$ enjoys nice bifunctorial properties. The affine group $\mathrm{Ext}_\rho(X,Y)$ is functorial in X by pull back but only restricted functorial in Y: Any equivariant rational isomorphism $g\colon Y \to Y'$, where Y' is a

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connected p-compact group locally isomorphic to Y equipped with a homotopy action ρ' by $\pi_0(X)$, induces a push forward map

$$g_*\colon \operatorname{Ext}_\rho(X,Y) \to \operatorname{Ext}_{\rho'}(X,Y')$$

which is affine [Lemma 3.8]. Given also a homomorphism $h\colon X \to X'$, pull back and push forward of extension classes provide obstructions to the existence of homomorphisms

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & G \arrow[r] \arrow[d, dashed] & X \arrow[d, "h"] \\ Y' \arrow[r] & G' \arrow[r] & X' \end{tikzcd}$$

under g and over h. Indeed, such a homomorphism exists [Theorem 3.9] if and only if $g_*(G) = h^*(G')$ in $\operatorname{Ext}(X,Y')$.

Fibrewise discrete approximations to fibered abelian p-compact groups are briefly discussed in Section 4. Lemma 4.1–4.3 show that provided the identity component X_0 of X is simply connected, there is a group isomorphism

$$\operatorname{Ext}_\zeta(X,Z) \cong H^2_\zeta(\pi_0(X);\check{Z})$$

where $\check{Z}$ is the discrete approximation to Z.

The above concepts are exploited in the final section for the classification of rational automorphisms of not necessarily connected p-compact groups. When combined with [10, Theorem 4.3] [12, Theorem 3.5], the short exact sequence of Theorem 5.2 could potentially lead to a fairly explicit classification of rational automorphisms of any given p-compact group.

2. Universal fibrations

Thanks to the homotopy equivalence [7, Theorem 1.3] between the center and the centralizer of the identity map of a p-compact group, the classification of fibrations with p-compact group classifying spaces as fibres is surprisingly manageable.

Let's first fix some notation. For any two p-compact groups, X and X', put $\operatorname{Hom}(X,X') = [BX,*;BX']$, the set of based homotopy classes of maps, and $\operatorname{Rep}(X,X') = [BX,BX']$, the set of unbased homotopy classes of maps. A homomorphism $h \in \operatorname{Hom}(X,X')$ is said to be a rational isomorphism if [10, Definition 2.1] the map

$$H^*(Bh_0;\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\colon H^*(BX_0';\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \to H^*(BX_0;\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p,$$

induced by the restriction $Bh_0\colon BX_0 \to BX_0'$ of h to the identity components, is an isomorphism. Let $\varepsilon_{\mathbb{Q}}(X,X') \subseteq \operatorname{Hom}(X,X')$ denote the subset of rational isomorphisms.

If $X = X'$, $\operatorname{End}(X) = \operatorname{Hom}(X,X)$ is a monoid (under composition) containing $\varepsilon_{\mathbb{Q}}(X) = \varepsilon_{\mathbb{Q}}(X,X)$ as a submonoid and having $\operatorname{Aut}(X)$ as its group of invertible elements. $\operatorname{Out}(X)$ denotes the invertible elements of the monoid $\mathrm{Rep}(X,X)$. If X is connected or abelian, BX is simple so there is no difference between the based or unbased case: $\mathrm{End}(X)=\mathrm{Rep}(X,X)$ and $\mathrm{Aut}(X)=\mathrm{Out}(X)$.

Turning to classifying fibrations, let Y be a p-compact group with center [13, 7] $Z(Y)$ and adjoint form $P(Y)=Y/Z(Y)$. Then $Z(Y)$ is an abelian p-compact toral group and there exists a fibration

$$BZ(Y)\to BY\to BP(Y)$$

of classifying spaces. Using a Borel construction as in the proof of [8, Proposition 8.3] this fibration may be extended one step further to the right to give a fibration

$$BY\to BP(Y)\xrightarrow{Bk} B^2Z(Y) \tag{1}$$

which is universal for fibrations with fibre BY over simply connected base spaces [7, Remark 1.11].

Assume from now on that Y is *connected* and let $g\colon Y\to Y'$ be a rational isomorphism into another connected p-compact group Y' locally isomorphic [10, Definition 2.7] to Y. Then g induces [10, Corollary 3.2, Theorem 3.3] a fibre map

$$\begin{tikzcd} BZ(Y) \arrow[r] \arrow[d, "BZ(g)"'] & BY \arrow[r] \arrow[d, "Bg"'] & BP(Y) \arrow[d, "BP(g)"'] \\ BZ(Y') \arrow[r] & BY' \arrow[r] & BP(Y') \end{tikzcd}$$

which also extends one step to the right.

Lemma 2.1. *Any rational automorphism g of Y extends to a fibre self map*

$$\begin{tikzcd} BY \arrow[r] \arrow[d, "Bg"'] & BP(Y) \arrow[r, "Bk"] \arrow[d, "BP(g)"'] & B^2Z(Y) \arrow[d, "B^2Z(g)"] \\ BY' \arrow[r] & BP(Y') \arrow[r, "Bk"'] & B^2Z(Y') \end{tikzcd}$$

of the universal fibration (1).

Proof. The claim is that $B^2Z(g)\circ Bk$ and $Bk\circ BP(g)$ are homotopic. Since looping provides a bijection $\Omega\colon [BP(Y),B^2Z(Y')]\to[P(Y),BZ(Y')]$, this follows from the extension one step to the left of the fibre map $(BZ(g),Bg,BP(g))$ shown above. $\square$

The fibration which is universal for fibrations with fibre BY over arbitrary base spaces has the form

$$BY\to BP(Y)_{\mathrm{hOut}(Y)}\xrightarrow{Bk_{\mathrm{hOut}(Y)}} B^2Z(Y)_{\mathrm{hOut}(Y)} \tag{2}$$

where $\mathrm{Out}(Y) = \pi_0 \operatorname{aut}(BY, *) = \pi_0 \operatorname{aut}(BY)$ is the group of homotopy classes of homotopy self-equivalences of BY and the homotopy orbit space $BP(Y)_{\mathrm{hOut}(Y)}$ ($B^2Z(Y)_{\mathrm{hOut}(Y)}$) denotes the classifying space of the group-like topological monoid $\operatorname{aut}(BY, *)$ ($\operatorname{aut}(BY)$) of based (free) homotopy self-equivalences of BY. The monodromy action associated to the homotopy orbit space $B^2Z(Y)_{\mathrm{hOut}(Y)}$ is induced from the conjugation action of $\mathrm{Out}(Y)$ on $BZ(Y) \simeq \operatorname{map}(BY, BY)_{B1}$, i.e. from the action $Z\colon \mathrm{Out}(Y) \to \mathrm{Out}(Z(Y))$ of [10, Corollary 3.2].

Suppose now that the locally isomorphic p-compact groups, Y and Y', are equipped with homotopy actions, $\rho\colon \pi \to \mathrm{Out}(Y)$ and $\rho'\colon \pi' \to \mathrm{Out}(Y')$, by discrete groups, π and π'.

Pulling back the universal fibration (2) along the maps $B\rho\colon B\pi \to \mathrm{BOut}(Y)$, $B\rho'\colon B\pi' \to \mathrm{BOut}(Y')$ produces fibrations

$$\begin{tikzcd}
BY \arrow[r] \arrow[d, "Bg"'] & BP(Y)_{h\rho\pi} \arrow[r, "Bk_{h\rho\pi}"] \arrow[d, dashed] & B^2Z(Y)_{h\rho\pi} \arrow[d, dashed] \\
BY' \arrow[r] & BP(Y')_{h\rho'\pi'} \arrow[r, "Bk_{h\rho'\pi'}"] & B^2Z(Y')_{h\rho'\pi'}
\end{tikzcd}$$

that are universal for fibrations with fibre BY and with monodromy action restricting to ρ, ρ'. The projection map $Bk_{h\rho\pi}$, $Bk_{h\rho'\pi'}$ is a map over $B\pi$, $B\pi'$ since the universal projection map $Bk_{\mathrm{hOut}(Y)}$ is a map over $\mathrm{BOut}(Y)$. Thus the first obstruction to extending Bg to a fibre map $Bk_{h\rho\pi} \to Bk_{h\rho'\pi'}$ is that g be χ-equivariant, i.e. $g \cdot \rho(\gamma) = \rho'(\chi(\gamma)) \cdot g$ in $\varepsilon_{\mathbb{Q}}(Y, Y')$ for all $\gamma \in \pi$, for some group homomorphism $\chi\colon \pi \to \pi'$. Provided the mapping space $\operatorname{map}(BP(Y), BP(Y'))_{BP(g)}$ is contractible, as is this case if Y and Y' are *completely reducible* [12, Definition 3.10] p-compact groups, this is in fact the only obstruction to extending.

Lemma 2.2. *Suppose that $g \in \varepsilon_{\mathbb{Q}}(Y, Y')$ is a χ-equivariant rational isomorphism between the locally isomorphic completely reducible p-compact groups Y and Y'.*

(1) *There exists, up to vertical homotopy, exactly one extension*

$$BP(g)_{h\chi}\colon BP(Y)_{h\rho\pi} \to BP(Y')_{h\rho'\pi'}$$

of $BP(g)\colon BP(Y) \to BP(Y')$ to a map over $B\chi$.

(2) *There exists, up to vertical homotopy, exactly one extension*

$$B^2Z(g)_{h\chi}\colon B^2Z(Y)_{h\rho\pi} \to B^2Z(Y')_{h\rho'\pi'}$$

of $B^2Z(g)\colon B^2Z(Y) \to B^2Z(Y')$ to a map over $B\chi$ such that

$$\begin{tikzcd} BP(Y)_{h\rho\pi} \arrow[r, "Bk_{h\rho\pi}"] \arrow[d, "BP(g)_{h\chi}"'] & B^2Z(Y)_{h\rho\pi} \arrow[d, "B^2Z(g)_{h\chi}"] \\ BP(Y')_{h\rho'\pi'} \arrow[r, "Bk_{h\rho'\pi'}"'] & B^2Z(Y')_{h\rho'\pi'} \end{tikzcd}$$

commutes up to vertical homotopy.

The proof is based on the fibred mapping space construction occuring e.g. in [2, 3]:

Let $p\colon U \to A$ and $q\colon V \to B$ be fibrations over connected and pointed base spaces. Suppose that $g\colon p^{-1}(*) \to q^{-1}(*)$ is a map between the fibres and $h\colon (A,*) \to (B,*)$ a map between the base spaces such that the pair (g,h) respects the monodromy action in the sense that $g \cdot \zeta = \pi_1(h)(\zeta) \cdot g$ holds in $[p^{-1}(*), q^{-1}(*)]$ for all $\zeta \in \pi_1(A,*)$. The question of whether (g,h) comes from a fibre map can be turned into a section problem.

Define the set

$$\mathrm{fibmap}(U,V)^g_h = \coprod_{a \in A} \mathrm{map}(p^{-1}(a), q^{-1}(h(a)))_{g_a}$$

where $g_a \in [p^{-1}(a), q^{-1}(h(a))]$ is the homotopy class making

$$\begin{tikzcd} p^{-1}(*) \arrow[r, "g"] \arrow[d, "\zeta"'] & q^{-1}(*) \arrow[d, "h(\zeta)"] \\ p^{-1}(a) \arrow[r, "g_a"'] & q^{-1}(h(a)) \end{tikzcd}$$

homotopy commutative for any path ζ from the base point $*$ to $a \in A$. Using the topology of [3], we obtain a fibration

$$\mathrm{map}(p^{-1}(*), q^{-1}(*))_g \to \mathrm{fibmap}(U,V)^g_h \to A$$

whose based section space, by the fibrewise exponential law [2, Theorem 1], is homeomorphic to the space of maps of U into V under g and over h. Of course, $\mathrm{fibmap}(U,V)^g_h = \mathrm{fibmap}(U,h^*V)^g_1$ where h^*V is the pull back of V along h and 1 denotes the identity map of A.

Proof of Lemma 2.2. Composition with the maps $Bk_{h\rho\pi}$ and $Bk_{h\rho'\pi'}$ induces, since $Bk \circ BP(g) \simeq B^2Z(g) \circ Bk$ by Lemma 2.1, fibre maps

$$\begin{tikzcd}
\mathrm{map}(BP(Y),BP(Y'))_{BP(g)} \arrow[r] \arrow[d, "\underline{Bk}"'] & \mathrm{fibmap}(BP(Y)_{h\rho\pi},BP(Y')_{h\rho'\pi'})^{BP(g)}_{B\chi} \arrow[r] \arrow[d] & B\pi \arrow[d, equal] \\
\mathrm{map}(BP(Y),B^2Z(Y'))_{B(kP(g))} \arrow[r] & \mathrm{fibmap}(BP(Y)_{h\rho\pi},B^2Z(Y')_{h\rho'\pi'})^{B(kP(g))}_{B\chi} \arrow[r] & B\pi \arrow[d, equal] \\
\mathrm{map}(B^2Z(Y),B^2Z(Y'))_{B^2Z(g)} \arrow[r] \arrow[u, "\overline{Bk}", "\simeq"'] & \mathrm{fibmap}(B^2Z(Y)_{h\rho\pi},B^2Z(Y')_{h\rho'\pi'})^{B^2Z(g)}_{B\chi} \arrow[r] \arrow[u] & B\pi
\end{tikzcd}$$

of fibred mapping spaces. The map $\overline{Bk}$ is easily seen to be a homotopy equivalence and the fibre $\mathrm{map}(BP(Y), BP(Y'))_{BP(g)}$ is contractible [7, Theorem 1.3] [12, Theorem 3.11] since Y and Y' and with them their adjoint forms $P(Y)$ and $P(Y')$ are completely reducible. Thus there exists up to vertical homotopy exactly one section of the upper fibration inducing a corresponding section of the lower fibration. $\square$

Note that the fibre map $(BP(g)_{h\pi}, B^2Z(g)_{h\pi})$ of point (ii) of Lemma 2.2 is an extension of the fibre map $(BP(g), B^2Z(g))$ of Lemma 2.1 and thus restricts to the map $Bg\colon BY \to BY'$ on the fibres.

The above constructions pertaining to the *connected* p-compact groups can also be carried out for *abelian* p-discrete or p-compact toral groups [8, Definition 6.3, Definition 6.5].

Let $\check{Z}$ be an abelian p-discrete toral group and Z its closure [8, Definition 6.6]. The group $\mathrm{Aut}(\check{Z})$ of abelian group automorphisms of $\check{Z}$ acts by based homeomorphisms on $B^2\check{Z}$ so we may apply the Borel construction to the path fibration $PB^2\check{Z} \to B^2\check{Z}$ to obtain the fibration

$$\Omega B^2\check{Z} \to (PB^2\check{Z})_{\mathrm{hAut}(\check{Z})} \xrightarrow{\check{\sigma}_0} (B^2\check{Z})_{\mathrm{hAut}(\check{Z})} \tag{3}$$

which is universal for fibrations with $B\check{Z}$ as fibre. Note that both the total space and the base space are spaces over and under $\mathrm{BAut}(\check{Z})$ and that the projection map $\check{\sigma}_0$ is a map over and under $\mathrm{BAut}(\check{Z})$. Since p-completion induces isomorphisms $\mathrm{End}(\check{Z}) \to \mathrm{End}(Z)$ and $\mathrm{Aut}(\check{Z}) \to \mathrm{Out}(Z)$ [13, Proposition 3.2], fibrewise completion of (3) results in the fibration

$$BZ \to \mathrm{BOut}(Z) \xrightarrow{\sigma_0} (B^2Z)_{\mathrm{hOut}(Z)} \tag{4}$$

which is universal for fibrations with BZ as fibre. The projection map σ_0 is a map of spaces over and under $\mathrm{BOut}(Z)$.

The abelian group structure on $\check{Z}$ induces on $B^2\check{Z}$ the structure of an abelian topological group. Let $\check{\nabla}\colon B^2\check{Z} \times B^2\check{Z} \to B^2\check{Z}$ be the addition map and $\check{\nu}\colon B^2\check{Z} \to B^2\check{Z}$ the inversion map such that

$$\check{\nabla} \circ \tau = \check{\nabla}, \quad \check{\nabla} \circ (\check{\nu} \times 1) \circ \Delta = 0$$

where τ is the switch and Δ the diagonal map. The p-completions of $\check{\nabla}$ and $\check{\nu}$, $\nabla\colon B^2Z \times B^2Z \to B^2Z$ and $\nu\colon B^2Z \to B^2Z$, promote B^2Z to an abelian group-like space. Moreover, since $\mathrm{Aut}(\check{Z})$ acts on $B^2\check{Z}$ through group isomorphisms, $\check{\nabla}$ and $\check{\nu}$ extend to maps over and under $\mathrm{BAut}(\check{Z})$

$$\check{\nabla}\colon \Delta^*(B^2\check{Z}_{\mathrm{hAut}(\check{Z})} \times B^2\check{Z}_{\mathrm{hAut}(\check{Z})}) \to B^2\check{Z}_{\mathrm{hAut}(\check{Z})} \tag{5}$$

$$\check{\nu}\colon B^2\check{Z}_{\mathrm{hAut}(\check{Z})} \to B^2\check{Z}_{\mathrm{hAut}(\check{Z})} \tag{6}$$

where Δ is the diagonal on $\mathrm{BAut}(\check{Z})$. The fibrewise p-completion of these maps are maps over and under $\mathrm{BOut}(Z)$

$$\nabla\colon \Delta^*(B^2Z_{\mathrm{hOut}(Z)} \times B^2Z_{\mathrm{hOut}(Z)}) \to B^2Z_{\mathrm{hOut}(Z)} \tag{7}$$

$$\nu\colon B^2Z \to B^2Z \tag{8}$$

extending the structure maps ∇ and ν on B^2Z.

Suppose now that $\check{Z}'$ is another p-discrete toral group and that $\check{Z}$ and $\check{Z}'$ support group actions $\zeta\colon \pi \to \mathrm{Aut}(\check{Z})$, $\zeta'\colon \pi' \to \mathrm{Aut}(\check{Z}')$. Any χ-equivariant abelian group homomorphism $\check{\jmath}\colon \check{Z} \to \check{Z}'$ extends to a topological group homomorphism $B^2\check{\jmath}\colon B^2\check{Z} \to B^2\check{Z}'$ and thus to a map

$$B^2\check{\jmath}_{h\chi}\colon B^2\check{Z}_{h\zeta\pi} \to B^2\check{Z}'_{h\zeta'\pi'} \tag{9}$$

over and under $B\chi$ such that

$$\check{\nabla}' \circ \Delta^*(B^2\check{\jmath}_{h\chi} \times B^2\check{\jmath}_{h\chi}) = B^2\check{\jmath}_{h\chi} \circ \check{\nabla}, \quad \check{\nu}' \circ B^2\check{\jmath}_{h\chi} = B^2\check{\jmath}_{h\chi} \circ \check{\nu}$$

where $\check{\nabla}'$ and $\check{\nu}'$ are the structure maps for $B^2\check{Z}'$.

Let Z' denote the ablian p-compact toral group which is the closure of $\check{Z}'$. Fibrewise p-completion of $B^2\check{\jmath}_{h\chi}$ is a map

$$B^2 j_{h\chi}\colon B^2Z_{h\zeta\pi} \to B^2Z'_{h\zeta'\pi'} \tag{10}$$

over and under $B\chi$ such that

$$\nabla' \circ \Delta^*(B^2j_{h\chi} \times B^2j_{h\chi}) = B^2j_{h\chi} \circ \nabla, \quad \nu' \circ B^2j_{h\chi} = B^2j_{h\chi} \circ \nu \tag{11}$$

where ∇' and ν' are the structure maps on B^2Z'.

3. Short exact sequences

This section contains information about fibrations of p-compact group classifying spaces.

Let X and Y be p-compact groups with classifying spaces BY and BX and let $\mathrm{cd}_{\mathbb{F}_p}(-)$ denote mod p cohomological dimension [8, Definition 6.13].

Lemma 3.1. *Let $BY \to BG \to BX$ be a fibration sequence. Then G is a p-compact group and* $\mathrm{cd}_{\mathbb{F}_p}(G) = \mathrm{cd}_{\mathbb{F}_p}(X) + \mathrm{cd}_{\mathbb{F}_p}(Y)$.

Proof. As the base space as well as the fibre are p-complete spaces, the Fibre lemma [5, II.5.1–5.2] implies that also the total space BG is p-complete.

Let Y_0 denote the identity component of Y. By pulling back the fibration $G \to X \to BY$ to the universal covering space BY_0 we obtain a fibration denoted $G \to X|BY_0 \to BY_0$. Extending this fibration one step to the left gives the fibration $Y_0 \to G \to X|BY_0$ with connected fibre. The action of the fundamental group of any component of the base on $H_i(Y_0; \mathbb{F}_p)$, $i \geq 0$, is nilpotent because it factors through the finite p-group $\pi_0(Y)$ (acting on $H_i(Y; \mathbb{F}_p)$). Hence [8, Lemma 6.16] the corresponding Serre spectral sequence is concentrated in a rectangle of dimensions $\mathrm{cd}_{\mathbb{F}_p}(X)$ by $\mathrm{cd}_{\mathbb{F}_p}(Y)$ and the group in the upper right corner is nontrivial. The fact that G is $\mathbb{F}_p$-finite and the formula for its mod p cohomological dimension now follows as in the proof of [8, Proposition 6.14].

This shows [8, Lemma 2.1, Definition 2.2] that G is a p-compact group. $\square$

It is a consequence of Lemma 3.1 that the composition of two epimorphisms [8, 3.2] is an epimorphism.

Definition 3.2. *An* extension *of X by Y is a fibration of based maps*

$$BY \to BG \to BX$$

over BX with fibre BY. Two extensions are equivalent *if there exists a fibre map of the form*

$$\begin{tikzcd} BY \arrow[r] \arrow[d, equal] & BG \arrow[r] \arrow[d] & BX \arrow[d, equal] \\ BY \arrow[r] & BH \arrow[r] & BX \end{tikzcd}$$

between them. $\mathrm{Ext}(X, Y)$ *denotes the set of all equivalence classes of extensions of X by Y.*

Since the total space BG is the classifying space of a p-compact group [Lemma 3.1], any extension of X by Y is a short exact sequence of p-compact groups [8, 3.2]. The extension $BY \to BG \to BX$ is often referred to simply as $Y \to G \to X$. In this notation, two extensions are equivalent if there exists a homomorphism of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Y \arrow[r] & H \arrow[r] & X \end{tikzcd}$$

between them.

Associated to the short exact sequence $Y \to G \to X$ is a homotopy action $\rho\colon \pi_0(X) \to \mathrm{Out}(Y)$. Observe that this monodromy action is an invariant of the equivalence class so that it makes sense to let $\mathrm{Ext}_\rho(X, Y)$ denote the subset of $\mathrm{Ext}(X, Y)$ represented by all short exact sequences realizing the action ρ.

Assume from now on that Y is a *connected* p-compact group.

Let $[BX, B^2Z(Y)_{\mathrm{hOut}(Y)}]_{B\rho}$ denote the set of vertical homotopy classes of lifts

$$\begin{tikzcd} & & B^2Z(Y)_{\mathrm{hOut}(Y)} \arrow[d] \\ BX \arrow[r, "B\pi_0"'] \arrow[urr, dashed] & B\pi_0(X) \arrow[r, "B\rho"'] & \mathrm{BOut}(Y) \end{tikzcd}$$

of the map $B\rho \circ B\pi_0$.

Similarly, if Z is an abelian p-compact group with discrete approximation $\check{Z} \to Z$ and $\zeta\colon \pi_0(X) \to \mathrm{Aut}(\check{Z}) = \mathrm{Out}(Z)$ an action, let $[BX, B^2\check{Z}_{\mathrm{hAut}(\check{Z})}]_{B\zeta}$ and $[BX, B^2Z_{\mathrm{hOut}(Z)}]_{B\zeta}$ denote the sets of vertical homotopy classes of lifts of $B\zeta \circ B\pi_0$.

Define $\mathrm{Ext}_\zeta(X, \check{Z})$ to be the set of equivalence classes (with respect to fibre homotopy equivalences under $B\check{Z}$ and over BX) of fibrations $B\check{Z} \to BG \to BX$ with monodromy action ζ.

Lemma 3.3. *Let Y be a connected and X any p-compact group. Then there are bijections*

$$[BX, B^2Z(Y)_{\mathrm{hOut}(Y)}]_{B\rho} \to \mathrm{Ext}_\rho(X, Y)$$
$$[BX, B^2\check{Z}_{\mathrm{hAut}(\check{Z})}]_{B\zeta} \to \mathrm{Ext}_\zeta(X, \check{Z})$$
$$[BX, B^2Z_{\mathrm{hOut}(Z)}]_{B\zeta} \to \mathrm{Ext}_\zeta(X, Z)$$

defined by pulling back the universal fibrations (2)*,* (3)*, and* (4)*, respectively.*

Proof. The base space of the fibration of based mapping spaces

$$\mathrm{map}_*(BX, B^2Z(Y)_{\mathrm{hOut}(Y)}) \to \mathrm{map}_*(BX, \mathrm{BOut}(Y))$$

is homotopically discrete [11, Lemma 2.2]. Therefore, the total space is homotopically equivalent to the disjoint union over all homomorphisms $\rho\colon \pi_0(X) \to \mathrm{Out}(Y)$ of the spaces of based lifts of $B\pi_0 \circ B\rho$.

By classification theory, pull back of the universal bundle (2) provides a bijection

$$\pi_0(\mathrm{map}_*(BX, B^2Z(Y)_{\mathrm{hOut}(Y)}) \to \mathrm{Ext}(X, Y)$$

which by the above remarks restricts to a bijection between the based and vertical homotopy classes of lifts of $B\pi_0 \circ B\rho$ and $\mathrm{Ext}_\rho(X, Y)$. However, since the fibre $B^2Z(Y)$ is simply connected, the clause that the lifts be based is superfluous.

Similar arguments apply in the remaining two cases. $\square$

The chosen equivalence relation [Definition 3.2] on BY-fibrations over BX (assumed to have a nondegenerate base point) corresponds by Allaud [1] to *based* homotopy classes of maps of BX into the classifying space $B^2Z(Y)_{\mathrm{hOut}(Y)}$. See [2] for an account of the relationship between the free and the based case.

For the following, assume that $Y \to G \to X$ and $Y \to H \to X$ are two short exact sequences realizing the same homotopy action $\rho\colon \pi_0(X) \to \mathrm{Out}(Y)$. Choose [Lemma 3.3] based lifts (also denoted) $G, H\colon BX \to B^2Z(Y)_{\mathrm{hOut}(Y)}$ of $B\rho \circ B\pi_0$ classifying the two fibrations. Define

$$\Omega B^2Z(Y) \to B\Delta(H,G) \to BX$$

to be the fibration whose fibre over any point $b \in BX$ is the space of vertical (i.e. having constant projection in $\mathrm{BOut}(Y)$) paths in $B^2Z(Y)_{\mathrm{hOut}(Y)}$ from $G(b)$ to $H(b)$. This fibration represents an element in $\mathrm{Ext}_{Z\rho}(X, Z(Y))$.

Theorem 3.4. *Let* $G \in \mathrm{Ext}_\rho(X,Y)$ *where* Y *is a connected and* X *an arbitrary* p*-compact group. Then the map*

$$\Delta(-,G)\colon \mathrm{Ext}_\rho(X,Y) \to \mathrm{Ext}_{Z\rho}(X,Z(Y))$$

is a bijection.

Proof. Pulling back to BX the two fibrations shown as downward pointing arrows in the diagram

$$\begin{tikzcd}
 & B^2Z(Y)_{\mathrm{hOut}(Y)} \arrow[d] & B^2Z(Y)_{\mathrm{hOut}(Z(Y))} \arrow[d, shift left] \\
BX \arrow[ur, "G"] \arrow[r, "B(\rho\pi_0)"'] & \mathrm{BOut}(Y) \arrow[r, "BZ"'] & \mathrm{BOut}(Z(Y)) \arrow[u, shift left, "\sigma_0"]
\end{tikzcd}$$

provides two sectioned fibrations, $B^2Z(Y)_{h\rho\pi_0 X} \to BX$ and $B^2Z(Y)_{hZ\rho\pi_0 X} \to BX$, with fibre $B^2Z(Y)$. (See (4) for the section σ_0 .) These two spaces over and under BX are equivalent in the sense that there exists up to homotopy over and under BX exactly one extension of the identity map of $B^2Z(Y)$ to a map

$$u_G\colon B^2Z(Y)_{h\rho\pi_0 X} \to B^2Z(Y)_{hZ\rho\pi_0 X}$$

over and under BX. This follows from the fact that the fibre of the *based* fibred mapping space

$$\mathrm{map}_*(B^2Z(Y), B^2Z(Y))_{B^21} \longrightarrow \mathrm{fibmap}_*(B^2Z(Y)_{h\rho\pi_0 X}, B^2Z(Y)_{hZ\rho\pi_0 X})^{B^21}_{B1} \longrightarrow BX$$

is contractible. Composition with u_G induces a bijection

$$(u_G)_*\colon \mathrm{Ext}_\rho(X,Y) \xrightarrow{\cong} \mathrm{Ext}_{Z\rho}(X,Z(Y))$$

identical to the map $\Delta(-,G)$. $\square$

It is a consequence of Theorem 3.4 that $\mathrm{Ext}_\rho(X, Y)$ is an affine group [4, §9, no 1].

To see this, note that the structure maps (7) and (8) make $\mathrm{Ext}_\zeta(X, Z)$ into an abelian group. The neutral element of this group is represented by the short exact sequence $Z \to Z \rtimes_\zeta X \to X$ classified by the map $\sigma_0 \circ B\zeta \circ B\pi_0$. The sum of two short exact sequences $Z \to A \to X$ and $Z \to B \to X$, with classifying maps $A, B \colon BX \to B^2 Z_{\mathrm{hOut}(Z)}$, is the short exact sequence classified by the lift $\nabla \circ (A \times B) \circ \Delta$ of $B\zeta \circ B\pi_0$. The inverse $-A$ is classified by $\nu \circ A$.

Note that the projection $B(Z \rtimes_\zeta X) \to BX$ admits a based section, i.e. that the short exact sequence

$$Z \longrightarrow Z \rtimes_\zeta X \rightleftarrows X \tag{12}$$

is a split short exact sequence, and that there exist homomorphisms

$$\begin{tikzcd} Z \times Z \arrow[r] \arrow[d, "\nabla"'] & \Delta^*(A \times B) \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Z \arrow[r] & A + B \arrow[r] & X \end{tikzcd} \tag{13}$$

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\nu"'] & A \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Z \arrow[r] & -A \arrow[r] & X \end{tikzcd} \tag{14}$$

of short exact sequences. These properties are characterizing.

Lemma 3.5. *Suppose that $Z \to C \to X$ is a short exact sequence representing an element of $\mathrm{Ext}_\zeta(X, Z)$.*

(1) *If there exists a splitting*

$$Z \longrightarrow C \rightleftarrows X$$

then $C = 0$ in $\mathrm{Ext}_\zeta(X, Z)$.

(2) *If there exists a short exact sequence homomorphism*

$$\begin{tikzcd} Z \times Z \arrow[r] \arrow[d, "\nabla"'] & \Delta^*(A \times B) \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Z \arrow[r] & C \arrow[r] & X \end{tikzcd}$$

then $C = A + B$ in $\mathrm{Ext}_\zeta(X, Z)$.

(3) *If there exists a short exact sequence homomorphism*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\nu"'] & A \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Z \arrow[r] & C \arrow[r] & X \end{tikzcd}$$

then $C = -A$ *in* $\mathrm{Ext}_\zeta(X, Z)$.

Proof. (i) The *based* fibred mapping space

$$\mathrm{map}_*(BZ, BZ)_{B1} \to \mathrm{fibmap}_*(B(Z \rtimes_\zeta X), BC)^{B1}_{B1} \to BX$$

admits a section because its fibre is contractible.
(ii) Precomposition with the short exact sequence homomorphism (13) determines a fibre map

$$\begin{tikzcd}
\mathrm{map}(BZ, BZ)_{B1} \arrow[r] \arrow[d, "\overline{\nabla}"', "\simeq"] & \mathrm{fibmap}(B(A+B), BC)^{B1}_{B1} \arrow[r] \arrow[d] & BX \arrow[d, equal] \\
\mathrm{map}(BZ \times BZ, BZ)_{\nabla} \arrow[r] & \mathrm{fibmap}(B\Delta^*(A \times B), BC)^{\nabla}_{B1} \arrow[r] & BX
\end{tikzcd}$$

which is a fibre homotopy equivalence since, Z being abelian, $\overline{\nabla}$ is a homotopy equivalence. As the lower fibration admits a section, so does the upper one.
(iii) Similar to (ii). $\square$

In case $Z = Z(Y)$ is the center of the connected p-compact group Y, the difference map Δ from Theorem 3.4 and the additive structure in $\mathrm{Ext}_{Z\rho}(X, Z(Y))$ are nicely related.

Lemma 3.6. *Let* $G, H, K \in \mathrm{Ext}_\rho(X, Y)$. *Then*

$$\Delta(K, G) = \Delta(K, H) + \Delta(H, G)$$

in $\mathrm{Ext}_{Z\rho}(X, Z(Y))$.

Proof. Since composition of paths defines a map

$$\Delta^*(B\Delta(H, G) \times B\Delta(K, H)) \to B\Delta(K, G)$$

over BX and under the H-space structure on $\Omega B^2 Z$, this formula follows from Lemma 3.5. $\square$

The formula of Lemma 3.6 implies that

$$\Delta(G, G) = Z(Y) \rtimes_{Z\rho} X, \quad -\Delta(H, G) = \Delta(G, H)$$

for all $G, H \in \mathrm{Ext}_\rho(X, Y)$. More formally

Corollary 3.7. $\mathrm{Ext}_\rho(X, Y)$ *is an affine group with the abelian group* $\mathrm{Ext}_{Z\rho}(X, Z(Y))$ *as its group of operators.*

Let's now look at functorial properties of the Ext-affine groups.

Let $Y' \to G' \to X'$ be another short exact sequence of p-compact groups with associated homotopy action $\rho' \colon \pi_0(X') \to \mathrm{Out}(Y')$. Any p-compact group homomorphism $h \colon X \to X'$ induces a map

$$h^* \colon \operatorname{Ext}_{\rho'}(X', Y') \to \operatorname{Ext}_{\rho' \pi_0(h)}(X, Y') \tag{15}$$

defined by pull back. Note that $h^*(G')$ is indeed a p-compact group by Lemma 3.1 and that h extends to a morphism

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, equal] & h^*(G') \arrow[r] \arrow[d] & X \arrow[d, "h"] \\ Y' \arrow[r] & G' \arrow[r] & X' \end{tikzcd} \tag{16}$$

of short exact sequences.

As to functorial properties in the second variable, assume now that Y' is connected, completely reducible, locally isomorphic to Y, and that $g \colon Y \to Y'$ is a rational isomorphism which is χ-equivariant for some group homomorphism $\chi \colon \pi_0(X) \to \pi_0(X')$. Let

$$g_* \colon \operatorname{Ext}_{\rho}(X, Y) \to \operatorname{Ext}_{\rho'\chi}(X, Y') \tag{17}$$

be the map induced by composing classifying maps with the essentially uniquely determined map $B^2Z(g)_{h\chi} \colon B^2Z(Y)_{h\rho\pi_0(X)} \to B^2Z(Y')_{h\rho'\chi\pi_0(X)}$ from Lemma 2.2. Note that the rational isomorphism g extends to a short exact sequence homomorphism

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & G \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Y' \arrow[r] & g_*(G) \arrow[r] & X \end{tikzcd} \tag{18}$$

where the middle arrow is induced from $BP(g)_{h\chi}$.

There are similar functorial properties in the abelian case. Let Z and Z' be abelian p-compact toral groups equipped with homotopy actions $\zeta \colon \pi_0(X) \to \mathrm{Out}(Z)$, $\zeta' \colon \pi_0(X') \to \mathrm{Out}(Z')$. Pull back along the map $Bh \colon BX \to BX'$ induces a map

$$h^* \colon \operatorname{Ext}_{\zeta'}(X', Z') \to \operatorname{Ext}_{\zeta'\pi_0(h)}(X, Z')$$

which clearly is a group homomorphism. Also, if $j \colon Z \to Z'$ is a χ-equivariant homomorphism, composition with the map $B^2j_{h\chi} \colon B^2Z_{h\zeta\pi_0(X)} \to B^2Z'_{h\zeta'\chi\pi_0(X)}$ over and under $B\pi_0(X)$ from (10) induces

$$j_* \colon \operatorname{Ext}_{\zeta}(X, Z) \to \operatorname{Ext}_{\zeta'\chi}(X, Z')$$

which is a group homomorphism by the identities (11).

The χ-equivariant rational isomorphism $g\colon Y \to Y'$ induces [10, Corollary 4.2] a χ-equivariant rational isomorphism $Z(g)\colon Z(Y) \to Z(Y')$.

Lemma 3.8. *Let $h\colon X \to X'$ be a homomorphism and $g\colon Y \to Y'$ a χ-equivariant rational isomorphism.*

(1) *The pull back (15) along h is an affine map with*

$$h^*\colon \operatorname{Ext}_{Z\rho'}(X', Z(Y')) \to \operatorname{Ext}_{Z\rho'\pi_0(h)}(X, Z(Y'))$$

as its corresponding operator group homomorphism.

(2) *The push forward (17) along g is an affine map with*

$$Z(g)_*\colon \operatorname{Ext}_{Z\rho}(X, Z(Y)) \to \operatorname{Ext}_{Z\rho'\chi}(X, Z(Y'))$$

as its corresponding operator group homomorphism.

Proof. (i) It is immediate that $\Delta(h_*G', h^*H') = h^*\Delta(G', H')$ for all $G', H' \in \operatorname{Ext}_{\rho'}(X', Y')$.
(ii) In the diagram

$$\begin{tikzcd}
B^2Z(Y)_{h\rho\pi_0X} \arrow[r, "u_G"] \arrow[d, "B^2Z(g)_{hX}"'] & B^2Z(Y)_{Z\rho\pi_0X} \arrow[d, "B^2Z(g)_{hX}"] \\
B^2(Y')_{h\rho'\chi\pi_0X} \arrow[r, "u_{g_*(G)}"'] & B^2Z(Y')_{hZ\rho'\chi\pi_0X}
\end{tikzcd}$$

the left vertical map is the one defined in Lemma 2.2 and the right vertical map is, despite the notational coincidence, the one defined in formula (10). However, all maps in this diagram are maps over and under BX and as such maps are essentially unique, cfr. the proof of Theorem 3.4, $B^2Z(g)_{hX} \circ u_G$ and $u_{g_*G} \circ B^2Z(g)_{hX}$ are homotopic over and under BX. Hence $Z(g)_*\Delta(-, G) = \Delta(g_*(-), g_*G)$. $\square$

For the final result of this section, suppose that the rational isomorphism $g\colon Y \to Y'$ is $\pi_0(h)$-equivariant such that push forward along g and pull back along h

$$\operatorname{Ext}_\rho(X, Y) \xrightarrow{g_*} \operatorname{Ext}_{\rho'\pi_0(h)}(X, Y') \xleftarrow{h^*} \operatorname{Ext}_{\rho'}(X', Y')$$

have the same target.

Theorem 3.9. *Assume that Y and Y' are locally isomorphic connected, completely reducible p-compact groups. Then there exists an extension homomorphism of the form*

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "g"'] & G \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
Y' \arrow[r] & G' \arrow[r] & X'
\end{tikzcd}$$

if and only if $g_*(G) = h^*(G')$ in $\operatorname{Ext}_{\rho'\pi_0(h)}(X, Y')$.

Proof. Precomposition with the map $BG \to B(g_*G)$ under Bg and over BX and postcomposition with the map $B(h^*(G')) \to BG'$ under BY' and over Bh induce a fibre map

$$\begin{tikzcd} \mathrm{map}(BY',BY')_{B1} \arrow[r] \arrow[d, "\overline{Bg}"', "\simeq"] & \mathrm{fibmap}(B(g_*G),B(h^*(G'))^{B1}_{B1} \arrow[r] \arrow[d] & BX \arrow[d, equal] \\ \mathrm{map}(BY,BY')_{Bg} \arrow[r] & \mathrm{fibmap}(BG,BG')^{Bg}_{Bh} \arrow[r] & BX \end{tikzcd}$$

of fibred mapping spaces. Since Y and Y' are completely reducible, this is a fibre homotopy equivalence [12, Theorem 3.11]. Hence one of the two fibrations admits a section if and only if the other one does. □

A Lie group version of the material contained in this section can be found in Notbohm [14].

4. Approximations

In this section obstruction theory is used to equate Ext-affine groups in certain advantageous situations.

As in the previous sections, X is any p-compact group, Y is a connected p-compact group, Z is an abelian p-compact (toral) group with discrete approximation $\check{Z}$, and $\rho\colon \pi_0(X) \to \mathrm{Out}(Y)$ and $\zeta\colon \pi_0(X) \to \mathrm{Aut}(\check{Z}) = \mathrm{Out}(Z)$ are homotopy actions.

Lemma 4.1. *Suppose that the identity component X_0 of X is simply connected. Then the component homomorphism $\pi_0\colon X \to \pi_0(X)$ induces bijections*

$$\pi_0^*\colon\ \mathrm{Ext}_\rho(\pi_0(X),Y) \to \mathrm{Ext}_\rho(X,Y)$$
$$\pi_0^*\colon\ \mathrm{Ext}_\zeta(\pi_0(X),Z) \to \mathrm{Ext}_\rho(X,Z)$$

of equivalence classes of extensions.

Proof. Since the composite map $BX_0 \to BX \to B\pi_0(X)$ is nonessential, the homotopy orbit space $B^2Z(Y)_{hX_0} \simeq B^2Z(Y) \times BX_0$ and the homotopy fixed point space $B^2Z(Y)^{hX_0} \simeq \mathrm{map}(BX_0, B^2Z(Y)) \simeq B^2Z(Y)$ because BX_0 is 3-connected by Browder [6] [13, Corollary 5.6]. Hence [8, Lemma 10.5, Remark 10.8]

$$B^2Z(Y)^{hX} \simeq (B^2Z(Y)^{hX_0})^{h\rho\pi_0(X)} \simeq B^2Z(Y)^{h\rho\pi_0(X)}$$

and these homotopy equivalences induce bijections

$$\mathrm{Ext}_\rho(X,Y) = \pi_0(B^2Z(Y)^{h\rho\pi_0X}) = \pi_0(B^2Z(Y)^{h\rho\pi_0(X)}) = \mathrm{Ext}_\rho(\pi_0(X),Y)$$

of Ext-sets. This proves the lemma for extensions of X by Y; extensions of X by Z are handled similarly. □

The referee pointed out the following

Corollary 4.2. *Suppose that X is a connected and simply connected p-compact group. Then every extension of X by Y is equivalent to the trivial extension $Y \to Y \times X \to X$.*

The structure maps (5) and (6) on $B^2\check{Z}_{\mathrm{hAut}(\check{Z})}$ make $\mathrm{Ext}_\zeta(X,\check{Z})$ into an abelian group and the map

$$e_*\colon \mathrm{Ext}_\zeta(X,\check{Z}) \to \mathrm{Ext}_\zeta(X,Z),$$

induced by fibrewise completion $e\colon B^2\check{Z}_{\mathrm{hAut}(\check{Z})} \to B^2Z_{\mathrm{hOut}(Z)}$, into an abelian group homomorphism.

The next lemma shows that extensions of X by Z have unique fibrewise discrete approximations if the identity component of X is semisimple.

A connected p-compact group is said to be semisimple if its fundamental group or, equivalently [13, Theorem 5.3], its center is finite.

Lemma 4.3. *The above group homomorphism e_* is surjective and also injective provided the identity component X_0 of X is semisimple.*

Proof. The sets $\mathrm{Ext}_\zeta(X,\check{Z})$ and $\mathrm{Ext}_\zeta(X,Z)$ correspond to vertical homotopy classes of the lifts indicated by dashed arrows in the diagram

$$\begin{tikzcd}
B\pi_0(X) & B^2Z_{h\zeta\pi_0(X)} \arrow[l] & B^2\check{Z}_{h\zeta\pi_0(X)} \arrow[l, "e"'] \\
BX \arrow[u, "B\pi_0"] \arrow[ur, dashed] \arrow[urr, dashed] & &
\end{tikzcd}$$

where the two spaces to the right are total spaces for the pull backs of the classifying fibrations (3) and (4) along $B\zeta\colon B\pi_0(X) \to \mathrm{BAut}(\check{Z}) = \mathrm{BOut}(Z)$.

The obstruction to lifting a map $BX \to B^2Z_{h\zeta\pi_0(X)}$ to $B^2\check{Z}(Y)_{h\zeta\pi_0(X)}$ lives in $H^3(BX;V)$ as the fibre of $e_{h\pi_0(X)}$ is B^2V for some rational vector space V [7, Proposition 3.2]. Since $\pi_3(BX) = \pi_2(X) = 0$ [6], [13, Corollary 5.6], there exists a 4-connected map $BX \to B$ to a 2-stage Postnikov tower B with fundamental group $\pi_1(B) \cong \pi_0(X)$ and $\pi_2(B) \cong \pi_1(X)$. Hence $H^3(BX;V) \cong H^3(B;V)$ and as $H^*(\pi_1(X),2;V)$ is a rational vector space, $H^3(B;V) \cong H^0(\pi_0(X);H^3(\pi_1(X),2;V))$ by the Serre spectral sequence with local coefficients. Universal Coefficients and [15, Theorem V.7.8] asserting that $H_3(\pi_1(X),2) = 0$ imply that the coefficient group $H^3(\pi_1(X),2;V) = 0$. We conclude that the obstruction group $H^3(BX;V)$ vanishes. This shows that $\mathrm{Ext}_\zeta(X,\check{Z})$ maps onto $\mathrm{Ext}_\zeta(X,Z)$.

The obstruction to lifting a vertical homotopy to $B^2\check{Z}_{h\zeta\pi_0(X)}$ lives in $H^2(BX;V) \cong H^2(B;V) \cong H^0(\pi_0(X);\mathrm{Hom}(\pi_1(X),V))$ which vanishes if the fundamental group $\pi_1(X)$ is finite. This shows that the map in the lemma is injective provided X_0 is semisimple. □

For a p-compact torus T of rank one [8, 6.3], $\mathrm{Ext}(T,\check{T}) \cong [BT, B\check{T}] \cong \check{T}$ while $\mathrm{Ext}(T,T) \cong [BT, B^2T] = 0$ so the map e_* of Lemma 4.3 is not injective in case $X = T = Z$.

There exists a version of Lemma 4.3 allowing the fibres to be arbitrary, not just abelian, p-compact toral groups.

We now know that in case the identity component X_0 is semisimple and $\mathrm{Ext}_\rho(X,Y) \neq \emptyset$, there are bijections

$$\mathrm{Ext}_\rho(X,Y) \xrightarrow[\cong]{\Delta(-,G)} \mathrm{Ext}_{Z\rho}(X, Z(Y)) \xleftarrow[\cong]{} \mathrm{Ext}_{Z\rho}(X, \check{Z}(Y))$$

where the right hand group is isomorphic to the cohomology group $H^2_{Z\rho}(BX; \check{Z}(Y))$. If X_0 is even simply connected, there are bijections

$$\begin{tikzcd}
\mathrm{Ext}_\rho(X,Y) & \mathrm{Ext}_{Z\rho}(\pi_0(X), \check{Z}(Y)) \arrow[d, "\cong"] \\
\mathrm{Ext}_\rho(\pi_0(X),Y) \arrow[u, "\cong", "\pi_0^*"'] \arrow[r, "\Delta(-,G)", "\cong"'] & \mathrm{Ext}_{Z\rho}(\pi_0(X), Z(Y))
\end{tikzcd}$$

where the upper right corner group is isomorphic to the cohomology group $H^2_{Z\rho}(\pi_0(X); \check{Z}(Y))$. Note, however, that the bijection $\Delta(-,G)$, depending on the choice of the extension G, is noncanonical.

Now follows an alternative description of the Theorem 3.4 difference $\Delta(H,G)$ between two short exact sequences $Y \to G \to X$ and $Y \to H \to X$ in $\mathrm{Ext}_\rho(X,Y)$.

Proposition 4.4. *There exists a homotopy equivalence*

$$\Lambda\colon B\Delta(H,G) \to \mathrm{fibmap}(BG, BH)^{B1}_{B1}$$

over BX.

Proof. Let $Bk = Bk_{\mathrm{hOut}(Y)}\colon BP(Y)_{\mathrm{hOut}(Y)} \to B^2Z(Y)_{\mathrm{hOut}(Y)}$ denote the projection map of and $\lambda\colon W \to \mathrm{map}(I, BP(Y)_{\mathrm{hOut}(Y)})$ a connection [15, p. 29] for the universal fibration (2); i.e. λ assigns to any element of

$$W = \{(x,u) \in BP(Y)_{\mathrm{hOut}(Y)} \times \mathrm{map}(I, B^2Z(Y)_{\mathrm{hOut}(Y)}) \mid Bk(x) = u(0)\}$$

a path $\lambda(x,u)$ in $BP(Y)_{\mathrm{hOut}(Y)}$ starting at $\lambda(x,u)(0) = x$ and lying over $Bk(\lambda(x,u)) = u$.

The fibres over any $b \in BX$ of $BG \to BX$, $BH \to BX$ are the fibres $Bk^{-1}(G(b))$, $Bk^{-1}(H(b))$ and the fibre of $B\Delta(G,H) \to BX$ is the space of vertical paths u in $B^2Z(Y)_{\mathrm{hOut}(Y)}$ from $G(b)$ to $H(b)$. Define

$$\Lambda\colon B\Delta(H,G) \to \mathrm{fibmap}(BG, BH)^{B1}_{B1}$$

as the map over BX taking u to the map $\lambda(-,u)(1)\colon Bk^{-1}(G(b)) \to Bk^{-1}(H(b))$. The restriction of Λ to the fibre over the basepoint (where the classifying maps G and H have the same value) is the monodromy $\Omega B^2Z(Y) \to$

$\mathrm{map}(BY, BY)_{B1}$ for the universal fibration (1), hence a homotopy equivalence. $\square$

Thus also

$$\mathrm{fibmap}(BG, -)^{B1}_{B1} \colon \mathrm{Ext}_\rho(X, Y) \to \mathrm{Ext}_{Z\rho}(X, Z(Y))$$

is a bijection.

The evaluation map $B\mu \colon BY \times \mathrm{map}(BY, BY)_{B1} \to BY$ is a left action $\mu \colon Y \times Z(Y) \to Y$ of $Z(Y)$ on Y. Using the alternative description of Proposition 4.4 of the difference $\Delta(H, G)$ it is immediate that μ extends to a morphism

$$\begin{tikzcd} Y \times Z(Y) \arrow[r] \arrow[d, "\mu"'] & \Delta^*(G \times \Delta(H,G)) \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Y \arrow[r] & H \arrow[r] & X \end{tikzcd}$$

of extensions. This property characterizes $\Delta(H, G)$ as an operator on $\mathrm{Ext}_\rho(X, Y)$.

Corollary 4.5. *Let $Z(Y) \to \Delta \to X$ be a short exact sequence representing an element $\Delta \in \mathrm{Ext}_{Z\rho}(X, Z(Y))$. Then $G + \Delta = H$ in $\mathrm{Ext}_\rho(X, Y)$ if and only if the action μ extends*

$$\begin{tikzcd} Y \times Z(Y) \arrow[r] \arrow[d, "\mu"'] & \Delta^*(G \times \Delta) \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Y \arrow[r] & H \arrow[r] & X \end{tikzcd}$$

to a morphism over X.

Proof. The fibrewise adjoint of such a fibre map is an equivalence between Δ and $\Delta(H, G) = \mathrm{fibmap}(BG, BH)^{B1}_{B1}$. $\square$

Corollary 4.5 concludes this section.

5. Rational automorphisms of non-connected p-compact groups

The purpose of this section is to investigate the monoid of rational automorphisms of not necessarily connected p-compact groups.

Let $Y \to G \to \pi$ be a short exact sequence of p-compact groups, where Y is connected and π is a finite p-group, representing an element $G \in \mathrm{Ext}_\rho(\pi, Y)$, $\rho \colon \pi \to \mathrm{Out}(Y)$ being the monodromy. (According to the remarks after Lemma 4.3, $\mathrm{Ext}_\rho(\pi, Y)$ is in bijection with the cohomology group $H^2_{Z\rho}(\pi; \check{Z}(Y))$.)

The pull back diagram

$$\begin{tikzcd} \mathrm{map}_{\mathrm{End}(\pi)}(BG,BG) \arrow[r] \arrow[d] & \mathrm{map}(BG,BG) \arrow[d] \\ \mathrm{End}(\pi) \arrow[r] & \mathrm{map}(BG,B\pi) \end{tikzcd}$$

where the bottom map takes $\chi\colon \pi \to \pi$ to $BG \to B\pi \xrightarrow{B\chi} B\pi$, serves as definition of the space in the upper left corner. Thus $\mathrm{map}_{\mathrm{End}(\pi)}(BG,BG)$ consists of self-maps of BG over maps $B\chi\colon B\pi \to B\pi$ induced from endomorphisms of the group π.

Recall that $\mathrm{End}(G) = [BG,*;BG]$ denotes the monoid of based homotopy classes of based self-maps of BG.

Lemma 5.1. *$\pi_0 \,\mathrm{map}_{\mathrm{End}(\pi)}(BG,BG) \cong \mathrm{End}(G)$.*

Proof. The above pull back diagram of free mapping spaces also has a based version

$$\begin{tikzcd} \mathrm{map}_{\mathrm{End}(\pi)}(BG,*;BG) \arrow[r] \arrow[d] & \mathrm{map}_*(BG,BG) \arrow[d] \\ \mathrm{End}(\pi) \arrow[r, "\simeq"'] & \mathrm{map}_*(BG,B\pi) \end{tikzcd}$$

defining the space in the upper left corner. Thus $\mathrm{map}_{\mathrm{End}(\pi)}(BG,*;BG)$ is the space of based self-maps of BG over maps $B\chi\colon B\pi \to B\pi$ induced from endomorphisms of the group π. Note that in this based version, the horizontal maps are homotopy equivalences [11, Lemma 2.2]. Thus we have monoid homomorphisms

$$\pi_0(\mathrm{map}_{\mathrm{End}(\pi)}(BG,BG)) \leftarrow \pi_0(\mathrm{map}_{\mathrm{End}(\pi)}(BG,*;BG)) \xrightarrow{\cong} \mathrm{End}(G)$$

where the right hand arrow actually is an isomorphism. Also the left hand arrow is an isomorphism for, as the fibre BY is simply connected, two vertically homotopic fibre maps are also based vertically homotopic. $\square$

The submonoid $\varepsilon_{\mathbb{Q}}(G) \subseteq \mathrm{End}(G) = \pi_0\,\mathrm{map}_{\mathrm{End}(\pi)}(BG,BG)$ of rational automorphisms of G thus consists of all vertical homotopy classes of fibre maps of the form $(f,B\chi)$ where $f\colon BG \to BG$ restricts to a rational automorphism $f|BY\colon BY \to BY$ on the fibre BY. Consider the monoid homomorphism

$$\lambda\colon \varepsilon_{\mathbb{Q}}(G) \to \mathrm{End}(\pi) \times \varepsilon_{\mathbb{Q}}(Y)$$

that to the fibre self-map $(f,B\chi)$ associates the pair consisting of $\chi \in \mathrm{End}(\pi)$ and the restriction $f|BY \in \varepsilon_{\mathbb{Q}}(Y)$.

Let $\mathrm{End}_{\mathbb{Q}}(\pi,Y) \subseteq \mathrm{End}(\pi) \times \varepsilon_{\mathbb{Q}}(Y)$ denote the submonoid consisting of pairs (χ,g) where g is χ-equivariant. $\mathrm{End}_{\mathbb{Q}}(\pi,Y)$ contains the submonoid $\mathrm{End}_{\mathbb{Q}}(\pi,Y)_G$ of elements (χ,g) for which $\chi^*(G) = g_*(G)$ in $\mathrm{Ext}_{\rho\chi}(\pi,Y)$.

Theorem 5.2. *Suppose that the connected component Y of G is completely reducible. Then there exists a short exact sequence*

$$0 \to H^1_{Z\rho}(\pi; \check{Z}(Y)) \to \varepsilon_{\mathbf{Q}}(G) \xrightarrow{\lambda} \mathrm{End}_{\mathbf{Q}}(\pi, Y)_G \to 1$$

of monoids. For any pair $(\chi, g) \in \mathrm{End}_{\mathbf{Q}}(\pi, Y)_G$, there is a bijection between the inverse image $\lambda^{-1}(\chi, g)$ and the cohomology group $H^1_{Z\rho\chi}(\pi; \check{Z}(Y))$.

Proof. The image of λ equals $\varepsilon_{\mathbf{Q}}(\pi, Y)_G$ by Theorem 3.9. The kernel of λ is the group of vertical homotopy classes of maps over $B\pi$ with restriction to BY homotopic to the identity. As a set, $\ker(\lambda)$ is in bijection with the vertical homotopy classes of sections, $\pi_0((BZ(Y))^{h\pi})$, of the fibration

$$\mathrm{map}(BY, BY)_{B1} \to BZ(Y)_{h\pi} \to B\pi \tag{19}$$

with total space $BZ(Y)_{h\pi} = \mathrm{fibmap}(BG, BG)^{B1}_{B1}$. If $f, g\colon BG \to BG$ are maps over $B\pi$ representing elements of $\ker(\lambda)$, let s_f, s_g denote the corresponding sections of fibration (19). Note that $s_{fg} = \underline{f} \circ s_g$ where $\underline{f}$ denotes the self-map over $B\pi$ of $\mathrm{fibmap}(BG, BG)^{B1}_{B1}$ given by composition with f. By Lemma 4.3, fibration (19) has a fibrewise discrete approximation

$$B\check{Z}(Y) \to B\check{Z}(Y)_{h\pi} \to B\pi. \tag{20}$$

and the fibre of the fibrewise discrete approximation $B\check{Z}(Y)_{h\pi} \to BZ(Y)_{h\pi}$ is BV for some rational vector space V. The vanishing of the cohomology groups $H^2(\pi; V)$ and $H^1(\pi; V)$ shows that in the situation

$$\begin{tikzcd} & B\check{Z}(Y)_{h\pi} \arrow[d] \\ B\pi \arrow[r] \arrow[ur, dashed] & BZ(Y)_{h\pi} \end{tikzcd}$$

there are no obstructions to lifting maps or (vertical) homotopies. Thus fibrewise completion induces a bijection $\pi_0((B\check{Z}(Y))^{h\pi}) \cong \pi_0((BZ(Y))^{h\pi})$ and we may view the section s_f of (19) as a section of the fibrewise discrete approximation (20). Furthermore, since $B\check{Z}(Y)_{h\pi}$ is the classifying space of a p-discrete toral group so that $H^1(B\check{Z}(Y)_{h\pi}; V) = 0 = H^2(B\check{Z}(Y)_{h\pi}; V)$, similar considerations applied to the situation

$$\begin{tikzcd} B\check{Z}(Y)_{h\pi} \arrow[rr, dashed] \arrow[d] & & B\check{Z}(Y)_{h\pi} \arrow[d] \\ BZ(Y)_{h\pi} \arrow[rr] \arrow[dr] & & BZ(Y)_{h\pi} \arrow[dl] \\ & B\pi & \end{tikzcd}$$

show that there is a bijection between vertical homotopy classes of self-maps over $B\pi$ of $BZ(Y)_{h\pi}$ and vertical homotopy classes of self-maps over $B\pi$ of $B\check{Z}(Y)_{h\pi}$. In particular, we may view the self-map $\underline{f}$ of $BZ(Y)_{h\pi}$ as a self-map of $B\check{Z}(Y)_{h\pi}$.

Associate to f the primary difference [15, p. 299] $\delta^1(s_f, s_1) \in H^1_{Z\rho}(\pi; \check{Z}(Y))$ between the sections (of fibration (20)) corresponding to f and to the identity map. Then

$$\begin{aligned}\delta^1(s_{fg}, s_1) &= \delta^1(s_{fg}, s_f) + \delta^1(s_f, s_1)\\ &= \delta^1(\underline{f} \circ s_g, \underline{f} \circ s_1) + \delta^1(s_f, s_1)\\ &= (\underline{f})_* \delta^1(s_g, s_1) + \delta^1(s_f, s_1)\\ &= \delta^1(s_g, s_1) + \delta^1(s_f, s_1)\end{aligned}$$

because $\underline{f}$ is homotopic to the identity map on the fibre $B\check{Z}(Y)$. This computation shows that the bijection $\ker(\lambda) \to H^1_{Z\rho}(\pi; \check{Z}(Y))\colon f \to \delta^1(s_f, s_1)$ is a group homomorphism.

For an arbitrary pair $(\chi, g) \in \mathrm{End}_{\mathbb{Q}}(\pi, Y)_G$, the inverse image $\lambda^{-1}(\chi, g)$ is in bijection with the vertical homotopy classes of sections of the fibration $\mathrm{fibmap}(BG, BG)^{Bg}_{B\chi} \to B\pi$, or, equivalently (see the proof of Theorem 3.9), the vertical homotopy classes of sections of the fibration $\mathrm{fibmap}(B(g_*G), B(\chi^*G))^{B1}_{B1} \to B\pi$. Taking primary differences, as above, with respect to some fixed section provides a (noncanonical) bijection $\lambda^{-1}(\chi, g) \to H^1_{Z\rho\chi}(\pi; \check{Z}(Y))$. $\square$

More explicitly, the elements of $\lambda^{-1}(\chi, g)$ are represented by maps of the form

$$BG \to B(g_*G) \to B(\chi^*G) \to BG$$

where the outer maps are fixed as the canonical ones [(16), (18)] and the middle arrow varies over all fibre homotopy equivalences over $B\pi$ and under the homotopy class of the identity map of BY.

The *space* of self-maps of BG over $B\chi$ and homotopic to Bg on the fibre BY is homotopy equivalent to the section space of $\mathrm{fibmap}(B(g_*G), B(\chi^*G))^{B1}_{B1} \to B\pi$, i.e. to the homotopy fixed point space

$$\mathrm{map}(BY, BY)^{h\pi}_{B1} \simeq BZ(Y)^{h\pi},$$

which, by obstruction theory, is a disjoint union of classifying spaces of p-compact toral groups.

Let $\mathrm{Aut}(\pi, Y)$ and $\mathrm{Aut}(\pi, Y)_G$ denote the subgroups of invertible elements of $\mathrm{End}_{\mathbb{Q}}(\pi, Y)$ and $\mathrm{End}_{\mathbb{Q}}(\pi, Y)_G$, respectively. The monoid short exact sequence of Theorem 5.2 restricts to a short exact sequence

$$0 \to H^1_{Z\rho}(\pi; \check{Z}(Y)) \to \mathrm{Aut}(G) \xrightarrow{\lambda} \mathrm{Aut}(\pi, Y)_G \to 1$$

of groups. The equivalence class of this group extension is unknown but it is perhaps worth noting that a somewhat similar group extension is determined by a differential in a Lyndon-Hochschield-Serre spectral sequence [11].

Corollary 5.3. *Suppose that Y is connected, completely reducible, and centerfree p-compact group. Then* $\mathrm{Ext}_\rho(\pi, Y) = \{Y \to G \to \pi\}$*,* $\varepsilon_{\mathbb{Q}}(G) = \mathrm{End}_{\mathbb{Q}}(\pi, Y)$*, and* $\mathrm{Aut}(G) = \mathrm{Aut}(\pi, Y)$*.*

Proof. Apply Lemma 3.3 and Theorem 5.2. □

Finally, a couple of examples to illustrate the use of Theorem 5.2.

Example 5.4. (1) The 2-compact group $\mathrm{SO}(2n+1)^\wedge_2$, $n \geq 2$, is centerfree. Hence $\mathrm{Ext}_\rho(\pi, \mathrm{SO}(2n+1)^\wedge_2)$ contains [Corollary 5.3] exactly one element G with $\varepsilon_{\mathbb{Q}}(G) = \mathrm{End}_{\mathbb{Q}}(\pi, \mathrm{SO}(2n+1)^\wedge_2)$ and $\mathrm{Aut}(G) = \mathrm{Aut}(\pi, \mathrm{SO}(2n+1)^\wedge_2)$ for any given homotopy action $\rho\colon \pi \to \mathrm{Out}(\mathrm{SO}(2n+1)^\wedge_2)$.
(2) The center of the 2-compact group $\mathrm{SO}(2n)^\wedge_2$, $n > 4$, is cyclic of order 2 so the affine group $\mathrm{Ext}_\rho(\pi, \mathrm{SO}(2n)^\wedge_2)$ of equivalence classes of short exact sequences of 2-compact groups

$$\mathrm{SO}(2n)^\wedge_2 \to G \to \pi$$

realizing a fixed homotopy action $\rho\colon \pi \to \mathrm{Out}(\mathrm{SO}(2n)^\wedge_2)$ has $H^2(\pi; \mathbb{Z}/2)$ as group of operators. Assume that the homotopy action ρ is injective. Since $\varepsilon_{\mathbb{Q}}(\mathrm{SO}(2n)^\wedge_2) = \mathrm{Aut}(\mathrm{SO}(2n)^\wedge_2)$ [10, Theorem 5.6] is abelian [9], it follows that $\mathrm{End}_{\mathbb{Q}}(\pi, \mathrm{SO}(2n)^\wedge_2) = \mathrm{Aut}(\pi, \mathrm{SO}(2n)^\wedge_2) = \mathrm{Aut}(\mathrm{SO}(2n)^\wedge_2)$ consists of all automorphisms. For any (equivariant) automorphism g of $\mathrm{SO}(2n)^\wedge_2$, $Z(g)$ is the identity map, so [Lemma 3.8] g_* fixes G if and only if it fixes all elements of $\mathrm{Ext}_\rho(\pi, \mathrm{SO}(2n)^\wedge_2)$. The short exact sequence of Theorem 5.2 implies that $\varepsilon_{\mathbb{Q}}(G) = \mathrm{Aut}(G)$.

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