DETERMINISTIC p-COMPACT GROUPS

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Abstract. We investigate the class of N-determined p-compact groups and the class of p-compact groups with N-determined automorphisms. A p-compact group is said to be N-determined if it is determined up to isomorphism by the normalizer of a maximal torus. The automorphisms of a p-compact group are said to be N-determined if they are determined by their restrictions to this maximal torus normalizer.

1. Introduction

The p-compact groups, introduced by Dwyer and Wilkerson [6] as homotopy theoretic Lie groups, have indeed turned out to possess an interesting and rich structure [18, 7, 8, 14]. For instance, any p-compact group is equipped with a maximal torus supporting a Weyl group action and the Borel construction of this action provides the normalizer of the maximal torus. The theme of this paper is the interrelation between this maximal torus normalizer and the p-compact group itself. Specifically, I'd like to discuss whether

- automorphisms of p-compact groups are determined by their restrictions to the maximal torus normalizer.
- p-compact groups are determined up to isomorphism by their maximal torus normalizer.

The first of these two problems will be reduced to a computation of two obstruction groups. The second problem, cf. [13, 5.2], [20, 5.20], will be related to the computation of yet another two obstruction groups. The main technical tools are the induction principle based on the homology decomposition theorem of Dwyer and Wilkerson [7, 8.1, 9.2] and the preferred lifts of [15] of monomorphisms of elementary abelian p-groups into p-compact groups.

To be more explicit, let X be a p-compact group (for the sake of this introduction assumed to be connected) and let $N \to X$ be the normalizer of a maximal torus.

Consider the groups $\mathrm{Out}(X)$ of homotopy classes of self-homotopy equivalences of BX and $\mathrm{Out}(N)$ of homotopy classes of self-homotopy equivalences of BN. There is a homomorphism

$$N\colon \mathrm{Out}(X) \to \mathrm{Out}(N) \tag{1.1}$$

defined by restricting automorphisms of X to N. This homomorphism is set up in Section 3. If (1.1) is injective, we say that X has N-determined automorphisms. Section 4 contains information about the class of p-compact groups with

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N-determined automorphisms. It is shown, for instance, that if the adjoint form $X/Z(X)$ has N-determined automorphisms, so does X itself. The conjecture that all p-compact groups have N-determined automorphisms is reduced to the computation of the two Wojtkowiak obstruction groups of (2.1).

Let now X' be another connected p-compact group and assume that also X' admits a monomorphism $N \to X'$ that is conjugate to the normalizer of a maximal torus . Does it follow that X and X' are isomorphic? If this is so for all choices of X', we say that X is N-determined. I think of this as an alternative and perhaps more geometric version of uniqueness than the cohomological uniqueness of e.g [3, 4, 19, 1]. Section 5 contains a discussion of determinism (at primes $p > 2$). It is shown, for instance, that if the adjoint form $X/Z(X)$ is N-determined, so is X itself. The conjecture that in fact all p-compact groups are N-determined is related to the computation of the two Wojtkowiak obstruction groups from (2.2).

This paper should clarify the very last sentence of my Research Report [14].

2. Obstruction theory

Let X be a p-compact group and $\mathbb{A}(\mathbb{X})$ the category whose objects are pairs (V, ν) where V is a nontrivial elementary abelian p-group and $\nu\colon V \to X$ a conjugacy class of monomorphisms. A morphism $g\colon (V_1, \nu_1) \to (V_2, \nu_2)$ in this category is an injection $g\colon V_1 \to V_2$ with $\nu_2 \circ g$ conjugate to ν_1.

Consider the functor BZ from $\mathbb{A}(\mathbb{X})$ to topological spaces given by

$$BZ(V, \nu) = \mathrm{map}(BC_X(\nu), BX)_{Be(\nu)}$$

where $Be(\nu)\colon BC_X(\nu) = \mathrm{map}(BV, BX)_{B\nu} \to BX$ is the evaluation monomorphism. By centricity [2], the space $BZ(V, \nu)$ is homotopy equivalent [7, 1.3] to the classifying space $\mathrm{map}(BC_X(\nu), BC_X(\nu))_{B1}$ of the center $Z(C_X(\nu))$ of the centralizer $C_X(\nu)$.

The $H\mathbb{F}_p$-equivalence $\mathrm{hocolim}_{\mathbb{A}(\mathbb{X})} BC_X(\nu) \to BX$ of the Homology Decomposition Theorem [7, 8.1] induces for any p-compact group X' a map

$$[BX, BX'] \to \lim\nolimits^0_{\mathbb{A}(\mathbb{X})}[BC_X(\nu), BX']$$

which, however, may be neither injective nor surjective. Nonetheless, Wojtkowiak's obstruction theory [22] provides some information.

Lemma 2.1. *Let $f\colon X \to X$ be an endomorphism of the p-compact group X such that $f \circ e(\nu)$ and $e(\nu)$ are conjugate homomorphisms $C_X(\nu) \to X$ for each object (V, ν) of $\mathbb{A}(\mathbb{X})$. If*

$$\lim\nolimits^1_{\mathbb{A}(\mathbb{X})}\pi_1(BZ(V, \nu)) = 0 = \lim\nolimits^2_{\mathbb{A}(\mathbb{X})}\pi_2(BZ(V, \nu))$$

then f is conjugate to the identity.

Lemma 2.2. *Let $(Bf(\nu))_{\nu \in \mathbb{A}(\mathbb{X})} \in \lim^0[BC_X(\nu), BX']$ be a collection of centric morphisms. If*

$$\lim\nolimits^2_{\mathbb{A}(\mathbb{X})}\pi_1(BZ(V, \nu)) = 0 = \lim\nolimits^3_{\mathbb{A}(\mathbb{X})}\pi_2(BZ(V, \nu))$$

then there exists a morphism $f\colon X \to X'$ such that $f \circ e(\nu) \simeq f(\nu)$ for all objects (V, ν) of $\mathbb{A}(\mathbb{X})$.

The morphism $f(\nu)\colon C_X(\nu) \to X'$ is centric if composition with $Bf(\nu)$ is a homotopy equivalence $\mathrm{map}(BC_X(\nu), BC_X(\nu))_{B1} \to \mathrm{map}(BC_X(\nu), BX')_{Bf(\nu)}$.

3. Normalizer homomorphisms

The aim of this section is to relate the automorphism group of a p-compact group to that of the normalizer of a maximal torus.

For any loop space BG, let $\mathrm{Aut}(G)$ denote the group of invertible elements of the monoid $[BG, *; BG]$ of based homotopy classes of self-maps and let $\mathrm{Out}(G)$ denote the group of invertible elements of the monoid $[BG; BG]$ of free homotopy classes of self-maps of BG. There is an exact sequence

$$\pi_0(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

reducing to an isomorphism $\mathrm{Aut}(G) \cong \mathrm{Out}(G)$ if G is connected.

Define $BZ(G) := \mathrm{map}(BG, BG)_{B1}$, the *center* of G. Any morphism $f\colon G \to H$ of loop spaces induces post- and precompostion maps

$$BZ(G) = \mathrm{map}(BG, BG)_{B1} \xrightarrow{\underline{Bf}} \mathrm{map}(BG, BH)_{Bf} \xleftarrow{\overline{Bf}} \mathrm{map}(BH, BH)_{B1} = BZ(H)$$

between mapping spaces. Provided f is *centric*, i.e. $\underline{Bf}$ is a homotopy equivalence, f determines a homomorphism of centers

$$Z(f)\colon Z(H) \to Z(G)$$

given by $\underline{Bf} \circ BZ(f) \simeq \overline{Bf}$. If the square of centric maps

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "f"'] & G \arrow[d, "f"] \\ H \arrow[r, "h"'] & H \end{tikzcd}$$

commutes up to conjugacy, then the induced square of centers

$$\begin{tikzcd} Z(G) & Z(G) \arrow[l, "Z(g)"'] \\ Z(H) \arrow[u, "Z(f)"] & Z(H) \arrow[l, "Z(h)"] \arrow[u, "Z(f)"'] \end{tikzcd}$$

also commutes up to conjugacy. In particular, we obtain when $H = G$ an antihomomorphism

$$Z\colon\ \mathrm{Out}(G) \to \mathrm{Out}(Z(G))$$

between groups of outer automorphisms.

Suppose that X is a *connected* p-compact group. Let $j\colon N \to X$ be the normalizer [7, 9.8] of a maximal torus $i\colon T \to X$ such that $Bj\colon BN \to BX$ is a fibration. Let $f \in \mathrm{Out}(X)$ be an outer automorphism and consider the space $(X/N)^{hN}$ of lifts

$$\begin{tikzcd} & & BN \arrow[d, "Bj"] \\ BN \arrow[r, "Bj"'] \arrow[urr, dashed] & BX \arrow[r, "Bf"'] & BX \end{tikzcd}$$

of $Bf \circ Bj$ over Bj.

Lemma 3.1. [17, 5.1] $(X/N)^{hN}$ *is contractible.*

According to this lemma, there exists a homomorphism

$$N\colon \operatorname{Out}(X) \to \operatorname{Out}(N) \tag{3.2}$$

which to any $f \in \operatorname{Out}(X)$ associates the unique outer automorphism $N(f) \in \operatorname{Out}(N)$ that makes

$$\begin{tikzcd} N \arrow[r, "N(f)"] \arrow[d, "j"'] & N \arrow[d, "j"] \\ X \arrow[r, "f"'] & X \end{tikzcd} \tag{3.3}$$

commute up to conjugacy. Note also that

$$\underline{Bj}\colon \operatorname{map}(BN, BN)_{BN(f)} \to \operatorname{map}(BN, BX)_{Bf\circ Bj}$$

is a homotopy equivalence and, in particular, that j is centric and induces a homomorphism $Z(j)$ between centers.

Lemma 3.4. [7, 7.1] *$Z(j)\colon Z(X) \to Z(N)$ is an isomorphism if p is odd.*

Proof. It follows from Shapiro's lemma,

$$\operatorname{map}(BN, BN) = \operatorname{map}(BT_{hW_T(X)}, BN) = \operatorname{map}(BT, BN)^{hW_T(X)},$$

and the homotopy equivalence [7, 10.6]

$$BC_T(T)^{hW_T(X)} \to BC_N(T)^{hW_T(X)} = \operatorname{map}(BT, BN)^{hW_T(X)}_{Bi}$$

induced from the $W_T(X)$-equivariant homotopy equivalence $BC_T(T) \to BC_N(T)$ [15, 2.6], [6, 9.1] that $BZ(N) = \operatorname{map}(BN, BN)_{B1}$ is homotopy equivalent to the identity component of $BC_T(T)^{hW_T(X)}$. Thus (the loop space of) the identity component of $BC_{\check{T}}(\check{T})^{hW_T(X)}$, where $\check{T} \to T$ is a discrete approximation [6, 6.4], is a discrete approximation to $Z(N)$ and also to $Z(X)$ by [7, §7]. □

We now turn to the case of a possibly *nonconnected* p-compact group.

Let π be a finite group and assume that G is a π-loop space, i.e. a fibration $BG \to BG_{h\pi} \to B\pi$ of based spaces over $B\pi$ with fibre BG. A π-automorphism of the π-loop space G is a fibre homotopy equivalence of the form

$$\begin{tikzcd} BG_{h\pi} \arrow[r, "Bf_{h\chi}"] \arrow[d] & BG_{h\pi} \arrow[d] \\ B\pi \arrow[r, "B\chi"'] & B\pi \end{tikzcd}$$

where χ is a group automorphism of π. We say that two such fibre maps, $(Bf_{h\chi}, B\chi)$ and $(Bf'_{h\chi'}, B\chi')$, are homotopic if $\chi = \chi'$ and $Bf_{h\chi}$ and $Bf'_{h\chi'}$ are homotopic over $B\chi = B\chi'$. Let $\operatorname{Aut}_{h\pi}(G)$ denote the group of homotopy classes of π-automorphisms of the π-loop space G.

This new automorphism group is related to the ones previously introduced.

Remark 3.5. Let G be a π-loop space.

- $BZ(G) := \mathrm{map}(BG, BG)_{B1}$ is a π-space and there is, cf. [16, 5.1, 5.2], a short exact sequence of groups
$$1 \to \pi_0(BZ(G)^{h\pi}) \to \mathrm{Aut}_{h\pi}(G) \to \mathrm{Aut}(\pi) \times \mathrm{Out}(G)$$
where $\pi_0(BZ(G))^{h\pi}$ is the group of fibre homotopy classes of self-maps of $BG_{h\pi}$ over $B\pi$ and under self-maps homotopic to the identity of BG. In case G is a p-compact group or an extended p-compact torus and $\check{Z}(G) \to Z(G)$ a discrete approximation, this group identifies to the cohomology group $H^1(\pi; \check{Z}(G))$.
- There exists a commutative diagram
$$\begin{tikzcd}
1 \arrow[r] & \pi_0(G) \arrow[r] \arrow[d, equal] & \pi_0(G_{h\pi}) \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d] & 1 \\
& \pi_0(G) \arrow[r] & \mathrm{Aut}(G_{h\pi}) \arrow[r] \arrow[d] & \mathrm{Aut}_{h\pi}(G) \arrow[r] \arrow[d] & 1 \\
& & \mathrm{Out}(G_{h\pi}) \arrow[r, equal] \arrow[d] & \mathrm{Out}(G_{h\pi}) \arrow[d] & \\
& & 1 & 1 &
\end{tikzcd}$$
with exact rows and columns. In particular, $\mathrm{Aut}(G_{h\pi}) \cong \mathrm{Aut}_{h\pi}(G)$ if G is connected.

Consider a p-compact group $X_{h\pi}$ with identity component X and component group π. The normalizer $N_{h\pi}$ of the maximal torus $T \to X \to X_{h\pi}$ is related to the normalizer N of the maximal torus $T \to X$ by the commmutative diagram
$$\begin{tikzcd}
N \arrow[r] \arrow[d, "j"'] & N_{h\pi} \arrow[r] \arrow[d, "j_{h\pi}"'] & \pi \arrow[d, equal] \\
X \arrow[r] & X_{h\pi} \arrow[r] & \pi
\end{tikzcd}$$
with exact rows. Thus both N and X are π-loop spaces and $j \colon N \to X$ is a π-loop space homomorphism.

Let $f_{h\chi} \in \mathrm{Aut}_{h\pi}(X)$ be a π-loop space automorphism and $(X_{h\pi}/N_{h\pi})^{hN_{h\pi}}$ the space of lifts
$$\begin{tikzcd}
& & BN_{h\pi} \arrow[d, "Bj_{h\pi}"] \\
BN_{h\pi} \arrow[r, "Bj_{h\pi}"'] \arrow[urr, dashed] & BX_{h\pi} \arrow[r, "Bf_{h\chi}"'] & BX_{h\pi}
\end{tikzcd}$$
of $Bf_{h\chi} \circ Bj_{h\pi}$ over $Bj_{h\pi}$.

Lemma 3.6. $(X_{h\pi}/N_{h\pi})^{hN_{h\pi}}$ *is contractible.*

Proof. We have
$$(X_{h\pi}/N_{h\pi})^{hN_{h\pi}} \simeq (X_{h\pi}/N_{h\pi})^{hN})^{h\pi} \simeq ((X/N)^{hN})^{h\pi} \simeq *$$
since $(X/N)^{hN}$ is contractible by (3.1). □

Lemma 3.6 implies that there exists a homomorphism

$$N_{h\pi}\colon \operatorname{Aut}_{h\pi}(X) \to \operatorname{Aut}_{h\pi}(N) \tag{3.7}$$

that to any $f_{h\chi} \in \operatorname{Aut}_{h\pi}(X)$ associates the unique $N(f)_{h\chi} \in \operatorname{Aut}_{h\pi}(N)$ such that

$$\begin{tikzcd} BN_{h\pi} \arrow[r, "BN(f)_{h\chi}"] \arrow[d, "Bj_{h\pi}"'] & BN_{h\pi} \arrow[d, "Bj_{h\pi}"] \\ BX_{h\pi} \arrow[r, "Bf_{h\chi}"'] & BX_{h\pi} \end{tikzcd}$$

commutes up to vertical homotopy. In (3.7), $\operatorname{Aut}_{h\pi}(X) \cong \operatorname{Aut}(X_{h\pi})$ and $\operatorname{Aut}_{h\pi}(N) \cong W_T(X_0)\backslash \operatorname{Aut}(N_{h\pi})$.

Ignoring the π-structure, (3.6) also determines a homomorphism

$$N\colon \operatorname{Out}(X_{h\pi}) \to \operatorname{Out}(N_{h\pi}) \tag{3.8}$$

such that $N(f)$ for any $f \in \operatorname{Out}(X_{h\pi})$ is the unique outer automorphism of $N_{h\pi}$ that makes

$$\begin{tikzcd} BN_{h\pi} \arrow[r, "BN(f)"] \arrow[d, "Bj_{h\pi}"'] & BN_{h\pi} \arrow[d, "Bj_{h\pi}"] \\ BX_{h\pi} \arrow[r, "Bf"'] & BX_{h\pi} \end{tikzcd}$$

commutative up to homotopy. The homomorphisms (3.7) and (3.8) are related by a commutative diagram

$$\begin{tikzcd} \pi \arrow[r] \arrow[d, equal] & \operatorname{Aut}_{h\pi}(X) \arrow[r] \arrow[d, "N_{h\pi}"'] & \operatorname{Out}(X_{h\pi}) \arrow[r] \arrow[d, "N"'] & 1 \\ \pi \arrow[r] & \operatorname{Aut}_{h\pi}(N) \arrow[r] & \operatorname{Out}(N_{h\pi}) \arrow[r] & 1 \end{tikzcd} \tag{3.9}$$

with exact rows. In particular, $N_{h\pi}$ mono $\Rightarrow N$ mono and N epi $\Rightarrow N_{h\pi}$ epi.

Definition 3.10. *The p-compact group $X_{h\pi}$ has N-determined automorphisms if the morphism $N_{h\pi}$ of (3.7) is injective.*

If the component group π is trivial, i.e. $X_{h\pi} = X$ is connected, then $N_{h\pi}$ and N are the same map.

There are p-compact groups with N-determined automorphisms.

Proposition 3.11. *If the Weyl group order of X is prime to p, then X has N-determined automorphisms.*

This follows from

Lemma 3.12. $H^*(Bj, \mathbb{F}_p)\colon \mathbb{H}^*(\mathbb{BX}; \mathbb{F}_p) \to \mathbb{H}^*(\mathbb{BN}; \mathbb{F}_p)$ *is an isomorphism if* $p \nmid |\pi_0(N)|$.

Proof. Since BX is p-complete, the map Bj factors through a map $(BN)^\wedge_p \to BX$. The domain of this map is a connected (Clark-Ewing) p-compact group with maximal torus normalizer N [15, 1.2] and the map itself is an isomorphism of p-compact groups since [18, 3.7] it is monomorphism (as the restriction to the p-normalizer of the maximal torus is) and a rational equivalence [6, 9.7]. $\square$

4. N-DETERMINED AUTOMORPHISMS OF p-COMPACT GROUPS

Let first $X_{h\pi}$ be a p-compact group with connected component X and component group π. We shall compare the two homomorphisms

$$N_{h\pi}\colon\ \mathrm{Aut}_{h\pi}(X) \to \mathrm{Aut}_{h\pi}(N)$$
$$N\colon\ \mathrm{Out}(X) \to \mathrm{Out}(N)$$

of automorphism groups. Let $\rho\colon \pi \to \mathrm{Out}(X)$ denote the monodromy action and

$$H^1_{Z^{-1}\rho}(\pi; \check{Z}(X)) \to H^1_{Z^{-1}N\rho}(\pi; Z(\check{N})) \tag{4.1}$$

the map induced on cohomology by the discrete approximation to $Z(N(i))$.

Proposition 4.2. *Let $X_{h\pi}$ be a p-compact group with identity component X and component group π. Suppose that X has N-determined automorphisms and that (4.1) is a monomorphism. Then $X_{h\pi}$ has N-determined automorphisms.*

Proof. The homomorphism $N_{h\pi}$ fits into a commutative diagram (3.5) with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^1_{Z^{-1}\rho}(\pi; \check{Z}(X)) \arrow[r] \arrow[d] & \mathrm{Aut}_{h\pi}(X) \arrow[r] \arrow[d, "N_{h\pi}"'] & \mathrm{Aut}(\pi) \times \mathrm{Out}(X) \arrow[d, "1\times N"'] \\ 0 \arrow[r] & H^1_{Z^{-1}N\rho}(\pi; Z(\check{N})) \arrow[r] & \mathrm{Aut}_{h\pi}(N) \arrow[r] & \mathrm{Aut}(\pi) \times \mathrm{Out}(N) \end{tikzcd}$$

where the outer vertical homomorphisms are injective. □

Recall (3.4) that at odd primes, (4.1) automatically is an isomorphism.

Example 4.3. If the order of the Weyl group of the identity component X is prime to p and (4.1) is an isomorphism (e.g. if $p > 2$), then $X_{h\pi}$ has N-determined automorphisms. Obviously, all p-compact toral groups have N-determined automorphisms.

Next, let X be a connected p-compact group with center $Z \to X$ and adjoint form X/Z. If $i\colon T \to X$ is a maximal torus for X with normalizer $j\colon N \to X$, then $i/Z\colon T/Z \to X/Z$ is a maximal torus for X/Z [18, 4.6] with normalizer $j/Z\colon N/Z \to X/Z$ [15, 3.8].

We shall, for the benefit of the prime $p = 2$, need a slight refinement of the homomorphism (3.2). The homomorphism

$$Z = Z(X) \xrightarrow{Z(j)} Z(N) \to N \tag{4.4}$$

is a central monomorphism of extended p-compact toral groups [7, 7.7] (that equals the center of N if p is odd (3.4)). Define $\mathrm{Out}(N, Z)$ to be the subgroup of $\mathrm{Out}(N)$ consisting of those conjugacy classes of automorphisms f of N that preserve the central monomorphism (4.4) in the sense that there exists some automorphism of Z such that

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z \arrow[d] \\ N \arrow[r, "f"'] & N \end{tikzcd}$$

commutes up to conjugacy. ($\mathrm{Out}(N,Z)=\mathrm{Out}(N)$ if p is odd.) Note, for instance by using discrete approximations [7, 3.13], that the automorphism of Z, if it exists, is uniquely determined by f, so that we have a restriction homomorphism $|Z\colon \mathrm{Out}(N,Z)\to\mathrm{Out}(Z)$. Now note that the homomorphism (3.2) actually takes values in the subgroup $\mathrm{Out}(N,Z)$ of $\mathrm{Out}(N)$ and that $N(f)|Z=Z(f)^{-1}$ for all $f\in\mathrm{Out}(X)$.

By naturality [6, 8.3] of the short exact sequence of p-compact groups

$$Z\to X\to X/Z$$

there is a homomorphism

$$(Z^{-1},/Z)\colon\ \mathrm{Out}(X)\to\mathrm{Out}(Z)\times\mathrm{Out}(X/Z) \tag{4.5}$$

taking $f\in\mathrm{Out}(X)$ to $(Z(f)^{-1},f/Z)$ where f/Z is the automorphism of the adjoint form X/Z induced by $Z(f)^{-1}$ and f. Observe that [17, 4.3] implies that (4.5) is injective.

Similarly, naturality of the short exact sequence of extended p-compact toral groups

$$Z\to N\to N/Z$$

allows us to define a group homomorphism

$$(|Z,/Z)\colon\ \mathrm{Out}(N,Z)\to\mathrm{Out}(Z)\times\mathrm{Out}(N/Z) \tag{4.6}$$

taking $f\in\mathrm{Out}(N,Z)$ to the pair consisting of restriction $f|Z$ of f to Z and the quotient automorphism f/Z as above.

The group homomorphisms (4.5) and (4.6) are related by a commutative diagram

$$\begin{tikzcd}
\mathrm{Out}(X) \arrow[r, "N"] \arrow[d, "{(Z^{-1},/Z)}"'] & \mathrm{Out}(N,Z) \arrow[d, "{(|Z,/Z)}"] \\
\mathrm{Out}(Z)\times\mathrm{Out}(X/Z) \arrow[r, "1\times N/Z"] & \mathrm{Out}(Z)\times\mathrm{Out}(N/Z)
\end{tikzcd} \tag{4.7}$$

and as left the vertical homomorphism (4.5) is injective, the next proposition is immediate.

Proposition 4.8. *Let X be a connected p-compact group with adjoint form X/Z. If X/Z has N-determined automorphisms, so does X.*

We now invoke the Homology Decomposition Theorem [7].

Theorem 4.9. *Let X be a connected p-compact group. Assume that*

(1) *$C_X(\nu)$ has N-determined automorphisms for each object (V,ν) of $\mathbb{A}(\mathbb{X})$.*
(2) $\lim^1_{\mathbb{A}(\mathbb{X})}\pi_1(BZ(V,\nu))=0=\lim^2_{\mathbb{A}(\mathbb{X})}\pi_2(BZ(V,\nu))$.

Then X has N-determined automorphisms.

Proof. Let $f\in\mathrm{Out}(X)$ be an outer automorphism with $N(f)=1$. Let $\nu\colon V\to X$ be a monomorphism of a nontrivial elementary abelian p-group V to X with [15] preferred lift $\mu\colon V\to N$ to the normalizer of the maximal torus.

Since $f \circ \nu = f \circ j \circ \mu = j \circ N(f) \circ \mu = j \circ \mu = \nu$, f restricts to an automorphism $C_f(\nu)$ of $C_X(\nu)$ such that $f \circ e(\nu) = e(\nu) \circ C_f(\nu)$ and such that the diagram

$$\begin{tikzcd} & C_N(\mu) \arrow[dl] \arrow[dr] & \\ C_X(\nu) \arrow[rr, "C_f(\nu)"] & & C_X(\nu) \end{tikzcd}$$

commutes up to conjugacy. By assumption (1), $C_f(\nu)$ is conjugate to the identity of $C_X(\nu)$ and $f \circ e(\nu) \simeq e(\nu)$. By assumption (2) and obstruction theory (2.1), f is conjugate to the identity of X. $\square$

Remark 4.10. If it can be shown that the obstruction groups of 4.9.(2) vanish for any connected, centerfree p-compact group X, then (4.2, 4.8, 4.9) will, by the induction principle of [7, §9], imply that all p-compact groups, $p > 2$, have N-determined automorphisms.

Example 4.11. Let $X = \mathrm{PU}(3)$ and $p = 3$. There are, up to isomorphism, two conjugacy classes, L_1 and L_2, of monomorphisms $\mathbb{Z}/\nvDash \rightarrowtail \mathbb{X}$, and two conjugacy classes, V_1 and V_2, of monomorphisms $(\mathbb{Z}/\nvDash)^{\nvDash} \rightarrowtail \mathbb{X}$. Their centralizers are

$$C_X(L_1) = \frac{\mathrm{U}(2) \times \mathrm{U}(1)}{\mathrm{U}(1)} \qquad C_X(L_2) = T \rtimes \langle \sigma \rangle$$
$$C_X(V_1) = T \qquad\qquad C_X(V_2) = V_2$$

where T is the maximal torus, V_1 consists of the elements of order 3 in T, σ is the permutation matrix of the 3-cycle $\sigma = (123)$, and $V_2 = \langle \sigma, \mathrm{diag}(1, \zeta, \zeta^2) \rangle$ where ζ is a primitive third root of unity. Note that these centralizers all have N-determined automorphisms (4.3).

Of the rank two elementary abelian p-groups in X, only V_2 has a centralizer with disconnected center and the automorphism group of V_2 as an object of the category $\mathbb{A}(\mathbb{X})$ can be shown to be the special linear group $\mathrm{SL}(2,3)$. Oliver's cochain complex [21] for the computation of the derived functors of the inverse limit functor now shows that $\lim^1_{\mathbb{A}(\mathbb{X})} \pi_1(BZ(V,\nu)) = 0$ since there are no nontrivial $\mathrm{SL}(2,3)$-homomorphisms from the Steinberg representation $\mathrm{St}(V_2)$ to V_2. Also $\lim^2_{\mathbb{A}(\mathbb{X})} \pi_2(BZ(V,\nu)) = 0$ since there are no monomorphisms from a rank three elementary abelian p-group to X.

Theorem 4.9 now implies that PU(3) (and hence also U(3) and SU(3)) has N-determined automorphisms at $p = 3$ (and in fact at all odd primes). (Alternatively, this follows from [9]; see Section 5)

For later use we note that Lemma 3.4 can be extended to nonconnected p-compact groups whose identity component has N-determined automorphisms.

Corollary 4.12. *Assume that X has N-determined automorphisms and that p is odd. Then the maps*

$$\mathrm{map}(BN_{h\pi}, BN_{h\pi})_{B1} \to \mathrm{map}(BN_{h\pi}, BX_{h\pi})_{Bj_{h\pi}} \leftarrow \mathrm{map}(BX_{h\pi}, BX_{h\pi})_{B1}$$

are homotopy equivalences.

Proof. The left map is a homotopy equivalence by Lemma 3.6. To see that also the right map is a homotopy equivalence, it suffices by Shapiro's lemma to see that the

π-map $\mathrm{map}(BN, BX_{h\pi})_{Bj} \leftarrow \mathrm{map}(BX, BX_{h\pi})_{Bj}$, where j denotes an appropriate inclusion homomorphism, is a homotopy equivalence.

Since X has N-determined automorphisms, the orbit $\pi \cdot B1 \subset \mathrm{Out}(X)$ of the π-action on the identity map of BX is taken bijectively onto the orbit $\pi \cdot Bj \subset [BN, BX]$ of the π-action on Bj. This means that the fibre map

$$\begin{tikzcd} \mathrm{map}(BN, BX_{h\pi})_{Bj} \arrow[d] & \mathrm{map}(BX, BX_{h\pi})_{Bj} \arrow[l] \arrow[d] \\ \mathrm{map}(BN, B\pi)_{B0} & \mathrm{map}(BX, B\pi)_{B0} \arrow[l, "\simeq"'] \end{tikzcd} \tag{4.13}$$

restricts to a bijection on the set of path components of the fibre. Since also the restriction to a typical component of the fibre, $\mathrm{map}(BN, BX)_{Bj} \leftarrow \mathrm{map}(BX, BX)_{B1}$, is a homotopy equivalence by Lemma 3.4, we may conclude that (4.13) is a (fibre) homotopy equivalence. $\square$

5. The Lie case

The purpose of this section is to verify that compact Lie groups have N-determined automorphisms (and that they do not contradict (7.2)).

Let G be any compact connected Lie group and let $\hat{G}$ denote G viewed as a p-compact group, i.e. $B\hat{G} = (BG)^\wedge_p$, where the prime p as in the previous section, is assumed to be *odd*. Let $N \to \hat{G}$ be the normalizer of a maximal torus $T \to \hat{G}$. (This maximal torus as well as its normalizer can be obtained from Lie group constructions.)

Proposition 5.1. [11, 3.5] $N\colon \mathrm{Out}(\hat{G}) \to \mathrm{Out}(N)$ *is an isomorphism.*

The proof of this proposition amounts to an interpretation of the fundamental results obtained by Jackowski, McClure, and Oliver.

The Weyl group $W = \pi_0(N)$ of the Lie group G or, what is the same, the p-compact group $\hat{G}$ acts on the short exact sequence

$$0 \longrightarrow \pi_1(T) \longrightarrow \pi_1(T) \otimes \mathbb{Q} \xrightarrow{\varepsilon} \check{\mathbb{T}} \longrightarrow \nvdash$$

where $\check{T}$ is the discrete approximation to T. We start by computing the first cohomology group of $\check{T}$ as a W-module.

Lemma 5.2. $H^1(W; \check{T}) = 0$.

Proof. [9, 3.5] The Weyl group has a presentation

$$W = \langle R_1, \ldots, R_n | R_i^2 = 1,\ (R_iR_j)^{m_{ij}} = 1,\ i \neq j \rangle$$

as a Coxeter group for certain integers m_{ij}. The equations

$$0 = (1 + R_i)(1 - R_i), \quad 1 = \frac{1}{2}(1 + R_i) + \frac{1}{2}(1 - R_i),$$

which hold in the group ring $\mathbb{Q}_l[\mathbb{W}]$, imply that $\mathrm{im}(1 - R_i) = \ker(1 + R_i)$ for all i in any $\mathbb{Q}_l[\mathbb{W}]$-module ($p > 2$).

Let now $f\colon W \to \check{T}$ be a 1-cocycle. It suffices to lift f to a 1-cocycle $W \longrightarrow \pi_1(T) \otimes \mathbb{Q}$, i.e. to find vectors $v_1, \ldots, v_n \in \pi_1(T) \otimes \mathbb{Q}$ such that $\varepsilon(v_i) = f(R_i)$ and

$$(1 + R_i)v_i = 0 \tag{5.3}$$

$$(1 + (R_iR_j) + \cdots + (R_iR_j)^{m_{ij}-1})(v_i + R_iv_j) = 0 \tag{5.4}$$

for all i and all $i \neq j$, respectively.

Since p is odd and the reflection R_i has order two, the cohomology group $H^1(\langle R_i\rangle; \check{T})$ is trivial. This means that f restricts to a 1-coboundary on the subgroup $\langle R_i\rangle$, i.e.

$$f(R_i) \in \operatorname{im}(1-R_i) = \operatorname{im}(1-R_i)\varepsilon = \varepsilon(\operatorname{im}(1-R_i)) = \varepsilon(\ker(1+R_i))$$

so that $\varepsilon(v_i) = f(R_i)$ for some $v_i \in \ker(1+R_i)$. Thus condition (5.3) is satisfied.

For $i \neq j$, the composition R_iR_j acts as rotation on the 2-dimensional subspace $\ker(1+R_i) + \ker(1+R_j)$ of $\pi_1(T) \otimes \mathbb{Q}$. Since

$$v_i + R_iv_j = v_i + v_j - (1-R_i)v_j$$

belongs to this subspace, also condition (5.4) is satisfied. □

Proof of Proposition 5.1. Let $\operatorname{Aut}(W, \check{T})$ denote the subgroup of $\operatorname{Aut}(W) \times \operatorname{Aut}(\check{T})$ consisting of pairs (χ, ϕ) such that ϕ is χ-equivariant. The natural map

$$\operatorname{Aut}(N) \longrightarrow \operatorname{Aut}(W, \check{T})$$

is injective since its kernel $H^1(W, \check{T}) = 0$ by (5.2), cf. [16, §5]. Thus also the induced map of quotient groups

$$\operatorname{Out}(N) = \frac{\operatorname{Aut}(N)}{W} \longrightarrow \frac{\operatorname{Aut}(W, \check{T})}{W}$$

is injective.

According to Jackowski, McClure, and Oliver [11, 3.5], the composition

$$\operatorname{Out}(\hat{G}) \xrightarrow{N} \operatorname{Out}(N) \rightarrowtail \frac{\operatorname{Aut}(W, \check{T})}{W}$$

is an isomorphism. Hence both maps in the above diagram are isomorphisms. □

6. Centralizers

This section contains preparatory material for the proof of Proposition 7.10 and Proposition 7.17.

Let X be a p-compact group, N an extended p-compact torus, and $j\colon N \to X$ a monomorphism conjugate to the normalizer of a maximal torus $T \longrightarrow X$. If V is a nontrivial elementary abelian p-group and $\nu\colon V \to X$ a conjugacy class of a monomorphisms, then a *preferred lift* of ν is a conjugacy class of monomorphisms $\mu\colon V \to N$ such that $j \circ \mu$ and ν are conjugate and the morphism $C_N(\mu) \longrightarrow C_X(\nu)$ induced by j is conjugate to the normalizer of a maximal torus.

For a preferred lift $\mu\colon V \to N$ of a monomorphism $\nu\colon V \to X$, let $A(\mu)$ denote the representation

$$V \xrightarrow{\pi_0(\mu)} \pi_0(N) \longrightarrow \operatorname{Aut}(\pi_1(N))$$

of V in the free $\mathbb{Z}_p$-module $\pi_1(N) = \pi_1(T)$.

Definition 6.1. *The preferred lift μ is a* special *preferred lift of ν if the representation $A(\mu)$ fails to be faithful.*

Note the following properties of special preferred lifts [15]:

- If $C_X(\nu)$ has maximal rank [7, §4], the unique preferred lift of ν is special (indeed, $A(\mu)$ is the trivial representation) [15, 4.8].

- If X is connected, the unique preferred lift of any monomorphism $\nu\colon \mathbb{Z}/\imath \to \mathbb{X}$ is special [15, 4.6, 4.9]
- If X is connected, any ν has a special preferred lift [15, Proof of 1.3].
- If X is connected, there exists for any monomorphism $f\colon V_1 \to V_2$ between nontrivial elementary abelian p-groups a special preferred lift μ_2 of ν_2 such that $\mu_2 f$ is a special preferred lift of $\nu_2 f$ [15, Proof of 1.3].

It is possible to enumerate the preferred lifts in the rank two case.

Proposition 6.2. *Let $\nu\colon (\mathbb{Z}/\imath)^{\nvDash} \to \mathbb{X}$ be a monomorphism of a rank two elementary abelian p-group into a connected p-compact group X.*

(1) *If the centralizer $C_X(\nu)$ has maximal rank, then ν admits a unique (special) preferred lift.*
(2) *If the centralizer $C_X(\nu)$ is not of maximal rank, then the map $\mu \longrightarrow \ker A(\mu)$ determines a bijection between the set of N-conjugacy classes of special preferred lifts μ of ν and the set of nontrivial, proper subgroups of V.*

Proof. The uniqueness of (special) preferred lifts in the maximal rank case is a general fact [15, 4.8].

Assume now that the rank of the centralizer $C_X(\nu)$ is less than the rank of X. Let μ_1 and μ_2 be two special preferred lifts of ν. The associated representations $A(\mu_1)$ and $A(\mu_2)$ have kernels of rank one since they are nontrivial and nonfaithful.

Suppose that $\ker A(\mu_1)$ and $\ker A(\mu_2)$ both equal the rank one subgroup $L < V$. Then the restrictions $\mu_1|L$ and $\mu_2|L$ are conjugate for they are both preferred lifts of $\nu|L$; put $\mu|L = \mu_1|L = \mu_2|L$. Write $(\mathbb{Z}/\imath)^{\nvDash} = \mathbb{L} \oplus \mathbb{L}^{\perp}$ as a direct sum of L and a complement $L^{\perp}$ and let $\nu^{\perp}\colon L^{\perp} \to C_X(L)$, $\mu_1^{\perp}, \mu_2^{\perp}\colon L^{\perp} \to C_N(\mu|L)$ denote the monomorphisms adjoint to ν, μ_1, μ_2, respectively, relative to this splitting of $(\mathbb{Z}/\imath)^{\nvDash}$. Then $C_N(\mu|L)$ is the normalizer of a maximal torus of $C_X(\nu|L)$ and $\mu_1^{\perp}$ and $\mu_2^{\perp}$ are preferred lifts of $\nu|L$. Since preferred lifts are unique in the rank one case, $\mu_1^{\perp}$ and $\mu_2^{\perp}$ are conjugate in $C_N(\mu|L)$ and, therefore, μ_1 and μ_2 are conjugate in N.

Given any nontrivial and proper subgroup $L < (\mathbb{Z}/\imath)^{\nvDash}$ there exists a preferred lift μ of ν such that $\mu|L$ is a preferred lift of $\nu|L$. We then have $\ker A(\mu) = L$. $\square$

Proposition 6.2 shows that, unless the centralizer is of maximal rank, any monomorphism of a rank two elementary abelian p-group into a connected p-compact group has exactly $p+1$ special preferred lifts.

Now follows a variation on the theme of centricity.

For any elementary abelian p-group V and any loop space Y, the mapping space $\mathrm{map}(BV, BY)$ as well as any of its components $\mathrm{map}(BV, BY)_{B\nu} = BC_Y(\nu)$ are spaces over BY by the evaluation maps. We say that a homotopy class of maps $BC_Y(\nu) \to \mathrm{map}(BV, BY)$ is a homotopy class over BY if

$$\begin{tikzcd}
BC_Y(\nu) \arrow[rr] \arrow[dr] & & \mathrm{map}(BV, BY) \arrow[dl] \\
& BY &
\end{tikzcd}$$

commutes up to homotopy.

Lemma 6.3. *Let V be a nontrivial elementary abelian p-group and $\nu\colon V \to X$ a monomorphism with preferred lift $\mu\colon V \to N$. Suppose that p is odd and that*

$C_X(\nu)$ has N-determined automorphisms. Then there is bijective correspondence between homotopy classes $BC_N(\mu) \to \mathrm{map}(BV, BN)$ over BN and homotopy classes $BC_X(\nu) \to \mathrm{map}(BV, BX)$ over BX.

The homotopy class $BC_N(\mu) \to \mathrm{map}(BV, BN)$ over BN and the homotopy class $BC_X(\nu) \to \mathrm{map}(BV, BX)$ over BX correspond to each other if and only if the diagram

$$\begin{array}{ccc} BC_N(\mu) & \longrightarrow & \mathrm{map}(BV, BN) \\ \downarrow & & \downarrow \\ BC_X(\nu) & \longrightarrow & \mathrm{map}(BV, BX) \end{array}$$

commutes up to homotopy.

Proof. The map $BC_Y(\lambda) \to \mathrm{map}(BV, BY)$ is a map over BY up to homotopy if and only if its adjoint $BV \to \mathrm{map}(BC_Y(\lambda), BY)_{Be}$ maps into the connected component of the evaluation map $Be\colon BC_Y(\lambda) = \mathrm{map}(BV, BY)_{B\lambda} \to BY$, $(Y, \lambda) = (N, \mu)$, (X, ν). Thus we want to compare homotopy classes of maps $BV \to \mathrm{map}(BC_N(\mu), BN)_{Be}$ to homotopy classes of maps $BV \to \mathrm{map}(BC_X(\nu), BX)_{Be}$.

By centricity [2], there are homotopy equivalences

$$\mathrm{map}(BC_N(\mu), BC_N(\mu))_{B1} \to \mathrm{map}(BC_N(\mu), BN)_{Be}$$
$$\mathrm{map}(BC_X(\nu), BC_X(\nu))_{B1} \to \mathrm{map}(BC_X(\nu), BX)_{Be}$$

and coupled with the general homotopy equivalences of (4.12) we see that $\mathrm{map}(BC_N(\mu), BN)_{Be}$ and $\mathrm{map}(BC_X(\nu), BX)_{Be}$ are homotopy equivalent in such a way so as to induce the asserted the correspondence. □

For the following lemma, let V_1 and V_2 be nontrivial elementary abelian p-groups and $\nu_1\colon V_1 \to X$, $\nu_2\colon V_2 \to X$ monomorphisms with preferred lifts $\mu_1\colon V_1 \to N$, $\mu_2\colon V_2 \to N$. Suppose that there exists a monomorphism $g\colon V_1 \to V_2$ such that $\mu_2 \circ g$ is conjugate to μ_1. Then also $\nu_2 \circ g$ is conjugate to ν_1 and g induces morphisms $g^*\colon C_N(\mu_2) \to C_N(\mu_1)$ and $g^*\colon C_X(\nu_2) \to C_X(\nu_1)$.

Lemma 6.4. *Suppose that p is odd and that $C_X(\nu_2)$ has N-determined automorphisms. Let $\gamma\colon C_X(\nu_2) \to C_X(\nu_1)$ be any morphism such that the diagram*

$$\begin{array}{ccc} BC_N(\mu_2) & \xrightarrow{Bg^*} & BC_N(\mu_1) \\ \downarrow & & \downarrow \\ BC_X(\nu_2) & \xrightarrow[B\gamma]{} & BC_X(\nu_1) \end{array}$$

commutes up to homotopy. Then $B\gamma$ is homotopic to $Bg^*\colon BC_X(\nu_2) \to BC_X(\nu_1)$ induced by Bg.

Proof. By (6.3) it suffices to show that $B\gamma$ is a map over BX.

Let $\overline{\nu}_2\colon V_2 \to C_X(\nu_2)$ denote the canonical central factorization of ν_2 through its centralizer [6, 8.2]. Using that $Bg^*\colon BC_N(\mu_2) \to BC_N(\mu_1)$ obviously is a map over

BN it follows that

$$\begin{tikzcd} & BV_2 \arrow[dl, "B\overline{\nu}_2"'] \arrow[dr, "B\nu_2"] & \\ BC_X(\nu_2) \arrow[rr, "Be\circ B\gamma"'] & & BX \end{tikzcd}$$

commutes up to homotopy, i.e. that $Be\circ B\gamma$ is a map under BV_2. Viewing $BC_X(\nu_2)$ as the total space of the fibration

$$BV_2 \xrightarrow{B\overline{\nu}_2} BC_X(\nu_2) \to B(C_X(\nu_2)/V_2),$$

the Shapiro lemma tells us that

$$\mathrm{map}(BC_X(\nu_2), BY) = \mathrm{map}((BV_2)_{h(C_X(V_2)/V_2)}, BY) = \mathrm{map}(BV_2, BY)^{h(C_X(V_2)/V_2)}$$

with, for instance, $Y = C_X(\nu_2)$ or $Y = X$. Noting that the homotopy equivalence $\mathrm{map}(BV_2, BC_X(\nu_2))_{B\overline{\nu}_2} \to \mathrm{map}(BV_2, BX)_{B\nu_2}$ induces a homotopy equivalence

$$\mathrm{map}(BV_2, BC_X(\nu_2))^{h(C_X(\nu_2)/V_2)}_{B\overline{\nu}_2} \to \mathrm{map}(BV_2, BX)^{h(C_X(\nu_2)/V_2)}_{B\nu_2}.$$

of homotopy fixed point spaces, we see that there exists a uniquely determined endomorphism $\overline{\gamma}$ of $C_X(\nu_2)$ such that

$$\begin{tikzcd} BC_X(\nu_2) \arrow[r, "B\overline{\gamma}"] \arrow[d, "B\gamma"'] & BC_X(\nu_2) \arrow[d, "Be"] \\ BC_X(\nu_1) \arrow[r, "Be"'] & BX \end{tikzcd}$$

commutes up to homotopy.

Similarly, there exists a uniquely determined endomorphism of $C_N(\mu_2)$ such that

$$\begin{tikzcd} BC_N(\mu_2) \arrow[r] \arrow[d, "Bg^*"'] & BC_N(\mu_2) \arrow[d, "Be"] \\ BC_N(\mu_1) \arrow[r, "Be"'] & BN \end{tikzcd}$$

commutes up to homotopy; in fact, this endomorphism of $C_N(\mu_2)$ can only be the identity.

By naturality of this construction, it follows that $\overline{\gamma}$ is an automorphism of $C_X(\nu_2)$ that is covered by the identity map of $C_N(\mu_2)$ and, since $C_X(\nu_2)$ is assumed to have N-determined automorphisms, γ itself must be conjugate to the identity. This shows that $B\gamma$ is a map over BX up to homotopy. □

We now observe that for a connected p-compact group, the nontrivial center of the p-normalizer of the maximal torus is contained in the maximal torus.

Lemma 6.5. *Suppose that X is connected and let N_p be the p-normalizer of the maximal torus $T \to X$. Then the center $Z(N_p)$ of N_p has the form $Z(N_p) \to T \to N_p$ and $Z(N_p)$ is a nontrivial abelian p-compact group.*

Proof. The center of N_p is nontrivial [5, 1.3] and the evaluation map $BZ(N_p) = \mathrm{map}(BN_p, BN_p)_{B1} \to BN_p$ factors through $\mathrm{map}(BT, BN_p)_{Bi} = BC_{N_p}(T) = BT$ [18, 4.1], [15, 2.2]. □

Finally a version of what is known as the "Zabrodsky lemma".

Lemma 6.6. *In the commutative diagram*

$$\begin{tikzcd} E_0 \arrow[rr, "u"] \arrow[dr, "f_0"'] & & E_1 \arrow[dl, "f_1"] \\ & B & \end{tikzcd}$$

suppose that B is connected and that f_0 and f_1 are fibrations with fibres F_0 and F_1, respectively. If $u|F_0\colon F_0 \to F_1$ is null homotopic and if constant maps provide a homotopy equivalence $F_1 \longrightarrow \mathrm{map}(F_0, F_1)_0$, then there exists a section v of f_1 such that $v \circ f_0$ and u are vertically homotopic.

Proof. Let $M \longrightarrow B$ be the fibration whose fibre over any point $b \in B$ is $\mathrm{map}(f_0^{-1}(b), f_1^{-1}(b))_0$. Constant maps provide a fibre homotopy equivalence

$$\begin{tikzcd} E_1 \arrow[rr, "\simeq"] \arrow[dr, "f_1"'] & & M \arrow[dl] \\ & B & \end{tikzcd}$$

so, since the adjoint of the lift u is a section of $M \longrightarrow B$, f_1 admits a corresponding section, v. $\square$

7. N-determined p-compact groups

This section contains an investigation of the class of p-compact groups that are determined by the normalizer of the maximal torus. *It is a standing assumption throughout the section that the prime p is odd.*

Definition 7.1. *The p-compact group X is* N-determined *if for any diagram of morphisms of the form*

$$X \xleftarrow{j} N \xrightarrow{j'} X'$$

where

- *X' is a p-compact group and N an extended p-compact torus*
- *j is conjugate to the normalizer of a maximal torus of X*
- *j' is conjugate to the normalizer of a maximal torus of X'*

there exists an isomorphism $f\colon X \to X'$ such that $fj \simeq j'$.

A p-compact group is totally N-determined *if it has N-determined automorphisms (3.10) and is N-determined.*

Note that if X has N-determined automorphisms, then the isomorphism f, if it exists, is unique up to conjugacy.

Total N-determinacy has strong consequences for automorphism groups.

Corollary 7.2. *If the p-compact group $X_{h\pi}$ with component group π and identity component is totally N-determined then the maps*

$$N_{h\pi}\colon \mathrm{Aut}_{h\pi}(X) \to \mathrm{Aut}_{h\pi}(N)$$
$$N\colon \mathrm{Out}(X_{h\pi}) \to \mathrm{Out}(N_{h\pi})$$

are isomorphisms.

Proof. It remains (3.9) to show surjectivity of the homomorphism N of (3.8). But that follows immediately from Definition 7.1 since any square of the form

$$\begin{tikzcd} N_{h\pi} \arrow[r, "\cong"] \arrow[d] & N_{h\pi} \arrow[d] \\ X_{h\pi} \arrow[r, dashed] & X_{h\pi} \end{tikzcd}$$

can be completed by an isomorphism. □

As shown in Section 5, compact connected Lie groups satisfy the conclusion of Corollary 7.2. Thus Lie groups have N-determined automorphisms and might be N-determined.

Now a few words about component groups to ensure that the π_0-image of the diagram of Definition 7.1 always can be completed.

Consider the Weyl representation $w\colon \pi_0(N) \to \mathrm{Aut}(\pi_1(N)\otimes\mathbb{Q})$ of the component group $\pi_0(N) = W_T(X)$ in the $\mathbb{Q}$-vector space $\pi_1(N)\otimes\mathbb{Q} = \pi_1(\mathbb{T})\otimes\mathbb{Q}$. Recall that an element $s \in \pi_0(N)$ is a *reflection* if $w(s)$ is nontrivial and pointwise fixes a hyperplane.

We want to characterize the Weyl group $\ker\pi_0(j) = W_T(X_0)$ of [18, 3.8] the identity component X_0 of X.

Lemma 7.3. *Let $s \in \pi_0(N)$ be a reflection. Then*

$$s \in \ker\pi_0(j) \Leftrightarrow \mathrm{ord}(s) = \mathrm{ord}(w(s)).$$

Proof. One direction follows from the fact [6, 9.7] that the Weyl representation is faithful on the Weyl group of the identity component.

Suppose, conversely, that $\mathrm{ord}(s) = \mathrm{ord}(w(s))$. Since s is a reflection (and p is odd) the order of $w(s)$ is prime to p. Hence s is mapped to the unit of the finite p-group $\pi_0(X)$. □

Corollary 7.4. *The subgroup* $\ker\pi_0(j)$ *of* $\pi_0(N)$ *is generated by the set of reflections* $s \in \pi_0(N)$ *with* $\mathrm{ord}(s) = \mathrm{ord}(w(s))$.

Proof. By [6, 9.7], the subgroup $\ker\pi_0(j)$, which is the Weyl group of the identity component, is generated by the reflections that it contains. □

Remark 7.5. Let $\nu\colon V \to X$ be a monomorphism which admits a factorization $\mu\colon V \to T$ through the maximal torus. The component group

$$\pi_0(C_N(\mu)) = W^\mu$$

is the isotropy subgroup of $B\mu$ for the W-action on $[BV, BT] = \mathrm{Rep}(V, T)$, and by (7.3), the component group $\pi_0(C_X(\nu))$ is isomorphic to the quotient of W^μ by the normal subgroup generated by the set of reflections it contains. In particular, $C_X(\nu)$ is connected if and only if W^μ is a reflection group.

It follows from (7.4) that $\ker\pi_0(j) = \ker\pi_0(j')$ so that in the situation of Definition 7.1 there exists a unique isomorphism

$$\varphi\colon \pi_0(X) = \pi_0(N)/\ker\pi_0(j) \to \pi_0(X') = \pi_0(N)/\ker\pi_0(j') \tag{7.6}$$

such that $\varphi\circ\pi_0(j) = \pi_0(j')$.

Note also that the class of N-determined p-compact groups is nonempty.

Proposition 7.7. *If p does not the Weyl group order of X, in particular if X is a p-compact torus, then X is N-determined.*

Proof. If the Weyl group order of X is prime to p, the p-completions of Bj and Bj' are homotopy equivalences (3.12). □

Proposition 7.8. *Suppose that the identity component X_0 of the p-compact group X is totally N-determined. Then X itself is totally N-determined.*

Proof. It remains (4.2) to show that X is N-determined.

Let BX_0, BN_0, BX_0' be the fibres of $BX \to B\pi_0(X)$, $BN \to B\pi_0(N) \to B\pi_0(X)$, $BX' \to B\pi_0(X')$, respectively, and let $X_0 \xleftarrow{j_0} N_0 \xrightarrow{j_0'} X_0'$ be the restrictions of j, j'. Since X_0 is assumed to be N-determined, there exists an isomorphism $f_0\colon X_0 \to X_0'$ with $Bf_0 \circ Bj_0 \simeq Bj_0'$.

The map Bf_0 is a $\pi_0(X)$-map in the sense that $Bf_0 \circ \xi \simeq \varphi(\xi) \circ Bf_0$ for any $\xi \in \pi_0(X)$ acting by monodromy on BX_0. To see this, note that the restriction map $[BX_0, BX_0'] \to [BN_0, BX_0']$ is injective on the subset of isomorphisms since X_0 is assumed to have N-determined automorphisms and that $Bf_0 \circ \xi \circ Bj_0 \simeq Bf_0 \circ Bj_0 \circ \xi \simeq Bj_0' \circ \xi \simeq \varphi(\xi) \circ Bf_0 \circ Bj_0$.

Because Bf_0 is a $\pi_0(X)$-map, the mapping space $\mathrm{map}(BX_0, BX_0')_{Bf_0}$ is a $\pi_0(X)$ space. Composition with Bj induces a homotopy equivalence (3.4)

$$\mathrm{map}(BX_0, BX_0')^{h\pi_0(X)}_{Bf_0} \to \mathrm{map}(BN_0, BX_0')^{h\pi_0(X)}_{Bj_0'}$$

and since the homotopy fixed point space to the right is nonempty – it contains Bj' – the homotopy fixed point space to the left is also nonempty: It contains a fibre homotopy equivalence $Bf\colon BX \to BX'$ over $B\pi$ such that $Bf \circ Bj$ is vertically homotopic to Bj'. □

Example 7.9. If the order of the Weyl group of the identity component X of $X_{h\pi}$ is prime to $p > 2$, then X is totally N-determined (3.11,7.7), so (7.8) $X_{h\pi}$ is also totally N-determined and (7.2) $\mathrm{Aut}_{h\pi}(X) \cong \mathrm{Aut}_{h\pi}(N)$ and $\mathrm{Out}(X_{h\pi}) \cong \mathrm{Out}(N_{h\pi})$ All p-compact toral groups, in particular, are totally N-determined.

Proposition 7.10. *Let X be a connected p-compact group with center $Z(X) \to X$. If the adjoint form $X/Z(X)$ is N-determined, X itself is N-determined.*

Proof. Let N be an extended p-compact torus, X' a connected p-compact group with center $Z(X') \to X'$ and suppose that there exist morphisms, $j\colon N \to X$ and $j'\colon N \to X'$, that are conjugate to normalizers of maximal tori. The induced morphisms of centers, $Z(j)\colon Z(X) \to Z(N)$ and $Z(j')\colon Z(X') \to Z(N)$, are isomorphisms since p is assumed to be odd. In the induced commutative diagram

$$\begin{tikzcd}
Z(X) \arrow[d] & Z(N) \arrow[l, "Z(j)^{-1}", "\cong"'] \arrow[r, "Z(j')^{-1}", "\cong"'] \arrow[d] & Z(X') \arrow[d] \\
X \arrow[d] & N \arrow[l, "j"'] \arrow[r, "j'"] \arrow[d] & X' \arrow[d] \\
X/Z(X) & N/Z(N) \arrow[l, "j/Z"] \arrow[r, "j'/Z"'] & X'/Z(X')
\end{tikzcd} \tag{7.11}$$

the morphisms j/Z and j'/Z are conjugate to normalizer of maximal tori so since $X/Z(X)$ is assumed to be N-determined there exists an isomorphism $f/Z\colon X/Z(X) \to X'/Z(X')$ such that $B(f/Z) \circ B(j/Z) \simeq B(j'/Z)$.

Our aim is to complete the diagram

$$\begin{tikzcd}
Z(X) \arrow[r, "Z(f)^{-1}", "\cong"'] \arrow[d] & Z(X') \arrow[d] \\
X \arrow[r, dashed, "f"] \arrow[d] & X' \arrow[d] \\
X/Z(X) \arrow[r, "\cong", "f/Z"'] & X'/Z(X')
\end{tikzcd}$$

by an isomorphism f such that $Bf \circ Bj \simeq Bj'$; here $Z(f)^{-1} := Z(j')^{-1} \circ Z(j)$.

We shall consider the following diagram of fibered mapping spaces derived from diagram (7.11)

$$\begin{tikzcd}
\mathrm{map}(BZ(X), BZ(X'))_{BZ(f)} \arrow[r] & BZ(X')_{h(X/Z(X))} \arrow[r] & B(X/Z(X)) \\
\mathrm{map}(BZ(X), BZ(X'))_{BZ(f)} \arrow[r] \arrow[u, equal] \arrow[d, "\simeq"'] & BZ(X')_{h(N/Z(N))} \arrow[r] \arrow[u] \arrow[d] & B(N/Z(N)) \arrow[u, "B(j/Z)"'] \arrow[d, equal] \\
\mathrm{map}(BZ(N), BZ(X'))_{BZ(j')^{-1}} \arrow[r] & BZ(X')_{h(N/Z(N))} \arrow[r] & B(N/Z(N))
\end{tikzcd}$$

where

- in the bottom horizontal fibration, the fibre over any point $b \in B(N/Z(N))$ is $\mathrm{map}(BN_b, BX'_{B(j'/Z)(b)})_{BZ(j')^{-1}}$
- in the middle horizontal horizontal fibration, the fibre over any point $b \in B(N/Z(N))$ is $\mathrm{map}(BX_{B(j/Z)(b)}, BX'_{B(j'/Z)(b)})_{BZ(f)}$
- in the top horizontal fibration, the fibre over any point $b \in B(X/Z(X))$ is $\mathrm{map}(BX_b, BX'_{B(f/Z)(b)})_{BZ(f)}$
- the map of the middle fibration to the bottom fibration is induced by Bj
- the map of the middle fibration to the top fibration is induced by B(j/Z).

Note that the bottom fibration admits a section adjoint to the fibre map Bj'. Thus also the middle fibration admits a section. By (6.6) this section will be the pull back of a section of the top fibration provided

(1) $X/N \to \mathrm{map}(BZ(X), BZ(X'))_{BZ(f)}$ is null homotopic
(2) $X/N \to \mathrm{map}(X/N, BZ(X'))_0$ is a homotopy equivalence.

The first condition is satisfied because the restriction to the fibre $X/N \subseteq B(N/Z(N))$ of the middle fibration is trivial and the section corresponds to the constant map of X/N into $\mathrm{map}(BZ(X), BZ(X'))_{BZ(f)}$. To see that the second condition is satisfied, recall that $\mathrm{map}(BZ(X), BZ(X'))_{BZ(f)} \simeq BZ(X') \simeq K(A,1) \times K(\mathbb{Z}_{\imath}, \nvDash)^{\frown}$ for some finite abelian p-group A and that $\pi_1(X/N) \cong W_T(X)$ is finite.

This section of the top fibration is adjoint to a map $Bf\colon BX \to BX'$ over $B(f/Z)$ and under $BZ(f)$ such that $Bf \circ Bj \simeq Bj'$. $\square$

From now on, let X and X' be two connected, centerfree p-compact groups and N an extended p-compact torus with monomorphisms

$$X \xleftarrow{j} N \xrightarrow{j'} X' \tag{7.12}$$

conjugate to normalizers of maximal tori. Also, make the induction hypothesis that all centralizers $C_X(\nu)$, $(V,\nu) \in \mathrm{Ob}(\mathbb{A}(\mathbb{X}))$, have N-determined automorphisms and are N-determined.

We shall first see that the special preferred lifts are the same for X and X'. (The proof was communicated to me by Bill Dwyer.)

Lemma 7.13. *Let $\mu\colon V \to N$ be a monomorphism of a nontrivial elementary abelian p-group V into N. Then μ is a special preferred lift of $j\mu$ if and only if μ is a special preferred lift of $j'\mu$.*

Proof. Put $\nu = j\mu$ and $\nu' = j'\mu$ and assume that μ is a special preferred lift of ν. Let $L < V$ denote the (nontrivial) kernel of the representation $A(\mu)$. Then $\mu|L$ is the unique preferred lift of $\nu|L$ and of $\nu'|L$. Let (if $L \neq V$) $L^\perp$ be a complement to L and $\mu^\perp\colon L^\perp \to C_N(\mu|L)$, $\nu^\perp\colon L^\perp \to C_X(\nu|L)$, $(\nu')^\perp\colon L^\perp \to C_{X'}(\nu'|L)$ the monomorphisms that are adjoint to μ, ν, ν', respectively. By the above induction hypothesis there exists an isomorphism $C_X(\nu|L) \to C_{X'}(\nu'|L)$ under $C_N(\mu|L)$ inducing an isomorphism

$$C_X(\nu) = C_{C_X(\nu|L)}(\nu^\perp) \to C_{C_{X'}(\nu'|L)}((\nu')^\perp) = C_{X'}(\nu')$$

under $C_{C_N(\mu|L)}(\mu^\perp) = C_N(\mu)$. Thus μ is also a preferred lift of ν'. □

For any object (V,ν) of $\mathbb{A}(\mathbb{X})$, let $\mathrm{SPL}(V,\nu) \subset [BV, BN]$ denote the set of conjugacy classes of special preferred lifts of (V,ν). Note that the automorphism group $\mathrm{Aut}(V,\nu)$ of (V,ν), consisting of those automorphisms α of V for which $\nu\alpha$ and ν are conjugate, acts on the set $\mathrm{SPL}(V,\nu)$.

Let $\mathbb{A}(\mathbb{N})$ be the category whose objects (V,μ) are conjugacy classes of monomorphisms $\mu\colon V \to N$ of nontrivial elementary abelian p-groups into N such that μ is a special preferred lift of $j\mu$ and $j'\mu$. A morphism $\alpha\colon (V_1,\mu_1) \to (V_2,\mu_2)$ in $\mathbb{A}(\mathbb{N})$ is a monomorphism $\alpha\colon V_1 \to V_2$ such that μ_1 and $\mu_2\alpha$ are conjugate. The morphisms j and j' induce full functors $\mathbb{A}(\mathbb{X}) \leftarrow \mathbb{A}(\mathbb{N}) \to \mathbb{A}(\mathbb{X}')$ that restrict (see the remarks immediately after Definition 6.1) to isomorphisms on the full subcategories $\mathbb{A}_{\not\Bbbk}(\mathbb{X}) \cong \mathbb{A}_{\not\Bbbk}(\mathbb{N}) \cong \mathbb{A}_{\not\Bbbk}(\mathbb{X}')$ of rank one objects.

Lemma 7.14. *Let $\alpha\colon (V_1,\mu_1) \to (V_2,\mu_2)$ be a morphism in $\mathbb{A}(\mathbb{N})$. Then the diagram*

$$\begin{tikzcd} C_X(j\mu_2) \arrow[r, "\alpha^*"] \arrow[d, "f(\mu_2)"'] & C_X(j\mu_1) \arrow[d, "f(\mu_1)"] \\ C_{X'}(j'\mu_2) \arrow[r, "\alpha^*"'] & C_{X'}(j'\mu_1) \end{tikzcd}$$

commutes up to conjugacy where $f(\mu_1)\colon C_X(j\mu_1) \to C_{X'}(j'\mu_1)$ is the isomorphism under $C_N(\mu_1)$ and $f(\mu_2)\colon C_X(j\mu_2) \to C_{X'}(j'\mu_2)$ the isomorphism under $C_N(\mu_2)$.

Proof. As the diagram

$$\begin{tikzcd}
C_X(j\mu_2) \arrow[rrr, "\alpha^*"] \arrow[dd, "f(\mu_2)"', "\cong"] & & & C_X(j\mu_1) \arrow[dd, "\cong"', "f(\mu_1)"] \\
& C_N(\mu_2) \arrow[ul] \arrow[dl] \arrow[r, "\alpha^*"] & C_N(\mu_1) \arrow[ur] \arrow[dr] & \\
C_{X'}(j'\mu_2) \arrow[rrr, "\alpha^*"'] & & & C_{X'}(j'\mu_1)
\end{tikzcd}$$

shows that $f(\mu_1)^{-1} \circ \alpha^* \circ f(\mu_2)$ is covered by α^*, we infer (6.4) that $\alpha^* \circ f(\mu_2)$ and $f(\mu_1) \circ \alpha^*$ are conjugate morphisms. $\square$

We shall now employ the fact that the cohomology of BX is determined by $\mathbb{A}_{\leq \not\succeq}(\mathbb{X})$ to find conditions under which the cohomological image of the diagram of Definition 7.1 can be completed.

Proposition 7.15. *In the situation of (7.12), assume (1) and either (2) or (3) from the list*

(1) $\mathrm{Aut}(V,\nu)$ *acts transitively on* $\mathrm{SPL}(V,\nu)$ *for all* $(V,\nu) \in \mathrm{Ob}(\mathbb{A}_{\not\succeq}(\mathbb{X}))$.
(2) $\mathrm{Aut}(V,\nu')$ *acts transitively on* $\mathrm{SPL}(V,\nu')$ *for all* $(V,\nu') \in \mathrm{Ob}(\mathbb{A}_{\not\succeq}(\mathbb{X}'))$.
(3) *Any two objects of* $\mathbb{A}_{\not\succeq}(\mathbb{X})$ *with isomorphic centralizers of less than maximal rank are isomorphic.*

Then $H^*(Bj)H^*(BX;\mathbb{F}_{\mathsf{l}}) = \mathbb{H}^*(\mathbb{B}\mathfrak{I}')\mathbb{H}^*(\mathbb{B}\mathbb{X}';\mathbb{F}_{\mathsf{l}})$ *in* $H^*(BN;\mathbb{F}_{\mathsf{l}})$.

Proof. Assume first that hypotheses (1) and (2) hold. We are going to apply Oliver's cochain complex [21] to the covariant functor $H(X)$ from $\mathbb{A}(\mathbb{X})$ to the category of $\mathbb{F}_{\mathsf{l}}$-modules given by $H(X)(V,\nu) = H^*(BC_X(V,\nu))$.

As noted above, there are bijective correspondences between the sets $\varepsilon_1(X) = \varepsilon_1(N) = \varepsilon_1(X')$ of isomorphism classes of objects of the rank one subcategories. Choose one representing object (L,μ) in each isomorphism class of $\mathbb{A}_{\not\Vdash}(\mathbb{N})$ and let $(L,\phi) = (L, j\mu)$ and $(L,\phi') = (L, j'\mu)$ be the corresponding objects of $\mathbb{A}_{\not\Vdash}(\mathbb{X})$ and $\mathbb{A}_{\not\Vdash}(\mathbb{X}')$. The isomorphisms $f(\mu)\colon C_X(L,\phi) \to C_{X'}(L,\phi')$ under $C_N(L,\mu)$ induce an isomorphism

$$\Phi = \prod f(\mu)^*\colon \prod_{(L,\phi')\in\varepsilon_1(X')} H(X')(L,\phi') \to \prod_{(L,\phi)\in\varepsilon_1(X)} H(X)(L,\phi)$$

which in turn induces an isomorphism of 0-cochains

$$\prod_{(L,\phi')\in\varepsilon_1(X')} \mathrm{Hom}(\mathrm{St}(L), H(X')(L,\phi')) \to \prod_{(L,\phi)\in\varepsilon_1(X)} \mathrm{Hom}(\mathrm{St}(L), H(X)(L,\phi))$$

where $\mathrm{St}(L)$ is the Steinberg representation (in this case the trivial representation) and Hom is taken in the category of $\mathrm{Aut}(L,\phi) = \mathrm{Aut}(\mathbb{L},\phi')$ representations.

The morphisms j and j' induce full functors $\mathbb{A}_{\not\succeq}(\mathbb{X}) \leftarrow \mathbb{A}_{\not\succeq}(\mathbb{N}) \to \mathbb{A}_{\not\succeq}(\mathbb{X}')$ but these full subcategories of rank two objects are not necessarily isomorphic since in the rank two case special preferred lifts are not in general uniqely determined (6.2). However, by hypotheses (1) and (2), special preferred lifts are unique up to isomorphism and hence j and j' induce bijections $\varepsilon_2(X) = \varepsilon_2(N) = \varepsilon_2(X')$ of isomorphism classes of rank two objects in these three Quillen categories.

Oliver defines in [21, Proposition 5] for each isomorphism class $(E,\psi)\in\varepsilon_2(X)$ the coboundary map

$$\delta(E,\psi)\colon \prod_{(L,\phi)\in\varepsilon_1(X)} \operatorname{Hom}(\operatorname{St}(L),H(X)(L,\phi)) \to \operatorname{Hom}(\operatorname{St}(E),H(X)(E,\psi))$$

as the homomorphism that to a collection $(c(L,\phi))$, $(L,\phi)\in\varepsilon_1(X)$, of $\operatorname{Aut}(L,\phi)$-homomorphisms of the domain associates the homomorphism

$$\operatorname{St}(E)\xrightarrow{R_E}\oplus_{[E:A]=p}\operatorname{St}(A)\xrightarrow{\oplus c(A,\psi|A)}\oplus_{[E:A]=p}H(X)(A,\psi|A)\xrightarrow{\oplus H(X)(i_A)}H(X)(E,\psi)$$

where R_E is a certain connecting homomorphism (which is surjective in this case where E has rank two) and $i_A\colon A\to E$ is the inclusion. I claim that $\Phi(\ker\delta(E,\psi'))=\ker\delta(E,\psi)$, where (E,ψ) and (E,ψ') correspond under the identification $j'j^{-1}\colon\varepsilon_2(X)=\varepsilon_2(X')$.

Consider first the case where the centralizer $C_X(E,\psi)$ has maximal rank. Then (E,ψ) has a unique preferred lift (E,μ) and for any rank one subgroup A of E, $(A,\mu|A)$ is the preferred lift of $(A,\psi|A)$. According to (7.14), the diagram

$$\begin{tikzcd}
\oplus H(X)(A,\psi|A) \arrow[r, "\oplus H(X)(i_A)"] & \oplus H(X)(E,\psi) \\
\oplus H(X')(A,\psi'|A) \arrow[u, "\oplus f(\mu|A)^*", "\cong"'] \arrow[r, "\oplus H(X')(i_A)"'] & \oplus H(X')(E,\psi') \arrow[u, "\cong", "\oplus f(\mu)^*"']
\end{tikzcd}$$

commutes. It follows that $\ker\delta(E,\psi)$ and $\ker\delta(E,\psi')$ correspond under the isomorphism Φ in this case.

Next, consider the case where the rank of the centralizer $C_X(E,\psi)$ is less than maximal. The $p+1$ special preferred lifts μ_A of ψ are indexed (6.2) by the rank one subgroups $A<E$. By hypothesis (1), $\mu_B=\mu_A\beta$ for some automorphism β of (E,ψ). Then $\beta(B)=A$, $(B,\mu_B|B)=(B,\mu_A\beta|B)$ is isomorphic to $(A,\mu_A|A)$, and $(B,j\mu_B)=(B,\psi|B)$ is isomorphic to $(A,j\mu_A)=(A,\psi|A)$. Hence the restrictions $(A,\psi|A)$ and $(B,\psi|B)$ are isomorphic objects for all rank one subgroups $A,B<E$. Let (L,ϕ_E) be this common isomorphism class and $f_A\colon(L,\phi_E)\to(A,\psi|A)$ an isomorphism. The commutative diagram

$$\begin{tikzcd}
\oplus\operatorname{St}(A) \arrow[r, "\oplus c(A,\psi|A)"] & \oplus H(X)(A,\psi|A) \arrow[r, "\oplus H(X)(i_A)"] & \oplus H(X)(E,\psi) \\
\oplus\operatorname{St}(L) \arrow[u, "\cong", "\oplus\operatorname{St}(f_A)"'] \arrow[r, "\oplus c(L,\phi_E)"'] & \oplus H(X)(L,\phi_E) \arrow[u, "\cong", "\oplus H(X)(f_A)"'] \arrow[r, "H(X)(f_Ai_A)"'] & \oplus H(X)(E,\psi) \arrow[u, equal]
\end{tikzcd}$$

shows that $\ker\delta(E,\psi)$ consists of those

$$(c(L,\phi))\in\prod_{(L,\phi)\in\varepsilon_1(X)}\operatorname{Hom}(\operatorname{St}_L,H(X)(L,\phi))$$

for which the coordinate $c(L,\phi_E)=0$. Similarly, $\ker\delta(E,\psi')$ is described by the condition that $c(L,\phi'_E)=0$ where (L,ϕ_E) corresponds to (L,ϕ'_E) under the identification $j'j^{-1}\colon\varepsilon_1(X)=\varepsilon_1(X')$. Therefore, $\Phi(\ker\delta(E,\psi'))=\ker\delta(E,\psi)$ also in this case of centralizers of less than maximal rank.

From Oliver [21] and Dwyer and Wilkerson [5, 1.2] we know that the intersection of the kernels $\ker \delta(E,\psi)$ as (E,ψ) ranges over $\varepsilon_2(X)$ equals

$$\lim^0_{\mathbb{A}(\mathbb{X})} H(X) = H^*(BX),$$

and similarly for X', and hence we obtain a commutative diagram of monomorphisms

$$\begin{tikzcd}
H^*(BX) \arrow[r, tail, "H^*(Bj)"] \arrow[d, tail] & H^*(BN) \arrow[d, tail] & H^*(BX') \arrow[l, tail, "H^*(Bj')"'] \arrow[d, tail] \\
\prod_{(L,\phi)\in\varepsilon_1(X)} H(X)(L,\phi) \arrow[r, tail] & \prod_{(L,\mu)\in\varepsilon_1(N)} H(N)(L,\mu) & \prod_{(L,\phi')\in\varepsilon_1(X')} H(X')(L,\phi') \arrow[l, tail] \arrow[ll, bend left, "\cong"', "\Phi"]
\end{tikzcd}$$

where the isomorphism Φ restricts to an isomorphism between $H^*(BX')$ and $H^*(BX)$. (Use (6.5) to get injectivity of the middle vertical arrow.)

Assume next that hypotheses (1) and (3) hold. As above, hypothesis (1) shows that the full functor $j\colon \mathbb{A}_{\not\models}(\mathbb{N}) \to \mathbb{A}_{\not\models}(\mathbb{X})$ induces a bijection $j\colon \varepsilon_2(N) \to \varepsilon_2(X)$ of objects and we obtain a surjection

$$j'j^{-1}\colon \varepsilon_2(X) \twoheadrightarrow \varepsilon_2(X') \tag{7.16}$$

by composing j^{-1} with the surjection $j'\colon \varepsilon_2(N) \to \varepsilon_2(X')$. Since $C_X(V,\nu)$ is assumed to be N-determined, $C_X(V,\nu)$ is isomorphic to $C_{X'}(j'j^{-1}(V,\nu))$ for all $(V,\nu) \in \mathrm{Ob}(\mathbb{A}_{\not\models}(\mathbb{X}))$. In particular, the surjection (7.16) takes objects with centralizers of less than full rank to objects of the same kind. By hypothesis (3), (7.16) is injective on this subset of $\varepsilon_2(X)$. But (7.16) is also injective on the set of isomorphism classes of objects with full rank centralizers since special preferred lifts are unique in this case. Hence, (7.16) is in fact a bijection.

Let (E,ψ) be an object of $\mathbb{A}_{\not\models}(\mathbb{X})$ and (E,ψ') an object of $\mathbb{A}(\mathbb{X}')$ representing isomorphism classes that correspond under the bijection (7.16). Suppose that the common rank of the centralizers $C_X(E,\psi)$ and $C_{X'}(E,\psi')$ is less then maximal. Let $A < E$ be any rank one subgroup of E. As before, the isomorphism class of $(A,\psi|A)$ is independent of the choice of A and since $(A,\psi'|A)$ corresponds to $(A,\psi|A)$ under the bijection $\varepsilon_1(X) = \varepsilon_1(X')$, also the restriction $(A,\psi'|A)$ is independent, up to isomorphism, of $A < E$.

Taking these observations into account, the above argument under the assumption of (1) and (2) also applies under the assumption of (1) and (3). $\square$

It can be shown that the p-compact group $BX = B\,\mathrm{PU}(n)^\wedge_p$, $n \geq 2$, $p > 2$, satisfies 7.15.(1) and 7.15.(3). In fact, there does not exist any monomorphism $(\mathbb{Z}/\shortmid)^{\not\models} \rightarrowtail \mathbb{X}$ with centralizer of rank $< n-1$ unless p divides n in which case there exists an essentially unique such monomorphism and its automorphism group in $\mathbb{A}(\mathbb{X})$ is $\mathrm{SL}(2,p)$ which acts transitively on the set (6.2) of special preferred lifts.

When $H^*(BX)$ and $H^*(BX')$ have the same image in $H^*(BN)$ there exists an isomorphism Φ of algebras over the Steenrod algebra that makes the diagram

$$\begin{tikzcd} & H^*(BN) & \\ H^*(BX) \arrow[ur, "H^*(Bj)"] & & H^*(BX') \arrow[ul, "H^*(Bj')"'] \arrow[ll, "\cong"', "\Phi"] \end{tikzcd}$$

commute. Since Quillen categories, by Lannes theory [12], only depend on cohomological information, this isomorphism induces an isomorphism $\Phi\colon \mathbb{A}(\mathbb{X}) \to \mathbb{A}(\mathbb{X}')$ of categories. On objects, $\Phi(V, j\mu) = (V, j'\mu)$ for all $(V, \mu) \in \mathrm{Ob}(\mathbb{A}(\mathbb{N}))$, and $\Phi(\alpha) = \alpha$ for any morphism.

Proposition 7.17. *In the situation of (7.12), assume that $H^*(Bj)H^*(BX) = H^*(Bj')H^*(BX')$ in $H^*(BN)$ and that*

1. *$\mathrm{Aut}(V, \nu)$ acts transitively on $\mathrm{SPL}(V, \nu)$ for all objects (V, ν) of $\mathbb{A}(\mathbb{X})$.*
2. *The isomorphism $f(\mu)\colon C_X(j\mu) \to C_{X'}(j'\mu)$ under $C_N(\mu)$ is $\mathrm{Aut}(j\mu)$-equivariant for any object (V, μ) of $\mathbb{A}(\mathbb{N})$ with centralizer of less than maximal rank.*
3. $\lim^2_{\mathbb{A}(\mathbb{X})} \pi_1(BZ(V, \nu)) = 0 = \lim^3_{\mathbb{A}(\mathbb{X})} \pi_2(BZ(V, \nu))$.

Then there exists an isomorphism $f\colon X \to X'$ such that $f \circ j$ is conjugate to j'.

Proof. Hypotheses (1) and (2) imply that all choices of special preferred lift (V, μ) of (V, ν) lead to the same isomorphism $f(\mu)\colon C_X(V, \nu) \to C_{X'}(\Phi(V, \nu))$: By hypothesis (1), all special preferred lifts of (V, ν) have the form $(V, \mu\alpha)$ for some $\alpha \in \mathrm{Aut}(V, \nu) = \mathrm{Aut}(\Phi(V, \nu))$ and (7.14) $f(\mu\alpha) = \alpha^* \circ f(\mu) \circ (\alpha^*)^{-1} = f(\mu)$ if we assume, as in hypothesis (2), that $f(\mu)$ is equivariant. Let $f(V, \nu)\colon C_X(V, \nu) \to C_{X'}(\Phi(V, \nu))$ denote $f(\mu)$ for any special preferred lift μ.

Then (7.14) the centric morphisms

$$C_X(V, \nu) \xrightarrow{f(V,\nu)} C_{X'}(\Phi(V, \nu)) \xrightarrow{e(\Phi(V,\nu))} X'$$

define an element of $\lim^0_{\mathbb{A}(\mathbb{X})}[BC_X(V, \nu), BX']$. By hypothesis (3), see also (2.2), there exists an isomorphism $f\colon X \to X'$ such that $f \circ e(V, \nu) = e(\Phi(V, \nu)) \circ f(V, \nu)$ for all objects (V, ν) of $\mathbb{A}(\mathbb{X})$.

It remains to show that $f \circ j \simeq j'$. By construction, $f \circ j|C_N(\mu) \simeq j'|C_N(\mu)$ for any special preferred lift $\mu\colon V \to N$. Take $V = \mathbb{Z}/\imath$ and take $\mu\colon V \to T$ to be central in the Sylow p-subgroup N_p (6.5). It then follows that $f \circ j|N_p \simeq j'|N_p$ and, in particular, that $f \circ j|T\colon T \to X'$ is the maximal torus of X'.

The set of homotopy classes of maps $BN \to BX'$ under BT,

$$\pi_0(\mathrm{map}(BT, BX')^{hW_T(X')}_{Bi'}) = H^2(W_T(X'); \pi_2(BT)),$$

injects into the set of homotopy classes of maps $BN_p \to BX'$ under BT,

$$\pi_0(\mathrm{map}(BT, BX')^{hW_T(X')_p}_{Bi'}) = H^2(W_T(X')_p; \pi_2(BT)).$$

In particular, $f \circ j$ and j' must be conjugate since their restrictons to N_p are. $\square$

It is clear that more work has to be done here before we can claim that the vanishing of the obstruction groups in condition 7.17.(3) for any centerfree, connected p-compact group will imply that all p-compact groups are N-determined.

Remark 7.18. By the Splitting Theorem of [8], it represents no loss of generality to assume that X and X' are simple p-compact groups in Proposition 7.17. (This is also necessary for hypothesis (1).) Note also that all assumptions (1), (2), and (3) of (7.15) as well as (1) and (2) of (7.17) are satisfied in case $H^*(BX)$ injects into $H^*(BT)$ for then all centralizers of elementary abelian p-groups have maximal rank and preferred lifts are unique.

Example 7.19. Let $X = \mathrm{PU}(3)$ and $p = 3$. All objects of $\mathbb{A}(\mathbb{X})$ have centralizers that have N-determined automorphisms and are N-determined since they are p-compact toral groups or (4.3,7.9) their Weyl groups have order prime to p. The automorphism group, $\mathrm{SL}(2,3)$, of $V_2 \in \mathrm{Ob}(\mathbb{A}_{\not\Vdash}(\mathbb{X}))$, the only isomorphism class with a centralizer of less then maximal rank, acts transitively on the set $\mathrm{SPL}(V_2)$ of (6.2) rank one subgroups of V_2. Therefore (7.15), $H^*(BX)$ and $H^*(BX')$ have the same image in $H^*(BN)$ in the situation of (7.12). Any special preferred lift $\mu\colon V_2 \to N$ of the centric morphism $V_2 \to X$ must also be centric. Using this, the diagram $C_X(V_2) \leftarrow C_N(\mu) \to C_{X'}(V_2)$ identifies to a diagram $V_2 \leftarrow V_2 \to V_2$ of $\mathrm{SL}(2,3)$-equivariant isomorphisms. Thus also $f(\mu)\colon C_X(V_2) \to C_{X'}(V_2)$ is $\mathrm{SL}(2,3)$-equivariant up to homotopy. Finally, the limits $\lim^2_{\mathbb{A}(\mathbb{X})} \pi_1(BZ(V,\nu)) = 0 = \lim^3_{\mathbb{A}(\mathbb{X})} \pi_2(BZ(V,\nu))$ for general reasons [21]. Hence PU(3) is N-determined at $p = 3$. (See [1] for a related result.)

When combined with (4.8, 4.11, 7.10) this example shows, independently of [9, 10], that U(3), SU(3), and PU(3) are totally N-determined at the prime $p = 3$; it follows (7.2) for instance that the automorphism group of PU(3) is $\mathrm{Out}(\mathrm{PU}(3)) = \mathbb{Z}^*_{\not\Vdash}$.

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