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EMBEDDINGS OF DI_2 IN F_4

CARLES BROTO AND JESPER M. MØLLER

ABSTRACT. We show that there is only one embedding of BDI_2 in BF_4 at the prime $p = 3$, up to self-maps of BDI_2. We also describe the effect of the group of self-equivalences of BF_4 at the prime $p = 3$ on this embedding and then show that the Friedlander exceptional isogeny composed with a suitable Adams map is an involution of BF_4 whose homotopy fixed point set coincide with BDI_2

1. INTRODUCTION

A p-compact group is a p-complete analogue in homotopy theory of the concept of compact Lie group. The notion was introduced by Dwyer and Wilkerson in [13]. A p-compact group is a p-complete space BX whose loop space is $\mathbf{F}_p$-finite, i.e. has finite mod p cohomology. The p-completion of the classifying space of a compact connected Lie group is an example of a p-compact group but there are other, exotic, examples. Among other notions that p-compact groups share with compact Lie groups are that of a maximal torus, Weyl group, and maximal torus normalizer (see [13] for the definitions). There is also a well defined concept of homomorphism or monomorphism between p-compact groups. Since the introduction of this concept one main issue has been the description of the exotic examples. These are simply connected and irreducible p-compact groups that do not appear as the p-completion of a compact Lie group. Contrary to what happens with compact Lie groups these exotic examples can only be defined as p-complete spaces at a certain given prime p. We will focus our attention on one of these examples, namely, BDI_2. This is a 3-compact group with mod 3 cohomology ring

$$H^*(\mathrm{BDI}_2; \mathbf{F}_3) \cong P[x_{12}, x_{16}], \qquad P^1 x_{12} = x_{16}, \tag{1.1}$$

the rank two Dickson algebra at the prime 3. All of our calculations and results will be 3-primary. In particular we are considering the 3-completion of all spaces involved, thus we will omit explicit notation for this fact.

Theorem 1.1. *Any non-trivial homomorphism from* DI_2 *to* F_4 *is a monomorphism and the group of automorphisms of* DI_2 *acts simply transitively on the non-empty set of monomorphisms from* DI_2 *to* F_4*.*

There is then a monomorphism $\mathrm{B}\alpha\colon \mathrm{BDI}_2 \to \mathrm{BF}_4$ and it is unique up to composition with automorphisms of DI_2. We can rephrase it by saying that F_4 contains a unique copy of DI_2.

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If we compose the Friedlander exceptional isogeny of F_4, $\mathrm{B}\varphi\colon \mathrm{BF}_4 \to \mathrm{BF}_4$ [15], with the Adams map $\psi^{1/\zeta}$, where $\zeta^2 = -2$, then $\mathrm{B}\tau = \mathrm{B}\psi^{1/\zeta} \circ \mathrm{B}\phi$ is a homotopy involution of BF_4 that fixes $\mathrm{B}\alpha\colon \mathrm{BDI}_2 \to \mathrm{BF}_4$ up to homotopy.

Theorem 1.2. *There is an action of* $\mathbf{Z}/2$ *on* BF_4 *induced up to homotopy by* $\mathrm{B}\tau$*, for which* $\mathrm{B}\alpha\colon \mathrm{BDI}_2 \to \mathrm{BF}_4$ *induces a homotopy equivalence*

$$\mathrm{BDI}_2 \simeq (\mathrm{BF}_4)^{h\mathbf{Z}/2}$$

that identifies BDI_2 *with a homotopy fixed point space of* BF_4*.*

Notice that in particular the Bousfield-Kan spectral sequence for the homotopy groups of $(\mathrm{BF}_4)^{h\mathbf{Z}/2}$ degenerates to

$$\pi_*(\mathrm{BDI}_2) \cong \pi_*(\mathrm{BF}_4)^{\mathbf{Z}/2}. \tag{1.2}$$

As a further corollary we obtain

Corollary 1.3. *For any space* X*,* $\mathrm{map}(X, \mathrm{BDI}_2) \simeq \mathrm{map}(X, \mathrm{BF}_4)^{h\mathbf{Z}/2}$.

And applying this result to the homogeneous space $\mathrm{F}_4/\mathrm{DI}_2$,

Corollary 1.4 (Harper [16])**.** *At the prime* 3 *there is a splitting*

$$\mathrm{F}_4 \simeq \mathrm{DI}_2 \times \mathrm{F}_4/\mathrm{DI}_2\,.$$

Some historical considerations about the construction of this space will clarify our motivation. The way exotic examples are obtained is as realizations of graded polynomial algebras over $\mathbf{F}_p$ as mod p cohomology rings of p-completed spaces BX which will turn out to be p-compact groups. Steenrod [32] already pointed out the importance of realizing polynomial algebras as cohomology rings. The idea was reconducted by Clark-Ewing [9], Wilkerson [34, 36], and Adams-Wilkerson [2] who noticed the relevance of invariant theory to the realization problem.

Dickson algebras at odd primes are invariant algebras $D(n) \cong \mathbf{F}_p[t_1, \dots, t_n]^{\mathrm{GL}_n(\mathbf{F}_p)}$ by the action of the full general linear group of n by n matrices over $\mathbf{F}_p$ on a polynomial algebra on n independent generators of degree 2. They are shown to be again polynomial algebras of n independent generators of degrees $2(p^n - p^i)$ [11, 29, 35]. In [29] the authors already proved that Dickson algebras at odd primes can only appear as a cohomology ring if $n = 1$, in which case we obtain the cohomology of a p-local sphere of dimension $2(p-1)$ or in case $n = 2$ and $p = 3$, when $D(2) = \mathbf{F}_3[t_1, t_2]^{\mathrm{GL}_2(\mathbf{F}_3)} \cong \mathbf{F}_3[x_{12}, x_{16}]$ is the algebra of equation (1.1).

The existence and properties of $\mathrm{DI}(2)$ have attracted the attention of experts in the subject since the seventies. Much published and unpublished work has been done. We will only sketch some main achievements that can be found in the literature.

A space BDI_2 realizing $D(2)$ as a mod 3 cohomology ring was first obtained by Zabrodsky in [38]. His method was technical and quite involved. It consisted in realizing geometrically the 3-local homotopy classes of BF_4, the classifying space of the exceptional Lie group F_4, that remain invariant under the action of the Friedlander exceptional isogeny. A more intelligible construction was provided by Aguadé [3]. The structure of BDI_2 as a 3-compact group was exposed in [23]. However one important feature of Zabrodsky's construction remained unexplained in these new approaches. Zabrodsky's construction came together with a map $\mathrm{BDI}_2 \to \mathrm{BF}_4$, after 3-completion, turning $H^*(\mathrm{BDI}_2; \mathbf{F}_3)$ into a finitely generated $H^*(\mathrm{BF}_4; \mathbf{F}_3)$-module, thus representing a monomorphism of 3-compact groups. The original motivation of this note was an attempt to explain not only the methods of Zabrodsky but the fact that BDI_2 was obtained as a sort of invariant gadget in BF_4. This is achieved in Theorem 1.1 and equation (1.2).

The paper is organized as follows. Sections 2 and 3 are devoted to a careful investigation of p-compact group homomorphisms from DI_2 to F_4. This leads to the proof of Theorem 1.1. In Section 4 we compute the cohomological structure of the homogeneous space $\mathrm{F}_4/\mathrm{DI}_2$. It turns out to coincide with that of Harper's molecule at the prime 3, and this, in turn implies that both spaces have the same homotopy type. Details are included in Section 7 for convenience of the reader. Sections 5 and 6 are devoted to the Friedlander exceptional isogeny of $(\mathrm{BF}_4)_{1/2}$. It is shown that suitably composed with an Adams map it becomes an involution that fixes DI_2. The effect of these maps in cohomological generators is determined and this leads to the proof of Theorem 1.2 and Corollaries 1.3 and 1.4. Some straightforward calculations, mainly in Section 2, were done with the aid of a computer using the *magma* system. It should be easy to perform the same calculations with any other suitable program.

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2. Admissible homomorphisms

Let X_1 and X_2 be p-compact groups [13] with maximal tori $i_1\colon T_1 = T(X_1) \to X_1$, $i_2\colon T_2 = T(X_2) \to X_2$, and corresponding Weyl groups $W_1 = W(X_1)$, $W_2 = W(X_2)$, respectively. Write $\mathrm{Rep}(X_1, X_2) = [\mathrm{B}X_1, \mathrm{B}X_2]$ for the set of conjugacy classes of morphisms $X_1 \to X_2$. The restriction map

$$\mathrm{Rep}(X_1, X_2) \xrightarrow{\overline{i_1}} \mathrm{Rep}(T_1, X_2) \xleftarrow[\cong]{\underline{i_2}} W_2\backslash \mathrm{Rep}(T_1, T_2)$$

where $\overline{i_1}$ and the bijection $\underline{i_2}$ [24, 3.4.(1)] are induced by i_1 and i_2 respectively, takes $f \in \mathrm{Rep}(X_1, X_2)$ to $f|\overline{T_1} = W_2T(f)$ where $T(f)$ is any lift

$$\begin{tikzcd} T_1 \arrow[r, "T(f)"] \arrow[d, "i_1"'] & T_2 \arrow[d, "i_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd} \tag{2.1}$$

of f to a morphism between the maximal tori.

It is sometimes useful to know that certain morphisms $T_1 \to T_2$ are *not* of the form $T(f)$ for any $f\colon X_1 \to X_2$.

Lemma 2.1 (Cf. [19, 1.8]). *Let $f\colon X_1 \to X_2$ be a p-compact group morphism where p is odd and X_1 is connected. Assume that*

- *$\pi_1(T(f))$ is injective, and*
- *p divides the order of the Weyl group W_1.*

Then p does not divide $\pi_1(T(f))$ in $\mathrm{Hom}(\pi_1(T_1), \pi_1(T_2))$.

Proof. By fixed point theory [14, 2.10, 2.14], f lifts to a morphism of the p-normalizers $N_p(f)\colon \mathrm{Syl}_p(N_1) \to \mathrm{Syl}_p(N_2)$. The assumption that $\pi_1(T(f))$ be injective implies, since W_1 is faithfully represented in $\pi_1(T_1)$ [13, 9.7], that $\pi_0(N_p(f))$ embeds the Sylow p-subgroup of W_1 into W_2.

Choose a monomorphism $\mu\colon \mathbf{Z}/p \to \mathrm{Syl}_p(N_1)$ such that $\pi_0(\mu)\colon \mathbf{Z}/p \to \mathrm{Syl}_p(W_1)$ is also injective. This is possible since the epimorphism $\mathrm{Syl}_p(N_1) \to \mathrm{Syl}_p(W_1)$ admits a section when p is odd [4]. Note that the composition $N_p(f)\mu$ is a monomorphism since it induces a monomorphism on component groups. Consider now the commutative diagram

$$\begin{tikzcd}
& \mathrm{Syl}_p(N_1) \arrow[r, "N_p(f)"] \arrow[d, "j_p"'] & \mathrm{Syl}_p(N_2) \arrow[d] \\
\mathbf{Z}/p \arrow[ur, "\mu"] \arrow[r, "j_p\mu"] \arrow[dr, "\mu'"'] & X_1 \arrow[r, "f"] & X_2 \\
& T_1 \arrow[u] \arrow[r, "T(f)"'] & T_2 \arrow[u, "i_2"']
\end{tikzcd}$$

where μ' is a lift of $j_p\mu$ [13, 4.7, 5.6]. Since $N_p(f)\mu$ is monomorphic, so is $i_2T(f)\mu'$ by commutativity of the diagram. However, this map would be trivial if $\pi_1(T(f))$ were divisible by p. $\square$

This lemma is also true for $p = 2$ provided the extra assumption

- there exists a monomorphism $\mathbf{Z}/2 \to \mathrm{Syl}_2(W_1)$ that factors through $\mathrm{Syl}_2(N_1)$

is added.

We now return to the general situation of (2.1). For any element w_1 of the Weyl group W_1 of X_1,

$$\underline{i}_2(T(f) \circ w_1) = i_2 \circ T(f) \circ w_1 = f \circ i_1 \circ w_1 = f \circ i_1 = i_2 \circ T(f) = \underline{i}_2(T(f))$$

so there must exist some Weyl group element $w_2 \in W_2$ such that $T(f) \circ w_1 = w_2 \circ T(f)$. Thus the lift $T(f)$ lies in the subset $W_2 \backslash \mathrm{Adm}(T_1, T_2)$ where

Definition 2.2. $\mathrm{Adm}(T_1, T_2) = \{a \in \mathrm{Rep}(T_1, T_2) \mid a \cdot W_1 \subseteq W_2 \cdot a\}$.

The fundamental group functor can be used to identify this set $\mathrm{Adm}(T_1, T_2)$ of *admissible representations* with the corresponding set $\mathrm{Adm}(\pi_1(T_1), \pi_1(T_2)) \subseteq \mathrm{Hom}(\pi_1(T_1), \pi_1(T_2))$ of *admissible homomorphisms*; that is, homomorphisms $a\colon \pi_1(T_1) \to \pi_1(T_2)$ such that $a \cdot W_1 \subseteq W_2 \cdot a$.

We shall now determine the admissible homomorphisms in the particular case where $p = 3$, $X_1 = \mathrm{DI}_2$ [23, 3], and $X_2 = \mathrm{F}_4$ viewed as a 3-compact group.

The construction of BDI_2 exhibits monomorphisms [23]

$$T_1 \to \mathrm{SU}(3) \to \mathrm{DI}_2 \tag{2.2}$$

of maximal rank where the morphism $T_1 = T(\mathrm{SU}(3)) \to \mathrm{SU}(3)$ is a maximal torus for $\mathrm{SU}(3)$. Take the composite morphism (2.2) as the maximal torus for DI_2.

Let $i_2\colon T_2 = T(\mathrm{F}_4) \to \mathrm{F}_4$ be a maximal torus for the simple compact Lie group F_4 (defining a maximal torus for the corresponding 3-compact group).

There is a standard identification

$$\pi_1(T_1) \xrightarrow{\cong} \Sigma_0({\mathbf{Z}_3}^3) = \{(x_1, x_2, x_3) \in {\mathbf{Z}_3}^3 \mid x_1 + x_2 + x_3 = 0\}$$

where $\mathbf{Z}_3$ stands for the ring of 3-adic integers. We shall use $\{(1,-1,0),(0,1,-1)\}$ as a basis for the free $\mathbf{Z}_3$-module $\Sigma_0({\mathbf{Z}_3}^3)$ thus identifying $\Sigma_0({\mathbf{Z}_3}^3)$ with ${\mathbf{Z}_3}^2$. Under this identification we may assume [31, 7.3, Example 1] [23] that the Weyl group $W_1 = W(\mathrm{DI}_2)$ of DI_2 corresponds to the subgroup

$$W(\mathrm{DI}_2) = \langle \alpha, \sigma, \tau \rangle \subseteq \mathrm{Aut}(\Sigma_0({\mathbf{Z}_3}^3)) = \mathrm{GL}(2, \mathbf{Z}_3)$$

with generators

$$\alpha = \frac{1}{1+\zeta}\begin{pmatrix} -1+\zeta & -\zeta \\ \zeta & 1-\zeta \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

where $\zeta^2 + 2 = 0$ and $\zeta \equiv 1 \bmod 3$. The subgroup $W(\mathrm{SU}(3)) = \langle \sigma, \tau \rangle \cong \Sigma_3$ is the Weyl group of $\mathrm{SU}(3)$.

Similarly, there is a standard identification [7] between the fundamental group $\pi_1(T_2)$ of the maximal torus for F_4 and the free abelian group

$$\Sigma_2(\mathbf{Z}^4) = \{(x_1, x_2, x_3, x_4) \in \mathbf{Z}^4 \mid x_1 + x_2 + x_3 + x_4 \in 2\mathbf{Z}\}.$$

Under this identification, the Weyl group $W_2 = W(\mathrm{F}_4)$ of F_4 is carried to the group (of order $1152 = 384 \cdot 3$)

$$W(\mathrm{F}_4) = W(\mathrm{B}_4)E \cup W(\mathrm{B}_4)H_1 \cup W(\mathrm{B}_4)H_2 \tag{2.3}$$

where $W(\mathrm{B}_4)$ is the reflection group (of order $384 = 2^4 \cdot 4!$) of all signed permutation matrices, and H_1 and H_2 are the matrices

$$H_1 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad H_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Note that the group $W(\mathrm{B}_4)$ and the matrices H_1 and H_2 preserve the submodule $\Sigma_2(\mathbf{Z}^4)$ of $\mathbf{Z}^4$. The operators H_1 and H_2 satisfy the relations $H_1^2 = E = H_2^2, H_2H_1 = -H_2, H_1H_2 = \mathrm{diag}(-1,1,1,1)H_1$. For F_4 viewed as a 3-compact group with maximal torus T_2, extensions of scalars provide an identification

$$\pi_1(T_2) \xrightarrow{\cong} \Sigma_2(\mathbf{Z}^4) \otimes \mathbf{Z}_3 \xrightarrow{\cong} \mathbf{Z}^4 \otimes \mathbf{Z}_3 \cong {\mathbf{Z}_3}^4$$

taking the Weyl group $W(\mathrm{F}_4)$ onto the reflection group $W(\mathrm{F}_4) < \mathrm{GL}(4, \mathbf{Z}_3)$ as defined in (2.3).

The linear map $A(v)\colon \Sigma_0({\mathbf{Z}_3}^3) \to {\mathbf{Z}_3}^4$ with matrix

$$A(v) = v \cdot \begin{pmatrix} -\zeta & 1 \\ \zeta & 1-\zeta \\ 0 & 1+\zeta \\ -2 & 1 \end{pmatrix}, \quad v \in \mathbf{Z}_3, \tag{2.4}$$

is admissible with respect to $W(\mathrm{DI}_2)$ and $W(\mathrm{F}_4)$ since $A(v)w = \chi(w)A(v)$ for all $w \in W(\mathrm{DI}_2)$ where $\chi\colon W(\mathrm{DI}_2) \to W(\mathrm{F}_4)$ is the group homomorphism with the values

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

on the generators α, σ, and τ, respectively.

Note that $A(v)$ and $-A(v)$ lie in the same orbit under the action of $W(\mathrm{F}_4)$ as $-E \in W(\mathrm{F}_4)$.

Lemma 2.3. *Let v be a 3-adic integer.*

1. *The linear homomorphism $A\colon \Sigma_0(\mathbf{Z}_3{}^3) \to \mathbf{Z}_3{}^4$ is admissible with respect to $W(\mathrm{DI}_2)$ and $W(\mathrm{F}_4)$ if and only if $A \in W(\mathrm{F}_4)A(v)$ for some 3-adic integer $v \in \mathbf{Z}_3$.*
2. *The linear map $A(v)$ is split injective if and only if $v \in \mathbf{Z}_3{}^*$ is a 3-adic unit.*
3. *The map*
$$\begin{aligned} \{\pm 1\}\backslash \mathbf{Z}_3{}^* &\to W(\mathrm{F}_4)\backslash \mathrm{Hom}_{\mathbf{Z}_3}(\Sigma_0(\mathbf{Z}_3{}^3), \mathbf{Z}_3{}^4) \\ \pm v &\to W(\mathrm{F}_4)A(v) \end{aligned}$$
is injective.

The proof, which we omit, is by direct (computer assisted) computation.

3. Constructing maps out of BDI_2

Let $\mathbf{I} = \mathbf{I}(W(\mathrm{DI}_2), W(\mathrm{SU}(3)))$ be the category with two objects, 0 and 1, and morphism sets $\mathbf{I}(0,0) = N_{W(\mathrm{DI}_2)}(W(\mathrm{SU}(3))/W(\mathrm{SU}(3))) \cong Z(W(\mathrm{DI}_2)) \cong \mathbf{Z}/2$, $\mathbf{I}(1,1) = W(\mathrm{DI}_2)$, $\mathbf{I}(0,1) = W(\mathrm{DI}_2)/W(\mathrm{SU}(3))$, and $\mathbf{I}(1,0) = \emptyset$.

The space BDI_2 can be constructed [23, 6.10] as the homotopy colimit of the $\mathbf{I}^{\mathrm{op}}$-space

$$\begin{tikzcd} \mathrm{B}\,\mathrm{SU}(3) \arrow[loop left, "(\mathbf{Z}/2)^{\mathrm{op}}"] & & \mathrm{B}T \arrow[ll, "W(\mathrm{SU}(3))^{\mathrm{op}}\backslash W(\mathrm{DI}_2)^{\mathrm{op}}"'] \arrow[loop right, "W(\mathrm{DI}_2)^{\mathrm{op}}"] \end{tikzcd} \tag{3.1}$$

where $\mathbf{Z}/2$ acts on $\mathrm{B}\,\mathrm{SU}(3)$ as $\{\psi^{\pm 1}\}$. We shall use diagram (3.1) in connection with Wojtkowiak obstruction theory [37] to prove existence and uniqueness of certain maps out of BDI_2.

The set of conjugacy classes of monomorphisms, $\mathrm{Mono}(\mathrm{DI}_2, \mathrm{F}_4)$ of monomorphisms $\mathrm{DI}_2 \rightarrowtail \mathrm{F}_4$ will turn out to be faithfully represented in the set $\mathrm{Mono}(\mathrm{SU}(3), \mathrm{F}_4)$ of conjugacy classes of monomorphisms $\mathrm{SU}(3) \rightarrowtail \mathrm{F}_4$ which we now describe [25]:

There exists a bijection
$$\begin{aligned} \{(u,v) \in (\mathbf{Z}_3)^2 \mid u+v \equiv 1 \bmod 3\} &\to \mathrm{Mono}(\mathrm{SU}(3), \mathrm{F}_4) \\ (u,v) &\to e\psi^{(u,v)} \end{aligned}$$
where $\psi^{(u,v)}$ is the composite morphism
$$\mathrm{SU}(3) \xrightarrow{\Delta} \mathrm{SU}(3) \times \mathrm{SU}(3) \xrightarrow{\psi^u \times \psi^v} \mathrm{SU}(3) \times \mathrm{SU}(3) \to \mathrm{SU}(3,3)$$
and $e\colon \mathrm{SU}(3,3) \to \mathrm{F}_4$ is the inclusion, described in [19, 3.3], of $\mathrm{SU}(3,3) = \mathrm{SU}(3) \times_{\mathbf{Z}/3} \mathrm{SU}(3)$ as a maximal rank subgroup of F_4. (The central $\mathbf{Z}/3$ in $\mathrm{SU}(3) \times \mathrm{SU}(3)$ is generated by $(zE, z^{-1}E)$ where $z \neq 1$ is a third root of unity.) Furthermore, the injective restriction map
$$\mathrm{Mono}(\mathrm{SU}(3), \mathrm{F}_4) \to \mathrm{Mono}(T(\mathrm{SU}(3)), \mathrm{F}_4) = W(\mathrm{F}_4)\backslash \mathrm{Hom}_{\mathbf{Z}_3}(\Sigma_0(\mathbf{Z}_3{}^3), \mathbf{Z}_3{}^4) \tag{3.2}$$
takes the monomorphism $e\psi^{(u,v)}$ to
$$W(\mathrm{F}_4)\begin{pmatrix} -u & v \\ u & -u+v \\ 0 & u+v \\ -2v & v \end{pmatrix} \tag{3.3}$$

Lemma 3.1. *Let $f \in \mathrm{Mono}(\mathrm{DI}_2, \mathrm{F}_4)$ be a monomorphism. Then the restriction to* $\mathrm{SU}(3)$,

$$f|_{\mathrm{SU}(3)} = e\psi^{(\zeta v, v)}$$

for a uniquely determined 3-adic unit $v \in \mathbf{Z}_3{}^*$, $v \equiv -1 \bmod 3$.

Proof. According to (2.3), the restriction of f to the maximal torus,

$$f|T(\mathrm{DI}_2) \in \mathrm{Mono}(T(\mathrm{DI}_2), \mathrm{F}_4) = W(\mathrm{F}_4)\backslash \mathrm{Hom}_{\mathbf{Z}_3}(\Sigma_0(\mathbf{Z}_3{}^3), \mathbf{Z}_3{}^4)$$

is of the form $f|T(\mathrm{DI}_2) = W(\mathrm{F}_4)A(v)$ for some 3-adic unit $v \in \mathbf{Z}_3{}^*$, uniquely determined up to sign. (The 3-adic number v must be a unit because $A(v)$ is split injective as $f|T(\mathrm{DI}_2)$ is a monomorphism [26, 23].) But this means that the restriction $f|\,\mathrm{SU}(3) = e\psi^{(\zeta v, v)}$ for a uniquely determined 3-adic unit $v \in \mathbf{Z}_3{}^*$, $v \equiv -1 \bmod 3$. □

Let now $v \in \mathbf{Z}_3{}^*$ be any 3-adic unit such that $v \equiv -1 \bmod 3$. Then the two homotopy classes $e\psi^{(\zeta v, v)} \in [\mathrm{B\,SU}(3), \mathrm{BF}_4]$ and $W(\mathrm{F}_4)A(v) \in [\mathrm{B}T(\mathrm{SU}(3)), \mathrm{BF}_4]$ form a homotopy coherent pair of maps out of the $\mathbf{I}^{\mathrm{op}}$-space (3.1) in the sense that

- $e\psi^{(\zeta v, v)}|T(\mathrm{SU}(3)) = W(\mathrm{F}_4)A(v)$,
- $W(\mathrm{F}_4)A(v)$ is $W(\mathrm{DI}_2)$-invariant,
- $e\psi^{(\zeta v, v)}$ is $\langle \psi^{-1} \rangle$-invariant,

where the last property follows from the computation

$$e\psi^{(\zeta v, v)}\psi^{-1} = e(\psi^{-1} \times \psi^{-1})\psi^{(\zeta v, v)} = e\psi^{(\zeta v, v)}$$

which uses the identity $e(\psi^{-1} \times \psi^{-1}) = e$ from [19, 3.3]. Therefore, there is an induced $\mathbf{I}$-space

(3.4)

$$\mathbf{Z}/2 \circlearrowright \mathrm{B}C_{\mathrm{F}_4}(e\psi^{(\zeta v, v)}) \xrightarrow{W(\mathrm{DI}_2)/W(\mathrm{SU}(3))} \mathrm{B}C_{\mathrm{F}_4}(W(\mathrm{F}_4)A(v)) \circlearrowleft W(\mathrm{DI}_2)$$

of connected mapping spaces where

$$\mathrm{B}C_{\mathrm{F}_4}(e\psi^{(\zeta v, v)}) = \mathrm{map}(\mathrm{BSU}(3), \mathrm{BF}_4)_{\mathrm{B}e\psi^{(\zeta v, v)}} \simeq \mathrm{B}Z(\mathrm{SU}(3)),$$
$$\mathrm{B}C_{\mathrm{F}_4}(W(\mathrm{F}_4)A(v))\,\mathrm{map}(\mathrm{B}T(\mathrm{SU}(3)), \mathrm{BF}_4)_{\mathrm{B}(W(\mathrm{F}_4)A(v))} \simeq \mathrm{B}T(\mathrm{F}_4)$$

as $e\psi^{(\zeta v, v)}$ is centric and p-toric [25]. These two spaces are simple so we may apply the homotopy functor π_t to (3.4) to obtain the $\mathbf{I}$-module $\underline{\pi}_t(v)$.

Lemma 3.2. $\lim_{\mathbf{I}}^{-s} \underline{\pi}_t(v) = 0$ *for all $s \le 0$, $t \ge 0$, $s + t \ge -1$ and for all 3-adic units $v \in \mathbf{Z}_3{}^*$ with $v \equiv -1 \bmod 3$.*

Proof. For the $\mathbf{I}$-module $\underline{\pi}_1(v)$,

$$\mathbf{Z}/2 \circlearrowright \mathbf{Z}/3 \xrightarrow{W(\mathrm{DI}_2)/W(\mathrm{SU}(3))} 0 \circlearrowleft W(\mathrm{DI}_2)$$

we have $\lim_{\mathbf{I}}^{-s} \underline{\pi}_1(v) = H^{-s}(\mathbf{Z}/2; \mathbf{Z}/3)$ which is trivial since $\mathbf{Z}/2 = \langle \psi^{-1} \rangle$ acts non-trivially on the center $\mathbf{Z}/3 = Z(\mathrm{SU}(3))$. For the $\mathbf{I}$-module $\underline{\pi}_2(v)$,

$$\mathbf{Z}/2 \circlearrowright 0 \xrightarrow{W(\mathrm{DI}_2)/W(\mathrm{SU}(3))} \mathbf{Z}_3^4 \circlearrowleft W(\mathrm{DI}_2)$$

we have $\lim_{\mathbf{I}}^0 \underline{\pi}_2(v) = 0$, $\lim_{\mathbf{I}}^1 \underline{\pi}_2(v) = (\mathbf{Z}_3{}^4)^{Z(W(\mathrm{DI}_2) \times W(\mathrm{SU}(3))} = 0$ as $-E$ belongs to the center $Z(W(\mathrm{DI}_2))$, and $\lim_{\mathbf{I}}^{-s} \underline{\pi}_2(v) = 0$ for $s \le -2$ by [23, 12.7]. □

With this lemma in place we are ready for one of the main results of this paper.

Theorem 3.3. *The following hold for the set* $\mathrm{Mono}(\mathrm{DI}_2,\mathrm{F}_4)$ *of conjugacy classes of monomorphisms* $\mathrm{DI}_2\to\mathrm{F}_4$*:*

1. *The restriction map*
$$\mathrm{Mono}(\mathrm{DI}_2,\mathrm{F}_4)\to\mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4)$$
is an injection with $\{e\psi^{(\zeta v,v)}\mid v\in\mathbf{Z}_3{}^*, v\equiv -1 \bmod 3\}$ *as its image.*
2. *The restriction map*
$$\mathrm{Mono}(\mathrm{DI}_2,\mathrm{F}_4)\to\mathrm{Mono}(T(\mathrm{DI}_2),\mathrm{F}_4)$$
is an injection with $\{W(\mathrm{F}_4)A(v)\mid v\in\mathbf{Z}_3{}^*, v\equiv -1 \bmod 3\}$ *as its image.*

Proof. 1. This follows from Wojtkowiak's obstruction theory [37] as $\lim_{\mathbf{I}}^{-s}\underline{\pi}_t(v)=0$ when $s+t=0$ and $s+t=-1$.

2. The map is a composition
$$\mathrm{Mono}(\mathrm{DI}_2,\mathrm{F}_4)\to\mathrm{Mono}(\mathrm{SU}(3),\mathrm{F}_4)\to\mathrm{Mono}(T(\mathrm{SU}(3)),\mathrm{F}_4)$$
of two injections (3.2). □

Let $f(v)\colon \mathrm{DI}_2\to\mathrm{F}_4$, $v\in\mathbf{Z}_3{}^*, v\equiv -1 \bmod 3$, denote the unique extension to DI_2 of the monomorphism $e\psi^{(\zeta v,v)}\colon\ \mathrm{SU}(3)\to\mathrm{F}_4$ on $\mathrm{SU}(3)$.

Corollary 3.4. 1. *The group* $\mathrm{Out}(\mathrm{DI}_2)$ *acts simply transitively on the set* $\mathrm{Mono}(\mathrm{DI}_2,\mathrm{F}_4)$ *of monomorphisms.*
2. *The centralizer of any monomorphism* $\mathrm{DI}_2\to\mathrm{F}_4$ *is trivial.*
3. *The Weyl group* [23, 9.5] *of any monomorphism* $\mathrm{DI}_2\to\mathrm{F}_4$ *is trivial.*

Proof. 1. This is clear since [23] $\mathrm{Out}(\mathrm{DI}_2)\cong\{v\in\mathbf{Z}_3{}^*\mid v\equiv 1 \bmod 3\}$ and
$$f(v)\psi^u=e\psi^{(\zeta v,v)}\psi^u=e\psi^{(\zeta vu,vu)}=f(vu)$$
so that $f(v)\psi^u=f(v)\Leftrightarrow u=1$.

2. The E_2-page of the Bousfield-Kan spectral sequence, $E_s^{st}=\lim_{\mathbf{I}}^{-s}\underline{\pi}_t(v)$, converging to $\pi_{s+t}(\mathrm{B}C_{\mathrm{F}_4}(f(v)\mathrm{DI}_2))$ vanishes completely (3.2).

3. The Weyl group $W_{\mathrm{F}_4}(f(v)\mathrm{DI}_2)$ is [23, 9.6] the stabilizer subgroup for $\mathrm{Out}(\mathrm{DI}_2)$ acting on $f(v)\in\mathrm{Mono}(\mathrm{DI}_2,\mathrm{F}_4)$. We have just seen that this stabilizer is trivial. □

In a way, (3.4.1) says that F_4 contains a unique copy of DI_2.

Finally, we determine the set $\mathrm{Rep}(\mathrm{DI}_2,\mathrm{F}_4)=[\mathrm{BDI}_2,\mathrm{BF}_4]$ of all maps up to homotopy.

Proposition 3.6. *Any non-trivial morphism of* DI_2 *to* F_4 *is a monomorphism.*

Proof. Let $f\colon\mathrm{DI}_2\to\mathrm{F}_4$ be any non-trivial morphism. Then $f|T(\mathrm{DI}_2)=W(\mathrm{F}_4)A(v)\in[\mathrm{B}T(\mathrm{DI}_2),\mathrm{BF}_4]$ for some 3-adic integer $v\in\mathbf{Z}_3$. This 3-adic integer, v, must be non-zero as f is non-trivial [24, 6.7] and even a 3-adic unit by (2.1). This shows that $A(v)$ is split injective and hence [26, 3.6] [23, 8.2] that f is a monomorphism. □

Proof of Theorem 1.1. We conclude from (3.3, 3.4, 3.6) that
$$\mathrm{Rep}(\mathrm{DI}_2,\mathrm{F}_4)=\{0\}\cup\mathrm{Mono}(\mathrm{DI}_2,\mathrm{F}_4)$$
is in bijection with the set $\{0\}\cup\mathbf{Z}_3{}^*/\{\pm1\}$. □

In the following, we let $\mathrm{B}\alpha\colon\mathrm{BDI}_2\to\mathrm{BF}_4$ denote the monomorphism $Bf(-1)$ corresponding (3.3) to the admissible homomorphism with matrix $A(-1)$ (2.4).

4. The homogeneous space $\mathrm{F}_4/\mathrm{DI}_2$

In this section we compute the mod 3 cohomology of the exotic homogeneous space $\mathrm{F}_4/\mathrm{DI}_2$.

The unstable mod 3 cohomology of F_4 is known since Borel [6]:

$$H^*(\mathrm{F}_4;\mathbf{F}_3) \cong E[z_3, z_7, z_{11}, z_{15}] \otimes P[z_8]/({z_8}^3)$$

with $P^1z_3 = z_7$, $\beta z_7 = z_8$ and $P^1z_{11} = z_{15}$.

The 3-complete space BDI_2 realizes the rank 2 mod 3 Dickson algebra meaning that

$$H^*(\mathrm{BDI}_2;\mathbf{F}_3) \cong P[x_{12}, x_{16}]$$

with $P^1x_{12} = x_{16}$.

The homogeneous space $\mathrm{F}_4/\mathrm{DI}_2$ is defined as the homotopy fibre of the map $\mathrm{BDI}_2 \xrightarrow{\mathrm{B}\alpha} \mathrm{BF}_4$, and we have a sequence of fibrations

$$\mathrm{DI}_2 \xrightarrow{\Omega\mathrm{B}\alpha} \mathrm{F}_4 \xrightarrow{\pi} \mathrm{F}_4/\mathrm{DI}_2 \xrightarrow{j} \mathrm{BDI}_2 \xrightarrow{\mathrm{B}\alpha} \mathrm{BF}_4\,. \tag{4.1}$$

We now investigate the value of the functor $H^*(-;\mathbf{F}_3)$ on this sequence.

Proposition 4.1. *The effects on* mod 3 *cohomology of the maps π and j of (4.1) are as follows:*

1. *The unstable* mod 3 *cohomology algebra of* $\mathrm{F}_4/\mathrm{DI}_2$ *is*
$$H^*(\mathrm{F}_4/\mathrm{DI}_2;\mathbf{F}_3) \cong E[z_3, z_7] \otimes P[z_8]/(z_8^3)$$
with $P^1z_3 = z_7$ and $\beta z_7 = z_8$.
2. *$H^*(\pi;\mathbf{F}_3)$ is the obvious inclusion*
$$\begin{aligned}H^*(\mathrm{F}_4/\mathrm{DI}_2;\mathbf{F}_3) &\cong E[z_3, z_7] \otimes P[z_8]/(z_8^3)\\ &\subseteq E[z_3, z_7, z_{11}, z_{15}] \otimes P[z_8]/(z_8^3) \cong H^*(\mathrm{F}_4;\mathbf{F}_3)\end{aligned}$$
of algebras.
3. *$H^{>0}(j;\mathbf{F}_3)$ is the trivial homomorphism $H^{>0}(\mathrm{BDI}_2;\mathbf{F}_3) \to H^{>0}(\mathrm{F}_4/\mathrm{DI}_2;\mathbf{F}_3)$.*

Proof. It consists of a calculation with the Serre spectral sequence for the fibration $\mathrm{F}_4 \xrightarrow{\pi} \mathrm{F}_4/\mathrm{DI}_2 \xrightarrow{j} \mathrm{BDI}_2$. This starts at $E_2^{p,q} \cong H^p(\mathrm{BDI}_2;\mathbf{F}_3) \otimes H^q(\mathrm{F}_4;\mathbf{F}_3)$.

For degree reasons the classes z_3, z_7 and z_8 in the vertical edge are permanent cycles. Assume that $z_{11} \in E_2^{0,*}$ is a permanent cycle. Then $z_{15} = P^1z_{11}$ is also a permanent cycle and therefore the spectral sequence collapses at E_2. But this is impossible because the fibre $\mathrm{F}_4/\mathrm{DI}_2$ of the monomorphism $\mathrm{B}\alpha$ is $\mathbf{F}_3$-finite [13].

Hence, the class z_{11} transgresses to $\pm x_{12} \in H^*(\mathrm{BDI}_2;\mathbf{F}_3) \cong E_2^{*,0}$ and then $z_{15} = P^1z_{11}$ trangresses to $\pm x_{16} = P^1(\pm x_{12})$. After this last, the spectral sequence collapses to $E_\infty^{*,*} \cong E_\infty^{0,*} \cong E[z_3, z_7] \otimes P[z_8]/({z_8}^3)$ and the edge homomorphisms are an injection

$$H^*(\mathrm{F}_4/\mathrm{DI}_2;\mathbf{F}_3) \cong E_\infty^{0,*} \cong E[z_3, z_7] \otimes P[z_8]/({z_8}^3) \rightarrowtail E_2^{0,*} \cong H^*(\mathrm{F}_4;\mathbf{F}_3)$$

and the trivial homomorphism

$$H^*(\mathrm{BDI}_2;\mathbf{F}_3) \cong E_2^{*,0} \rightarrowtail E_\infty^{0,*} \cong \mathbf{F}_3 \subset H^*(\mathrm{F}_4/\mathrm{DI}_2;\mathbf{F}_3)$$

which proves the proposition. □

Remark 4.2. The mod 3 cohomology of $\mathrm{F}_4/\mathrm{DI}_2$ coincides with that of Harper's molecule [17] at the prime 3 and this implies that they are actually homotopy equivalent up to 3-completion. Details will be worked out in Section 7.

5. Friedlander's exceptional isogeny

In [15], E.M. Friedlander showed the existence of a self-homotopy equivalence $\mathrm{B}\varphi$ of $(\mathrm{BF}_4)_{1/2}$ that restricts to the maximal torus to the isogeny

$$\mathrm{B}T(\varphi)\colon \mathrm{B}T(\mathrm{F}_4) \to \mathrm{B}T(\mathrm{F}_4)$$

determined by the matrix

$$\pi_2(\mathrm{B}T(\varphi)) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

acting on $\pi_2(\mathrm{B}T(\mathrm{F}_4)) \cong \mathbf{Z}_3{}^4$. The automorphism $\mathrm{B}\varphi$ of the 3-compact group BF_4 satisfies the relation $\mathrm{B}\varphi \circ \mathrm{B}\varphi \simeq \mathrm{B}\psi^2$. To see this, note that $W(\mathrm{F}_4)T(\varphi)^2 = W(\mathrm{F}_4)T(\psi^2)$ and recall that BF_4 has N-determined automorphisms; that is, restriction to the maximal torus normalizer provides a bijective map $\mathrm{Out}(\mathrm{F}_4) \to \mathrm{Out}(N_{\mathrm{F}_4}(T))$ [23, 20].

Proposition 5.1. *There is a homotopy commutative diagram*

$$\begin{tikzcd} \mathrm{BDI}_2 \arrow[r, "\mathrm{B}\alpha"] \arrow[d, "\mathrm{B}\psi^\zeta"'] & \mathrm{BF}_4 \arrow[d, "\mathrm{B}\varphi"] \\ \mathrm{BDI}_2 \arrow[r, "\mathrm{B}\alpha"'] & \mathrm{BF}_4 \end{tikzcd} \tag{5.1}$$

where $\zeta^2 = -2$ *and* $\mathrm{B}\psi^\zeta \in [\mathrm{BDI}_2, \mathrm{BDI}_2]$ *is the corresponding Adams map.*

Proof. According to Theorem 3.3 it is enough to check the restriction to the maximal torus of BDI_2, and in fact,

$$\begin{aligned} \pi_2(\mathrm{B}T(\varphi)) \cdot \pi_2(\mathrm{B}T(\alpha)) &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \zeta & -1 \\ -\zeta & -1+\zeta \\ 0 & -1-\zeta \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2\zeta & -\zeta \\ 0 & -2+\zeta \\ -2 & -\zeta \\ 2 & -2-\zeta \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 \\ 0 & -\zeta-1 \\ -\zeta & 1 \\ -\zeta & -\zeta+1 \end{pmatrix} \cdot \zeta \in W(\mathrm{F}_4)(\pi_2(\mathrm{B}T(\alpha)) \cdot \zeta); \end{aligned}$$

hence $\mathrm{B}\varphi \circ \mathrm{B}\alpha = \mathrm{B}\alpha \circ \mathrm{B}\psi^\zeta$. ∎

Proposition 5.2. *The automorphism of* $H^*(\mathrm{BF}_4; \mathbf{Q}_3) \cong P[x_4, x_{12}, x_{16}, x_{24}]$ *induced by* $\mathrm{B}\varphi$ *is up to decomposables, determined by*

$$\mathrm{B}\varphi^*(x_4) = 2x_4\,, \quad \mathrm{B}\varphi^*(x_{12}) = -8x_{12}\,, \quad \mathrm{B}\varphi^*(x_{16}) = 16x_{16}\,, \quad \mathrm{B}\varphi^*(x_{24}) = -64x_{24}\,.$$

Proof. This is a long but straightforward calculation, easier done with the aid of a computer. This result coincides also with calculation done by Adams and Mahmud [1, Table 2.14]. ∎

Define $\mathrm{B}\tau\colon \mathrm{BF}_4 \to \mathrm{BF}_4$ to be the automorphism $\mathrm{B}\tau = \mathrm{B}\psi^{1/\zeta} \circ \mathrm{B}\varphi$. Then $(\mathrm{B}\tau)^2$ is homotopic to the identity and $\mathrm{B}\tau \circ \mathrm{B}\alpha = \mathrm{B}\alpha$ by (5.1).

Corollary 5.3. *$H^4(\mathrm{B}\tau;\mathbf{Z}_3)$ is multiplication by -1 on $H^4(\mathrm{BF}_4;\mathbf{Z}_3)$.*

Proof. The effect of $H^4(\mathrm{B}\tau;\mathbf{Q}_3)$ on $H^4(\mathrm{BF}_4;\mathbf{Q}_3)$ is given by

$$H^4(\mathrm{B}\tau;\mathbf{Q}_3) = H^4(\mathrm{B}\varphi;\mathbf{Q}_3) \circ H^4(\mathrm{B}\psi^{1/\zeta};\mathbf{Q}_3) = 2\cdot\zeta^{-2} = -1$$

by (5.2) and because the unstable Adams operation ψ^λ induces multiplication by λ^i in degree $2i$. □

Let $\tau\colon \mathrm{F}_4/\mathrm{DI}_2 \to \mathrm{F}_4/\mathrm{DI}_2$ denote the self-homotopy equivalence of the exotic homogeneous space $\mathrm{F}_4/\mathrm{DI}_2$ induced by $\mathrm{B}\tau$ on BF_4 and the identity on BDI_2. This map makes the diagram

$$\begin{tikzcd}
\mathrm{F}_4/\mathrm{DI}_2 \arrow[r, "j"] \arrow[d, "\tau"] & \mathrm{BDI}_2 \arrow[r, "\mathrm{B}\alpha"] \arrow[d, equal] & \mathrm{BF}_4 \arrow[d, "\mathrm{B}\tau"] \\
\mathrm{F}_4/\mathrm{DI}_2 \arrow[r, "j"'] & \mathrm{BDI}_2 \arrow[r, "\mathrm{B}\alpha"'] & \mathrm{BF}_4
\end{tikzcd}$$

commute up to homotopy.

Corollary 5.4. *The involution $H^*(\tau;\mathbf{F}_3)$ of $H^*(\mathrm{F}_4/\mathrm{DI}_2;\mathbf{F}_3)$ sends the generators z_3, z_7, and z_8 to $-z_3$, $-z_7$, and $-z_8$, respectively.*

Proof. That $H^*(\tau;\mathbf{F}_3)(z_3) = -z_3$ follows from (5.3) since (4.1.2) the mod 3 cohomology of $\mathrm{F}_4/\mathrm{DI}_2$ embeds in the mod 3 cohomology of F_4. As the two other generators are linked to z_3 by Steenrod operations (4.1.1), $H^*(\tau;\mathbf{F}_3)(z_3)$ must also act as multiplication by -1 on them. □

We shall return to the homotopy involution $\mathrm{B}\tau$ in Section 6.

6. BDI_2 as a homotopy fixed point space

The automorphism $\mathrm{B}\tau = \mathrm{B}\psi^{1/\zeta}\circ\mathrm{B}\phi$ of BF_4 is a homotopy involution, $(\mathrm{B}\tau)^2 \simeq 1$, that homotopy fixes BDI_2 in the sense that the diagram

(6.1)
$$\begin{tikzcd}
\mathrm{BDI}_2 \arrow[r, tail, "\mathrm{B}\alpha"] \arrow[d, equal] & \mathrm{BF}_4 \arrow[d, "\mathrm{B}\tau"] \\
\mathrm{BDI}_2 \arrow[r, tail, "\mathrm{B}\alpha"] & \mathrm{BF}_4
\end{tikzcd}$$

commutes up to homotopy (5.1). This suggests that BDI_2 should be fixed under $\mathrm{B}\tau$ in some sense. To make this precise, we need to rigidify the above two properties.

Proposition 6.1. *We may assume that the maps $\mathrm{B}\alpha$ and $\mathrm{B}\tau$ satisfy the identities $\mathrm{B}\tau\circ\mathrm{B}\alpha = \mathrm{B}\alpha$ and $\mathrm{B}\tau\circ\mathrm{B}\tau = 1$.*

Proof. Let $\mathbf{D}$ be the category

$$\begin{tikzcd} \cdot \arrow[r, "a"] & \cdot \arrow[loop right, "b"] \end{tikzcd}$$

with two objects and two non-identity morphisms, a and b, subject to the relations $ba = a$ and $bb = 1$. The $\mathbf{D}$-diagram

(6.2)
$$\begin{tikzcd} \mathrm{BDI}_2 \arrow[r, "\mathrm{B}\alpha"] & \mathrm{BF}_4 \arrow[loop right, "\mathrm{B}\tau"] \end{tikzcd}$$

in the homotopy category of spaces is centric; in fact, all the relevant mapping spaces are weakly contractible (3.4.2) [23, 14]. Hence (6.2) has an essentially unique realization in the category of spaces [12, 1.1]. $\square$

With such specific realizations of $\mathrm{B}\alpha$ and $\mathrm{B}\tau$, let $\mathrm{F}_4/\mathrm{DI}_2$ denote the homotopy fibre of $\mathrm{B}\alpha$ over any point fixed by $\mathrm{B}\tau$, e.g. any point in the image of $\mathrm{B}\alpha$. Then $\mathrm{F}_4/\mathrm{DI}_2$ is $\mathbf{Z}/2$-space and there is a homotopy fibre sequence

$$\mathrm{F}_4/\mathrm{DI}_2 \xrightarrow{j} \mathrm{B}DI_2 \xrightarrow{\mathrm{B}\alpha} \mathrm{BF}_4$$

of $\mathbf{Z}/2$-spaces and $\mathbf{Z}/2$-maps which induces [8, XI.7.1] a homotopy fibre sequence

$$(\mathrm{F}_4/\mathrm{DI}_2)^{h\mathbf{Z}/2} \xrightarrow{j^{h\mathbf{Z}/2}} \mathrm{BDI}_2^{h\mathbf{Z}/2} \xrightarrow{(\mathrm{B}\alpha)^{h\mathbf{Z}/2}} \mathrm{BF}_4{}^{h\mathbf{Z}/2} \tag{6.3}$$

of homotopy fixed point spaces. Here,

$$\mathrm{BDI}_2^{h\mathbf{Z}/2} \simeq \mathrm{map}(\mathrm{B}\mathbf{Z}/2, \mathrm{BDI}_2) \simeq \mathrm{BDI}_2$$

since $\mathbf{Z}/2$ acts trivially on the 3-complete space BDI_2. As to the base space, the Bousfield-Kan spectral sequence for the homotopy fixed point space $\mathrm{BF}_4{}^{h\mathbf{Z}/2}$ degenerates to the formula $\pi_*(\mathrm{BF}_4{}^{h\mathbf{Z}/2}) = \pi_*(\mathrm{BF}_4)^{\mathbf{Z}/2}$. In particular, this space is non-empty and connected and its homotopy consists of the invariant part of the homotopy for BF_4, cf. [38]. The aim is to show that the fibre of (6.3) is contractible in order to obtain a homotopy equivalence between BDI_2 and $\mathrm{BF}_4{}^{h\mathbf{Z}/2}$.

Proposition 6.2. *The fibre,* $(\mathrm{F}_4/\mathrm{DI}_2)^{h\mathbf{Z}/2}$*, of the fibration sequence (6.3) is contractible.*

Proof. By the general result at the end of this section (6.3), it suffices to show that the space E sitting in the homotopy pull back diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{F}_4/\mathrm{DI}_2 \arrow[d, "{(1,\tau)}"] \\ \mathrm{F}_4/\mathrm{DI}_2 \arrow[r, "\Delta"'] & (\mathrm{F}_4/\mathrm{DI}_2)^2 \end{tikzcd} \tag{6.4}$$

is contractible. The Eilenberg-Moore spectral sequence $E_r^{**} \Rightarrow H^*(E;\mathbf{F}_3)$ associated to (6.4) is a second quadrant cohomological spectral sequence with

$$E_2^{-pq} = \mathrm{Tor}_p^{R\otimes R}(R,R)^q$$

where $R = H^*(\mathrm{F}_4/\mathrm{DI}_2;\mathbf{F}_3) \cong E[z_3,z_7]\otimes P[z_8]/(z_8^3)$ and where R is a right $R\otimes R$-module through the cup product $\mu\colon R\otimes R\to R$ and a left $R\otimes R$-module through $\mu\circ(1\otimes\tau^*)$. We shall show that $E_2^{**} = \mathbf{F}_3$.

Let κ be the algebra isomorphism of $R\otimes R$ given by

$$\kappa(z_i\otimes 1) = z_i\otimes 1 - 1\otimes z_i, \quad \kappa(1\otimes z_i) = 1\otimes z_i, \qquad i = 3,7,8\,.$$

Then κ satisfies the identity $\varepsilon\otimes 1 = \mu\kappa$, where $\varepsilon\colon R\to\mathbf{F}_3$ is the augmentation homomorphism, and hence (6.5)

$$E_2^{-p*} = \mathrm{Tor}_p^R(\mathbf{F}_3,R)$$

where R is a left R-module via the algebra morphism

$$R \xrightarrow{1\otimes\eta} R\otimes R \xrightarrow{\kappa} R\otimes R \xrightarrow{1\otimes\tau^*} R\otimes R \xrightarrow{\mu} R$$

which takes z_i to $-z_i$, $i = 3, 7, 8$. In particular, R is a free R-module and the vanishing of the E_2-page for the Eilenberg-Moore spectral sequence follows. $\square$

Proof of Theorem 1.2. By (6.3, 6.2), the monomorphism $\mathrm{B}\alpha\colon \mathrm{BDI}_2 \to \mathrm{BF}_4$ induces a homotopy equivalence $\mathrm{BDI}_2 \simeq (\mathrm{BF}_4)^{h\mathbf{Z}/2}$. $\square$

Proof of (1.2) and Corollary 1.3. Since homotopy fixed points might be seen as a homotopy limit, it is a general rule that $\mathrm{map}(X, Y^{hG}) = \mathrm{map}(X, Y)^{hG}$ and also using the Bousfield-Kan spectral sequence for the homotopy groups of a homotopy limit [8], that $\pi_i(M^{hG}) = \pi_i(M)^G$ when M is p-complete and simply connected, M^{hG} is not empty, and $p \nmid |G|$. $\square$

Proof of Corollary 1.4. We will prove that the map $j\colon \mathrm{F}_4/\mathrm{DI}_2 \to \mathrm{BDI}_2$ is null-homotopic. For this aim we observe that both j and the constant map c represent classes in $\pi_0\,\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, \mathrm{BDI}_2) \cong \pi_0\,\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, {\mathrm{BF}_4}^{h\mathbf{Z}/2})$ that map trivially down to $\pi_0\,\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, \mathrm{F}_4)$. Thus we only need to check that the map

$$\pi_0(\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, \mathrm{BF}_4)_c^{h\mathbf{Z}/2}) \longrightarrow (\pi_0\,\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, \mathrm{BF}_4)_c)^{\mathbf{Z}/2}$$

is injective. The obstructions to injectivity are given by the Bousfield-Kan spectral sequence [8, XI,§7]: $H^i(\mathbf{Z}/2, \pi_i(\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, \mathrm{BF}_4)_c))$. The groups

$$\pi_i = \pi_i(\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, \mathrm{BF}_4)_c)$$

are 3-local and nilpotent. In fact, applying $\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, -)$ to the Postnikov tower of BF_4, we get a spectral sequence with E_2-page $E_2^{i,j} \cong H^i(\mathrm{F}_4/\mathrm{DI}_2; \pi_j(\mathrm{BF}_4))$ converging to the homotopy of $\mathrm{map}(\mathrm{F}_4/\mathrm{DI}_2, \mathrm{BF}_4)_c$. Since the cohomology of $\mathrm{F}_4/\mathrm{DI}_2$ vanishes above certain dimension, this gives a finite filtration on π_1 with filtration quotients that are finitely generated modules over the 3-adic integers.

It follows that the groups $H^i(\mathbf{Z}/2, \pi_i)$ are trivial. Notice that $H^1(\mathbf{Z}/2, \pi_1)$ might be a non-abelian first cohomology group. For this we refer to [27, §5, Proposition 38], and use induction over the nilpotence class of π_1.

The splitting is now constructed using a section of $\pi\colon \mathrm{F}_4 \to \mathrm{F}_4/\mathrm{DI}_2$ and the H-space structure of F_4. $\square$

In the homotopy splitting from Corollary 1.4 of F_4, $\mathrm{F}_4/\mathrm{DI}_2 \simeq K(3)$ is one of Harper's H-spaces (7.5) and $\mathrm{DI}_2 \simeq B_5(3)$ is one of the Mimura-Toda bundles of completed spheres over spheres [22].

We finish this section with the general results that we used above to compute the homotopy fixed point space $(\mathrm{F}_4/\mathrm{DI}_2)^{h\mathbf{Z}/2}$.

First, let G be a finite group and K a G-space. Define E to be the homotopy pullback

(6.5)
$$\begin{tikzcd} E \arrow[r] \arrow[d] & K \arrow[d, "\mu"] \\ K \arrow[r, "\Delta"'] & \mathrm{map}(G, K) \end{tikzcd}$$

where Δ takes $x \in K$ to the constant map with value x and the action map μ takes x to the map $\mu(k)(h) = h^{-1}x$, $h \in G$. These two maps are equivariant if we equip $\mathrm{map}(G, K)$ with the action $(gu)(h) = u(g^{-1}h)$, $g, h \in G$, $u \in \mathrm{map}(G, K)$, and K in the lower left corner with the trivial action. Thus (6.5) is a diagram of G-spaces and G-maps.

Theorem 6.3. *If K is $\mathrm{B}G$-null, the G-map $E \to K$ induces a weak homotopy equivalence $E^{hG} \to K^{hG}$ of homotopy fixed point spaces.*

Proof. Since the homotopy fixed point functor commutes with homotopy pull-backs [8, XI.4.3], there is an induced homotopy pull-back diagram

$$\begin{tikzcd} E^{hG} \arrow[r] \arrow[d] & K^{hG} \arrow[d, "\mu^{hG}"] \\ K^{hG} \arrow[r, "\Delta^{hG}"'] & \mathrm{map}(G,K)^{hG} \end{tikzcd}$$

of homotopy fixed point spaces. The map $\Delta^{hG}\colon K^{hG} \to \mathrm{map}(G,K)^{hG}$ can be identified to the evaluation map $\mathrm{map}(\mathrm{B}G,K) \to K$. Indeed, $K^{hG} = \mathrm{map}(\mathrm{B}G,K)$, since the action is trivial in the lower left corner, and $\mathrm{map}(G,K)^{hG} = \mathrm{map}(G_{hG},K) = \mathrm{map}(*,K) = K$. If K is $\mathrm{B}G$-null, Δ^{hG} is a homotopy equivalence and so is then the top horizontal map $E^{hG} \to K^{hG}$. $\square$

By an argument dual to that of [21, 5.1] [28, p. 282], we may identify E and the homotopy limit of the diagram $K \overset{G}{\to} K$ consisting of the $|G|$ maps $K \to K$ given by $x \to gx$ for $g \in G$.

The final result that we used above was an algebraic computation ensuring the vanishing of an Eilenberg-Moore spectral sequence.

Let R be a unital and augmented (graded) algebra over a field k. The structure maps

$$\mu\colon R\otimes R \to R, \qquad \eta\colon k \to R, \qquad \varepsilon\colon R \to k$$

make R a right $R\otimes R$-module and k a right R-module.

For any right R-module A, the tensor product $A \otimes R$ is a right $R\otimes R$-module with right $R\otimes R$-multiplication given by

$$(a\otimes r)\cdot(s\otimes t) = as \otimes rt$$

for all elements $a \in A$, $r,s,t \in R$ (modified by the usual sign in the graded case). When $A = k$, in particular, right multiplication in $k \otimes R = R$ by $s\otimes t \in R\otimes R$ is right multiplication in R by $(\varepsilon\otimes 1)(s\otimes t) = \varepsilon(s)t$.

Lemma 6.4. *Let κ be an algebra isomorphism of $R\otimes R$ and $\otimes_R(R\otimes R)$ the left adjoint functor to the forgetful functor induced by the algebra homomorphism $\kappa\circ(1\otimes\eta)\colon R \to R\otimes R$. Then*

1. *$A\otimes_R (R\otimes R) = A\otimes R$ with right $R\otimes R$-multiplication $(a\otimes r)(s\otimes t) = (a\otimes r)\cdot\kappa^{-1}(s\otimes t)$*
2. *$\mathrm{Tor}_p^R(A,B) = \mathrm{Tor}_p^{R\otimes R}(A\otimes_R (R\otimes R), B)$ and*
$$\mathrm{Ext}^p_R(A,B) = \mathrm{Ext}^p_{R\otimes R}(A\otimes_R (R\otimes R), B)$$

for all right R-modules A, all left $R\otimes R$-modules B, and all $p \geq 0$.

Proof. We have

$$U(\kappa(1\otimes\eta)) = U(1\otimes\eta)U(\kappa) \quad \text{and} \quad U(\kappa)L(\kappa(1\otimes\eta)) = L(1\otimes\eta)$$

where $L(f)$ denotes the left adjoint to the forgetful functor $U(f)$ induced by the algebra homomorphism f. Note that $L(1\otimes\eta)(A) = A\otimes R$. It follows that $L(1\otimes\eta)$ is exact; it also takes R-projectives to $R\otimes R$-projectives because it is left adjoint to a (right) exact functor [33, 2.3.10]. The same is true of the functor $L(\kappa(1\otimes\eta))$

which differs from $L(1 \otimes \eta)$ by an isomorphism. Similarly, $U(\kappa(1 \otimes \eta))$ is an exact functor that takes injectives to injectives. Therefore, the identities

$$A \otimes_R UB = LA \otimes_{R \otimes R} B, \quad \mathrm{Hom}_R(A, UB) = \mathrm{Hom}_{R \otimes R}(LA, B)$$

where $L = L(\kappa(1 \otimes \eta))$ and $U = U(\kappa(1 \otimes \eta))$, prolong to the identities of (6.4.2). ☐

Corollary 6.5. *Suppose in addition to the assumptions of (6.4) that $\varepsilon \otimes 1 = \mu\kappa$. Then $\kappa \circ (1 \otimes \eta)$ induces an isomorphism*

$$\mathrm{Tor}_p^R(k, B) = \mathrm{Tor}_p^{R \otimes R}(R, B)$$

for all left $R \otimes R$-modules B and all $p \geq 0$.

Proof. Note that $L(1 \otimes \eta)(k) = U(\varepsilon \otimes 1)(R)$ where R is considered a module over itself. Therefore $L(\kappa(1 \otimes \eta))(k) = U(\kappa^{-1})L(1 \otimes \eta)(k) = U(\kappa^{-1})U(\varepsilon \otimes 1)(R) = U((\varepsilon \otimes 1)\kappa^{-1})(R) = U(\mu)(R)$ which is R as an $R \otimes R$-module. ☐

7. Harper's molecule

This section contains some comments on John Harper's work on finite H-spaces. Our starting point will be the theorem below and our main result (7.3) says that the space $K(p)$ is cohomologically unique in that it is determined up to homotopy equivalence by cohomological information. This will enable us (7.5) to identify $K(3)$ and the fibre $\mathrm{F}_4/\mathrm{DI}_2$ of the embedding $\mathrm{B}\alpha\colon \mathrm{BDI}_2 \to \mathrm{BF}_4$.

Theorem 7.1 ([17, Theorem B])**.** *For each odd prime p there exists a simply connected finite complex $K(p)$ whose p-localization is an H-space and with*

$$H^*(K(p); \mathbf{F}_p) \cong E[x_3, x_{2p+1}] \otimes P[x_{2p+2}]/(x_{2p+2})^p,$$
$$P^1 x_3 = x_{2p+1},\ \beta x_{2p+1} = x_{2p+2}.$$

It turns out that the p-completed homotopy type of the space $K(p)$ is determined by its mod p cohomology as an algebra over the Steenrod algebra. This is proved by means of classical homotopy theory methods and we are sure that it is known to J. Harper and perhaps other people. We will provide a proof for the sake of completeness.

For this aim we sketch the construction of this p-completed homotopy type by G.E. Cooke and L. Smith [10]. They first introduce the stable two-stage Postnikov system

$$\begin{tikzcd}
A(p) \arrow[rr] \arrow[d] & & * \arrow[d] \\
K(\mathbf{Z}/p, 2p^2+1) \arrow[rr, "\Psi"] & & K(\mathbf{Z}/p, 2p^2+2p-1) \times K(\mathbf{Z}, 2p^2+2p+1)
\end{tikzcd}$$

where $\Psi = (P^1\iota, \beta P^1 \beta\iota)$ (Notice that $\beta P^1 \beta \iota \in H^{2p^2+2p+1}(\mathbf{Z}/p, 2p^2+1; \mathbf{Z}/p)$ is a mod p reduction of an integral class.) This is constructed in such a way that the composition

$$K(\mathbf{Z}, 3) \xrightarrow{P^p P^1 \iota} K(\mathbf{Z}/p, 2p^2+1) \xrightarrow{\Psi} K(\mathbf{Z}/p, 2p^2+2p-1) \times K(\mathbf{Z}, 2p^2+2p+1)$$

is null-homotopic and therefore the first map lifts (non-uniquely) to φ_p:

$$\begin{tikzcd} & A(p) \arrow[d] \\ K(\mathbf{Z},3) \arrow[ur, "\varphi_p"] \arrow[r, "P^pP^1\iota"'] & K(\mathbf{Z}/p, 2p^2+1) \end{tikzcd}.$$

Now define $Y(p)$ as the homotopy fibre

$$\begin{tikzcd} Y(p) \arrow[r] \arrow[d, "q_p"'] & * \arrow[d] \\ K(\mathbf{Z},3) \arrow[r, "\varphi_p"'] & A(p) \end{tikzcd}$$

of the map φ_p.

The cohomology of the two-stage Postnikov system $A(p)$ is computed from results of [30]:

$$H^*(A(p);\mathbf{F}_p) \cong E[\iota] \otimes P[\beta\iota, P^1\beta\iota] \otimes H$$

where ι is restricted from the fundamental class of $K(\mathbf{Z}/p, 2p^2+1)$ and, in particular, $\deg \iota = 2p^2+1$ and H is a Hopf algebra that is $(2p^2+4p-2)$-connected. Now, $\varphi_p^*(\iota) = P^pP^1\iota_3$, $\varphi_p^*(\beta\iota) = \beta P^pP^1\iota_3$ and $\varphi_p^*(P^1\beta\iota) = P^1\beta P^pP^1\iota_3 = (\beta P^1\iota_3)^p$, from which it is computed the cohomology of $Y(p)$ in low dimensions:

Proposition 7.2 ([10, 1.2])**.** *In dimensions less than or equal to $2p^2+2p+2$ the mod p cohomology of $Y(p)$ coincides with $E[u, P^1u] \otimes P[\beta P^1u]/(\beta P^1u)^p$ where $u = q_p^*\iota_3$.*

It then follows that stage $2p^2+2p+2$ in the homology decomposition tower [18, Chap. 8] [5] for $Y(p)$ is a space $K(p) = Y(p)^{2p^2+2p+2}$ with

$$H^*(K(p);\mathbf{F}_p) \cong E[u, P^1u] \otimes P[\beta P^1u]/(\beta P^1u)^p\,.$$

Dually, the Postnikov tower for $K(p)$ has the form

$$\begin{tikzcd} & \vdots \arrow[d] & \\ & K(p)_{2p^2+2p+3} \arrow[d] & \\ K(p) \arrow[ur] \arrow[r] \arrow[dr] & K(p)_{2p^2+2p+2} \arrow[r] \arrow[d] & K(\pi_{2p^2+2p+3}(K(p)), 2p^2+2p+4) \\ & Y(p) \arrow[r] & K(\pi_{2p^2+2p+2}(K(p)), 2p^2+2p+3) \end{tikzcd}$$

because $K(p)_{2p^2+2p+1} = Y(p)^{2p^2+2p+2}_{2p^2+2p+1} = Y(p)_{2p^2+2p+1} = Y(p)$ appears at stage $2p^2+2p+1$.

We now come to our main result of this section which is homotopy uniqueness at the prime p for $K(p)^\wedge_p$.

Theorem 7.3. *Let X be a p-complete space with $H^*(X;\mathbf{F}_p) \cong H^*(K(p);\mathbf{F}_p)$ as algebras over the Steenrod algebra. Then $X \simeq K(p)^\wedge_p$.*

Proof. Assume that X is a p-complete space with

$$H^*(X;\mathbf{F}_p) \cong E[x_3, x_{2p+1}] \otimes P[x_{2p+2}]/(x_{2p+2})^p$$

satisfying $P^1 x_3 = x_{2p+1}$ and $\beta x_{2p+1} = x_{2p+2}$. Notice that X is 2-connected and since the mod p cohomology is finite dimensional, each $\pi_i(X)$ and $H^i(X;\mathbf{Z}_p)$ is a finitely generated $\mathbf{Z}_p$-module.

The Bockstein spectral sequence with p-adic coefficients applies. It collapses at

$$B_\infty \cong B_2 \cong E[x_3, x_{2p+1}(x_{2p+2})^{p-1}]$$

and shows that the top integral class is $H_{2p^2+2p+2}(X;\mathbf{Z}_p) \cong H^{2p^2+2p+2}(X;\mathbf{Z}_p) \cong \mathbf{Z}_p$. (The cohomological dimension $\mathrm{cd}_{\mathbf{Z}_p}(X) = 2p^2+2p+2$.) In particular, $H^{>2p^2+2p+2}(X;M) = 0$ for any $\mathbf{Z}_p$-module M and thus $[X, K(p)^\wedge_p] = [X, Y(p)^\wedge_p]$ by obstruction theory.

We now use the p-completed version of the Cooke-Smith construction of $K(p)$. Let

$$X \xrightarrow{x_3} K(\mathbf{Z}_p, 3)$$

represent a generator of $H^3(X;\mathbf{Z}_p) \cong \mathbf{Z}_p$.

Claim 7.4. $[X, A(p)^\wedge_p] = \{*\}$.

Proof. This follows from the exact sequence of sets

$$0 = H^{2p^2+2p-2}(X;\mathbf{F}_p) \times H^{2p^2+2p}(X;\mathbf{F}_p) \to [X, A(p)^\wedge_p] \to H^{2p^2+1}(X;\mathbf{F}_p) = 0$$

obtained by mapping X into the fibration $K(\mathbf{Z}/p, 2p^2+2p-2) \times K(\mathbf{Z}_p, 2p^2+2p) \to A(p)^\wedge_p \to K(\mathbf{Z}/p, 2p^2+1)$. $\square$

Since $\varphi_p x_3$ is null-homotopic by Claim 7.4, $x_3 \colon X \to K(\mathbf{Z}_p, 3)$ lifts to a map

$$f \colon X \to Y(p)^\wedge_p$$

that satisfies $f^*(u) = x_3$. Hence f^* is an isomorphisms in dimensions $\leq 2p^2+2p+2$ and the corresponding map $X \to K(p)^\wedge_p$ an isomorphism on mod p cohomology. $\square$

In the special case where $p = 3$, we note that Harper's molecule $K(3)^\wedge_3$ and the exotic homogeneous space $\mathrm{F}_4/\mathrm{DI}_2$ have isomorphic mod 3 cohomology algebras over the Steenrod algebra. Thus we may conclude that these two spaces are homotopy equivalent.

Proposition 7.5. $\mathrm{F}_4/\mathrm{DI}_2$ *and* $K(3)^\wedge_3$ *are homotopy equivalent* 3*-complete spaces.*

We do not know if $K(p)$ for $p > 3$ is an exotic homogeneous space with respect to some monomorphism of p-compact groups.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain

E-mail address: broto@mat.uab.es

Matematisk Institut, Universitetsparken 5, DK-2100 København Ø, Denmark

E-mail address: moller@math.ku.dk