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Fundamental Groups of Section Spaces

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1. Introduction

There are two standard ways, both used with success in the literature, of decomposing a mapping space into a sequence of fibrations: One is induced by a cell decomposition of the source and the other by a Moore–Postnikov factorization of the target space. We shall here apply the latter method to compute the first interesting homotopy group, i.e. the first one not given by elementary obstruction theory, of a general section space.

Throughout this note, let $F \xrightarrow{i} Y \xrightarrow{p} B$ be a fibration of connected spaces, X a connected CW complex, and $u : X \to Y$ a continuous map. Put $u_1 = pu$ and write $F_u(X; Y, B)$ for the space of all lifts of u_1 and $F_u^0(X; Y, B)$ for the component of $F_u(X; Y, B)$ that contains u. These section spaces are considered as having u as base point. In order to assure continuity of evaluation maps we shall work in the category of compactly generated spaces; so for example all mapping spaces are equipped with the compactly generated topology associated to the compact-open topology.

2. The case of a simple fiber

In this section we shall assume that F is a simple space and that there exist integers $1 \leq k < n$ such that $\pi_i(F) = 0$ for $0 \leq i < k$ and $k < i < n$. Put $d = \dim X$.

Elementary obstruction theory shows that if $d < n$, then

$$\pi_i F_u(X; Y, B) = H^{k-i}(x; u_1^* \pi_k(\mathcal{F}))$$

for $0 \leq i < n - d$. We shall here go one step futher and compute $\pi_{n-d} F_u(X; Y, B)$.

Since F is simple, $\pi_*(F)$ is a $\pi_1(B)$-module. Let $\varphi_n : \pi_1(B) \to \to \operatorname{Aut} \pi_n(F)$ be this local coefficient system and

$$L : L(\pi_n(F), n+1; \varphi_n) \rightleftarrows K(\pi_1(B), 1) =: D$$

the classifying sectioned fibration for $(n+1)$-dimensional cohomology with coefficients in φ. There exists a Postnikov factorization [8]

$$\begin{array}{ccccc} & & Y & & \\ & & \downarrow q & & \\ K(\pi_n(F), n) & \longrightarrow & Y_{n+1} & \longrightarrow & \overline{P}L \\ & & \downarrow p_{n+1} & & \downarrow \\ K(\pi_k(F), k) & \longrightarrow & Y_{k+1} & \xrightarrow{k} & L \\ & & \downarrow p_{k+1} & & \downarrow \\ & & B & \longrightarrow & D \end{array}$$

where the upper square is a pullback; $\overline{P}L \to L$ is the path space fibration over and under D [8]; $B \to D$ is some map inducing the identity on π_1. Let

$$\underline{k} : F_u(X; Y_{k+1}, B) \to F_{ku}(X; L, D)$$

by the map defined by composition with the k-invariant k.

Lemma 1. *There are weak homotopy equivalences*

$$\Phi_X : F_u(X; Y_{k+1}, B) \to \prod_{\alpha+\beta=k} K(H^\alpha(X; u_1^* \pi_k(\mathcal{F})), \beta)$$

$$\Phi_X : F_{ku}(X; L, D) \to \prod_{\alpha+\beta=n+1} K(H^\alpha(X; u_1^* \pi_n(\mathcal{F})), \beta).$$

This was proved in [11]; see also [4], [12].

Put $H^d := H^d(X; u_1^*\pi_n(\mathcal{F}))$. Let

$$[pr_{n+1-d} \circ \Phi_X \circ \underline{k}] \in H^{n+1-d}(F_u^0(X; Y_{k+1}, B); H^d)$$

be the homotopy class of $\Phi_X \underline{k}$ followed by the projection onto the factor $K(H^d, n+1-d)$.

Apply the composite homomorphism

$$\begin{aligned} &H^{n+1-d}(F_u^0(X; Y_{k+1}, B); H^d) \to \\ &\xrightarrow{(\Phi_X^*)^{-1}} H^{n+1-d}\left(\prod_{i=1}^{k} K(H^{k-i}(X; u_1^*\pi_k(\mathcal{F})), i); H^d\right) \to \\ &\xrightarrow{i_{n+1-d}^* \oplus i_{n-d}^*} H^{n+1-d}(H^{k-n+d-1}(X; u_1^*\pi_k(\mathcal{F})), n+1-d; H^d) \oplus \\ &\qquad \oplus H^{n+1-d}(K^{k-n+d}(X; u_1^*\pi_k(\mathcal{F})), n-d; H^d), \end{aligned}$$

where i_{n+1-d} and i_{n-d} are inclusions, to $[pr_i \circ \Phi_X \circ \underline{k}]$ and call the image $(\partial_u, \varepsilon_u)$. Note that

$$\partial_u \in \operatorname{Hom}(H^{k-n+d-1}(X; u_1^*\pi_k(\mathcal{F})), H^d(X; u_1^*\pi_n(\mathcal{F})))$$

and that

$$\varepsilon_u \in H^{n+1-d}(H^{k-n+d}(X; u_1^*\pi_k(\mathcal{F})), n-d; H^d(X; u_1^*\pi_n(\mathcal{F})))$$

classifies central extensions if $n-d=1$ and abelian extensions if $n-d>1$ of $H^{k-n+d}(X; u_1^*\pi_n(\mathcal{F}))$ by $H^d(X; u_1^*\pi_n(\mathcal{F}))$; see [9].

Theorem 2. *Suppose that F is simple and that $d < n$. There exists an $(n-d)$-connected map*

$$F_u(X; Y, B) \to \prod_{i=0}^{n-d} K(H^{k-i}(X; u_1^*\pi_k(\mathcal{F})), i)$$

and an exact sequence

$$\begin{aligned} H^{k-n+d-1}(X; u_1^*\pi_k(\mathcal{F})) \xrightarrow{\partial_u} H^d(X; u_1^*\pi_n(\mathcal{F})) \to \pi_{n-d} F_u(X; Y, B) \to \\ \to H^{k-n+d}(X; u_1^*\pi_k(\mathcal{F})) \to 0 \end{aligned}$$

involving ∂_u. The extension

$$0 \to \operatorname{Cokern} \partial_u \to \pi_{n-d} F_u(X; Y, B) \to H^{k-n+d}(X; u_1^*\pi_k(\mathcal{F})) \to 0$$

is central if $n-d=1$ and abelian if $n-d>1$ and classified by ε_u (or rather the image of ε_u under an obvious coefficient group homomorphism).

Proof. Since $q: F_u(X;Y,B) \to F_u(X;Y_{n+1},B)$ is $(n+1-d)$-connected we may assume that $Y = Y_{n+1}$. Then $F_u(X;Y,B)$ is the pullback of the diagram

$$\begin{array}{ccc} & & PF_{ku}(X;L,D) \\ & & \downarrow \\ F_u(X;Y_{k+1},B) & \xrightarrow{\underline{k}} & F_{ku}(X;L,D) \end{array}$$

where P is the path space functor based at ku. The $(n-d)$-dimensional homotopy group is, because of the structure of $F_{ku}(X;L,D)$ as a product of Eilenberg–MacLane spaces, equal to π_{n-d} of the pullback of

$$\begin{array}{ccc} & & PK(H^d, n+1-d) \\ & & \downarrow \\ F_u(X;Y_{k+1},B) & \xrightarrow{\underline{k}} & F_{ku}(X;L,D) \end{array}$$

and it is well known that this homotopy group is determined as asserted in the theorem. In particular we see that the extension is central because the path space fibration to the right is orientable. ∎

More surprising than the actual statement of Theorem 2 is perhaps the fact that ∂_u and ε_u are explicitly computable entities given sufficient information on the k-invariant. I shall illustrate this with a few examples thus giving a unified approach to a number of short exact sequences occuring in the recent literature [1], [5], [7].

The computations of the examples are based on two observations:

A ([11], Remark 3.2). The splitting Φ_X of Lemma 1 of $F_{ku} := F_{ku}(X;Y,B)$ is determined [11] by an isomorphism (also denoted Φ_X)

$$\Phi_X : H^{n+1}(? \times X; pr_2^* u_1^* \pi_n(\mathcal{F})) \to \bigoplus_{i=0}^{n+1} H^i(?; H^{n+1-i}(X; pr_1^* \pi_n(\mathcal{F})))$$

of contravariant functors. The vertical homotopy class of the adjoint

$$K : F_u^0(X;Y,B) \times X \to L$$

of $\underline{k} : F_u^0(X;Y,B) \to F_{ku}$ is a cohomology class

$$[K] \in H^{n+1}(F_u^0(X;Y,B) \times X; pr_2^* u_1^* \pi_n(\mathcal{F}))$$

and $[pr_{n+1-d} \circ \Phi_X \circ \underline{k}] = pr_{n+1-d}\, \Phi_X K$.

B. For any space Z (e.g. $Z = F_u^0(X;Y,B)$) the following diagram commutes

$$\begin{array}{ccc} & & H^{n+1}(Z \times X; pr_2^* u_1^* \pi_n(\mathcal{F})) \\ & \otimes \nearrow & \\ H^{n+1-d}(Z) \otimes H^d(X; u_1^* \pi_n(\mathcal{F})) & & \downarrow pr_{n+1-d} \circ \Phi_X \\ & \searrow & \\ & & H^{n+1-d}(Z; H^d(X; u_1^* \pi_n(\mathcal{F})) \end{array}$$

where $\otimes$ is exterior cross product, as in the Künneth sequence, and the unmarked arrow is the first homomorphism of the Universal Coefficient Theorem ([9], Exercise 2, p. 172).

Example 3. Let $Y = P(V) \to B$ be the projective bundle associated to a complex vector bundle of dimension $n+1$ and suppose $d = \dim X \le 2n$. In the Postnikov factorization $P(V)_3 = K(\mathbb{Z},2) \times B$ and ([10], Lemma 2.1)

$$k : K(\mathbb{Z},2) \times B \to K(\mathbb{Z},2n+2)$$

is the Euler class $k = c_{n+1}(\overline{\lambda} \otimes V)$ where λ is the tautological line bundle. According to Theorem 2 there is an exact sequence of the form

$$H^0(X) \xrightarrow{\partial_u} H^{2n}(X) \xrightarrow{\kappa} \pi_1 F_u(X; P(V), B) \to H^1(X) \to 0.$$

We shall now determine ∂_u and ε_u. Choose a free $\mathbb{Z}$-basis $\{x_j\}$ for $H^1(X)$ and let $\{x_j'\}$ be the dual basis for $H^1(H^1(X);\mathbb{Z}) = \mathrm{Hom}(H^1(X),\mathbb{Z})$. (We assume that $H^1(X)$ is finitely generated). Under the identification

$$F_u^0(X; P(V), B) = K(H^0(X),2) \times K(H^1(X),1)$$

the adjoint of $\underline{k} : F_u^0(X; P(V), B) \to K(\mathbb{Z},2n+2)^X$ is the map

$$K : K(H^0(X),2) \times K(H^1(X),1) \times X \to K(\mathbb{Z},2n+2)$$

given by ([10], Lemma 2.2), ([7], p. 236; [1], Lemma 3.2)

$$K = \sum_{i=0}^{n+1} 1 \otimes 1 \otimes c_i \cup \Big(\iota \otimes 1 \otimes 1 + \sum_j 1 \otimes x_j' \otimes x_j\Big)^{n+1-i}$$

where $c_i = c_i(u^*(\lambda)\otimes V)$ and $\iota \in H^2(H^0(X),2;\mathbb{Z})$ is the fundamental class. By observations A and B,

$$pr_2\Phi\underline{k} : K(H^0(X),2)\times K(H^1(X),1)\to K(H^{2n}(X),2)$$

is then represented by

$$pr_2\Phi K = \iota\otimes 1\otimes c_n - \sum_{s<t} 1\otimes x_s' x_t' \otimes 2x_s x_t c_{n-1}\,.$$

Consequently,

$$\partial_u(1) = c_n(u^*(\lambda)\otimes V)$$

and if $\overline{x}_s, \overline{x}_t \in \pi_1 F_u(X;P(V),B)$ project onto $x_s, x_t \in H^1(X)$, $s<t$, then the commutator

$$[\overline{x}_s,\overline{x}_t] = \kappa(2x_s x_t c_{n-1}(u^*(\lambda)\otimes V)).$$

This formula was also obtained by a different method by Crabb and Sutherland ([1], Theorem 2.12); see Hansen [5] and Larmore and Thomas [7] for the case $n=1$.

Example 4. Let V, B and X be as in Example 3. Form the associated lens space bundle

$$L^{2n+1}(m)\to L(v)\to b$$

relative to the action of a cyclic group $\mathbb{Z}/m$ of order $m>1$. Let $u : X\to$ $\to L(V)$ be a map. There is then a central extension

$$0\to H^{2n}(X)\xrightarrow{\kappa}\pi_1 F_u(X;L(V),B)\to H^0(X;\mathbb{Z}/m)\to 0$$

classified by the extra information that there exists an element $t\in pi_1 F_u(X;$ $L(V),B)$ which projects onto $1\in H^0(X;\mathbb{Z}/m)$ and satisfies

$$t^m = \kappa c_n(u^*(\lambda)\otimes V)$$

where λ is the tautological complex line bundle over $L(V)$.

Example 5. Let $V\to B$ be an $(n+2)$-dimensional real vector bundle and $Y = G_2^+(V)\to B$ the associated bundle of oriented 2-planes. If $n=2r+1$ is odd, then the only homotopy groups in dimensions $\le n$ of the fibre $G^+_{n+2,2} = O(n+2)/O(n)\times SO(2)$ are ([13], Theorem IV.10.13) $\pi_2 = \mathbb{Z}$

and $\pi_n = \mathbb{Z}/2$. Let $u : X \to G_2^+(V)$ be a map on an $(n-1)$-dimensional complex. Then

$$\pi_1 F_u(X; G_2^+(V), B) \cong H^1(X) \oplus H^{n-1}(X; \mathbb{Z}/2)/\partial_u$$

where $\partial_u = \sum_{i=0}^{r} (r+1-i) u^* w_2(\lambda)^{r-i} \cup w_{2i}(u^* V)$; again λ is the tautological 2-dimensional real vector bundle over $G_2^+(V)$.

I omit the detailed computations of the last two examples since they are very similar to those of Example 3.

As shown by these examples, and in fact already by Federer [2], $\pi_1 F_u$ need not be abelian even though F is simple. What is true, though, is that this fundamental group is given by a central extension if $d < n$. In the next section we shall see that non-central extensions may arise if F is no longer assumed to be simple.

3. The case of a non-simple fiber

We now assume that $\pi_i(F) = 0$ for $1 < i < n$ and we allow $\pi_1(F)$ to act non-trivially on $M := \pi_n(F)$. Then M is no longer a $\pi_1(B)$-module but $M \cong \pi_{n+1}(B, Y)$ is still a $\pi_1(Y)$-module and a $\pi_1(X)$-module through $u_* : \pi_1(X) \to \pi_1(Y)$. Also $\pi_1(F) \cong \pi_2(B, Y)$ is a $\pi_1(X)$-module in this way and we let $\pi_1(F)^u$ be the subgroup of $\pi_1(X)$-invariant elements.

This time the Moore–Postnikov factorization has the form [6], [8]

$$\begin{array}{ccccc} & & Y & & \\ & & \downarrow & & \\ K(M,n) & \to & Y_{n+1} & \longrightarrow & \overline{P}L(M, n+1; \varphi) \\ & & \downarrow & & \downarrow \\ K(\pi_1(F),1) & \to & Y_2 & \xrightarrow{k} & L(M, n+1; \varphi) =: L \\ & & \downarrow & & \hat{k} \downarrow\uparrow \check{k} \\ & & B & & K(\pi_1(Y),1) =: D \end{array}$$

where the square is a pullback and φ the $\pi_1(Y)$-module structure of M. Note that the k-invariant is no longer a map over $K(\pi_1(B), 1)$ but over $K(\pi_1(Y), 1)$ and that as a consequence, the induced map

$$\underline{k} : F_{u_2}(X; Y_2, B) \to F_{ku_2}(X; L) = L^X$$

goes into the full space $F_{ku}(X;L)$ and not just the subspace $F_{ku}(X;L,D)$ as in Section 2. It is however still possible to determine the source as well as the target space for $\underline{k}$.

Assume from now on that X is a finite complex of dimension $< n$.

Lemma 6 ([11], Theorem 6.3). *$F^0_{u_2}(X;Y_2,B)$ is an aspherical space with fundamental group $\pi_1 F^0_{u_2}(X;Y_2,B) = \pi_1(F)^u$.*

Consider the sectioned fibration

$$F^0_{ku}(X;L) \underset{\check{k}}{\overset{\hat{k}}{\rightleftarrows}} F^0_{\hat{k}ku}(X;D)$$

induced by $\hat{k}$ and $\check{k}$. By Gottlieb [3], or Lemma 6 above, the base space

$$F_{\hat{k}ku}(X;D) = K(\pi_1(Y)^u, 1)$$

is an aspherical space with fundamental group equal to the centralizer $\pi_1(Y)^u$ of $u_*(\pi_1(X))$ in $\pi_1(Y)$. For any $\xi \in \pi_1(Y)^u$, let $\varphi_*(\xi)$ be the coefficient group automorphism on cohomology induced from the automorphism $\varphi(\xi)$ of the local coefficient system $u_2^* M$. Form the associated classifying fibrations

$$L(H^{n+1-i}(X;u_2^*M), i; \varphi_*) \rightleftarrows K(\pi_1(Y)^u, 1), \quad 2 \le i \le n+1,$$

and their pullback $\bigoplus_{i=2}^{n+1} L(H^{n+1-i}(X;u_2^*M), i; \varphi_*)$ along the diagonal.

Lemma 7 ([11], Theorem 5.1). *As a space over and under $F^0_{\hat{k}ku}(X;D) = K(\pi_1(Y)^u, 1)$,*

$$F^0_{ku}(X;L) = \bigoplus_{i=2}^{n+1} L(H^{n+1-i}(X;u_2^*M), i; \varphi_*).$$

The composite map $pr_2 \circ \underline{k}$ of the diagram

$$\begin{array}{ccccc} & & F^0_{ku}(X;L) & \overset{pr_2}{\rightarrow} & L(H^{n-1}(X;u_2^*M), 2; \varphi_*) \\ & \underline{k} \nearrow & \downarrow\uparrow & & \downarrow\uparrow \\ K(\pi_1(F)^u, 1) = F^0_u(X;Y_2,B) & \longrightarrow & F^0_{\hat{k}ku}(X;D) & = & K(\pi_1(Y)^u, 1) \end{array}$$

is a lift of $i_* : K(\pi_1(F)^u, 1) \to K(\pi_1(Y)^u, 1)$ and hence its vertical homotopy class is an element

$$\varepsilon_u \in H^2_{\varphi_* i_*}(\pi_1(F)^u; H^{n-1}(X;u_2^*M))$$

of the second cohomology group of $\pi_1(F)^u$ with coefficients in the $\pi_1(F)^u$-module $H^{n-1}(X;u_2^*M)$.

Theorem 8. *Assume that $\pi_1(F) = 0$ for $1 < i < n$ and that X is a finite CW complex of dimension $\leq n-1$. Then there is a group extension*

$$0 \to H^{n-1}(X; u_2^* M) \to \pi_1 F_u^0(X; Y, B) \to \pi_1(F)^u \to 1$$

with operators

$$\pi_1(F)^u \xrightarrow{i_*} \pi_1(Y)^u \xrightarrow{\varphi_*} \operatorname{Aut} H^{n-1}(X; u_2^* M)$$

classified by $\varepsilon_u \in H^2_{\varphi_* i_*}(\pi_1(F)^u; H^{n-1}(X; u_2^* M))$.

With this theorem, proved formally as Theorem 2, it is easy to give examples of non-central extensions. On the other hand, Theorem 8 shows that the arising operators in all cases must be coefficient group automorphisms.

Example 9. Let $Y = P(V)$ be the projective bundle of a real vector bundle $V^{n+1} \to X$ and X a simply connected complex with top cohomology $H^{n-1}(X) = \mathbb{Z}/m$ cyclic of order $m \equiv 2 \bmod 4$. The extension

$$0 \to \mathbb{Z}/m \to \pi_1 F_u(X; P(V), B) \to \mathbb{Z}/2 \to 0$$

is central if n is odd but $\mathbb{Z}/m$ acts as multiplication by -1 if n is even. In either case, reduction mod2 induces an isomorphism

$$\rho_* : H^2(\mathbb{Z}/2; H^{n-1}(X)) \to H^2(\mathbb{Z}/2; H^{n-1}(X; \mathbb{Z}/2)) = H^{n-1}(X, \mathbb{Z}/2)$$

and, cf. Example 3,

$$\rho_*(\varepsilon_u) = u^* w_{n-1}(V) =: w_{n-1}.$$

Consequently

$$\pi_1 F_u(X; P(V), B) = \begin{cases} \mathbb{Z}/m \times \mathbb{Z}/2 & n \text{ odd}, w_{n-1} = 0 \\ \mathbb{Z}/2m & n \text{ odd}, w_{n-1} \neq 0 \\ \mathbb{Z}/m \rtimes \mathbb{Z}/2 & n \text{ even}, w_{n-1} = 0 \\ Q & n \text{ even}, w_{n-1} \neq 0 \end{cases}$$

where $Q = \langle a, b |\, a^m = 1,\ b^2 = a^k,\ b^{-1}ab = a^{-1} \rangle$, $2k = m$, is a sort of generalized quaternion group.

Remark 10. The short exact sequence of Theorem 8 also follows from the fibration

$$F_u(X; Y, Y_2) \to F_u(X; Y, B) \to F_u(X; Y_2, B)$$

where each component of the base is aspherical. This also shows that the higher homotopy groups of $F_u(X; Y, B)$ agree with those of $F_u(X; Y, Y_2)$ which are given in Theorem 2 through a range of dimensions.

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