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Realizability and tameness of fusion systems

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Abstract

A saturated fusion system over a finite p-group S is a category whose objects are the subgroups of S and whose morphisms are injective homomorphisms between the subgroups satisfying certain axioms. A fusion system over S is realized by a finite group G if S is a Sylow p-subgroup of G and morphisms in the category are those induced by conjugation in G. One recurrent question in this subject is to find criteria as to whether a given saturated fusion system is realizable or not. One main result in this paper is that a saturated fusion system is realizable if all of its components (in the sense of Aschbacher) are realizable. Another result is that all realizable fusion systems are tame: a finer condition on realizable fusion systems that involves describing automorphisms of a fusion system in terms of those of some group that realizes it. Stated in this way, these results depend on the classification of finite simple groups, but we also give more precise formulations whose proof is independent of the classification.

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INTRODUCTION

Let p be a prime. The *fusion system* of a finite group G over a Sylow p-subgroup S of G is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and whose morphisms are the homomorphisms between subgroups induced by conjugation in G, thus encoding G-conjugacy relations among

subgroups and elements of S. With this as starting point and also motivated by questions in representation theory, Puig defined the concept of abstract fusion systems (see [41] and Definition 1.1) and showed that they behave in many ways like finite groups.

By analogy with finite groups, a *component* C of a fusion system $\mathcal{F}$ is a subnormal fusion subsystem that is quasi-simple (i.e., $O^p(C) = C$ and $C/Z(C)$ is simple). The basic properties of components were shown by Aschbacher [4, Theorem 6] (see also Lemma 4.1 below).

A fusion system $\mathcal{F}$ over a finite p-group S is *realized* by a finite group G if $S \in \mathrm{Syl}_p(G)$ and $\mathcal{F} \cong \mathcal{F}_S(G)$, and is *realizable* if it is realized by some finite group. One of our main theorems is the following.

Theorem A. *Let p be a prime, let $\mathcal{F}$ be a saturated fusion system over a finite p-group, and let $\mathcal{E} \trianglelefteq \mathcal{F}$ be a normal fusion subsystem that contains all components of $\mathcal{F}$. If $\mathcal{E}$ is realizable, then $\mathcal{F}$ is also realizable.*

The following is an immediate corollary to Theorem A.

Corollary B. *Let p be a prime, and let $\mathcal{F}$ be a saturated fusion system over a finite p-group. If all components of $\mathcal{F}$ are realizable, then $\mathcal{F}$ is realizable.*

Corollary B is just the special case of Theorem A where $\mathcal{E}$ is the generalized Fitting subsystem of $\mathcal{F}$: the central product of the components of $\mathcal{F}$ and $O_p(\mathcal{F})$. Note, however, that a fusion system can be realizable even when some of its components are not.

For each component C of $\mathcal{F}$, $C/Z(C)$ is simple, and is a composition factor of $\mathcal{F}$ (see [5, § II.10]). Hence, one consequence of Corollary B is that $\mathcal{F}$ is realizable if all of its composition factors are realizable. However, the converse of this is not true either: $\mathcal{F}$ can be realizable without all of its composition factors being realizable.

In order to prove Theorem A, we need to work with linking systems and tameness. The concept of *linking systems* associated to fusion systems was first proposed by Benson in [8] and in unpublished notes, and was developed in detail by Broto, Levi, and Oliver [11]. See Definition 1.7 for precise definitions. This was originally motivated by questions involving classifying spaces of fusion systems and of the finite groups that they realize, but also turns out to be important when studying many of the purely algebraic properties of fusion systems.

A fusion system $\mathcal{F}$ is *tamely realized* by G if it is realized by G, and in addition, the natural homomorphism from $\mathrm{Out}(G)$ to $\mathrm{Out}(\mathcal{L}^c_S(G))$ is split surjective (Definitions 2.8 and 2.9). Here, $\mathcal{L}^c_S(G)$ is the linking system associated to G and to $\mathcal{F}$. We say that $\mathcal{F}$ is *tame* if it is tamely realized by some finite group.

Tameness was originally defined in [2, §2], motivated by questions of realizability and extensions of fusion systems, and that is how it is used here in the proof of Theorem A. In this way, it also plays a role in Aschbacher's program for classifying simple fusion systems over 2-groups and reproving certain parts of the classification of finite simple groups. See [6, §2.4] for more detail.

Tameness can also be interpreted topologically. For a finite group G, let $BG^\wedge_p$ be the classifying space of G completed at p in the sense of Bousfield and Kan, and let $\mathrm{Out}(BG^\wedge_p)$ be the set of homotopy classes of self-homotopy equivalences of $BG^\wedge_p$. Then for $S \in \mathrm{Syl}_p(G)$, the fusion system $\mathcal{F}_S(G)$ is tamely realized by G if and only if the natural map from $\mathrm{Out}(G)$ to $\mathrm{Out}(BG^\wedge_p)$ is split surjective. We refer to [10, Theorem B], [11, Lemma 8.2], and [2, Lemma 1.14] for the proof that $\mathrm{Out}(\mathcal{L}^c_S(G)) \cong \mathrm{Out}(BG^\wedge_p)$.

We can now state our second main theorem.

Theorem C. *For each prime p, every realizable fusion system over a finite p-group is tame.*

One of the original motivations for defining tameness in [2] was the hope that it might provide a new way to construct exotic fusion systems; that is, fusion systems not realized by any finite group. By [2, Theorem B], if $\mathcal{F}$ is a reduced fusion system that is not tame, then there is an extension of $\mathcal{F}$ whose reduction is isomorphic to $\mathcal{F}$ and is exotic. However, Theorem C tells us that this procedure does not give us any new exotic examples, since if $\mathcal{F}$ is not tame, then it is itself exotic.

A saturated fusion system $\mathcal{F}$ is *reduced* if $O_p(\mathcal{F}) = 1$ and $O^p(\mathcal{F}) = \mathcal{F} = O^{p'}(\mathcal{F})$ (see Definitions 1.3 and 1.14). The *reduction* $\mathfrak{red}(\mathcal{F})$ of an arbitrary saturated fusion system $\mathcal{F}$ is the fusion system obtained by taking $C_{\mathcal{F}}(O_p(\mathcal{F}))/Z(O_p(\mathcal{F}))$, and then alternately taking $O^p(-)$ or $O^{p'}(-)$ until the sequence becomes constant. By [2, Theorem A], $\mathcal{F}$ is tame if $\mathfrak{red}(\mathcal{F})$ is tame. So, one immediate consequence of Theorem C is as follows.

Corollary D. *If $\mathcal{F}$ is a saturated fusion system over a finite p-group S, and $\mathfrak{red}(\mathcal{F})$ is realizable, then $\mathcal{F}$ is also realizable.*

The proofs of Theorems A and C as formulated above, as well as those of Corollaries B and D, require the classification of finite simple groups. But they will be reformulated in Section 5 in a way so as to be independent of the classification. Our main theorem there, Theorem 5.4, is independent of the classification and includes Theorems A and C as special cases (the latter is reformulated as Theorem 5.6).

The first two sections of the paper contain mostly background material: some basic definitions and properties of fusion and linking systems are in Section 1, and those of automorphism groups and tameness in Section 2. We then deal with products in Section 3 and components of fusion systems in Section 4. Theorems A and C, as well as some other applications, are shown in Section 5, as Theorems 5.4 and 5.6.

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Notation. The notation used in this paper is mostly standard, with a few exceptions. Composition of functions and functors is always from right to left. Also, C_n denotes a (multiplicative) cyclic group of order n. When G is a (multiplicative) group, $1 \in G$ always denotes its identity element.

When $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, then for objects c, c' in $\mathcal{C}$, we let $f_{c,c'}$ be the induced map from $\mathrm{Mor}_{\mathcal{C}}(c, c')$ to $\mathrm{Mor}_{\mathcal{D}}(f(c), f(c'))$, and also set $f_c = f_{c,c}$ for short.

When G is a group, we indicate conjugation by setting ${}^g x = c_g(x) = gxg^{-1}$ and ${}^g H = c_g(H) = gHg^{-1}$ for $g, x \in G$ and $H \leqslant G$. Also, for $P, Q \leqslant G$, we let $\mathrm{Hom}_G(P, Q)$ be the set of (injective) homomorphisms from P to Q induced by conjugation in G, and set $\mathrm{Aut}_G(P) = \mathrm{Hom}_G(P, P)$.

Throughout the paper, p will always be a fixed prime.

1 | FUSION SYSTEMS AND LINKING SYSTEMS

This is a background section intended to provide the reader with the necessary basic definitions and properties of fusion and linking systems that will be used throughout the paper. Fusion systems and saturation were originally introduced by Puig, first in unpublished notes, and then in [41]. Abstract linking systems were defined in [11]. As general references for the subject, we refer to [5] and [15].

1.1 | Fusion systems

For a prime p, a *fusion system* over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are injective homomorphisms between subgroups such that for each $P, Q \leqslant S$:

- $\mathrm{Hom}_{\mathcal{F}}(P,Q) \supseteq \mathrm{Hom}_S(P,Q)$; and
- for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P),P)$.

Here, $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ denotes the set of morphisms in $\mathcal{F}$ from P to Q. We also write $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ for the set of isomorphisms, $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Iso}_{\mathcal{F}}(P,P)$, and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$. For $P \leqslant S$ and $g \in S$, we set

$$P^{\mathcal{F}} = \{\varphi(P) \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)\} \qquad \text{and} \qquad g^{\mathcal{F}} = \{\varphi(g) \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle g\rangle, S)\}$$

(the sets of subgroups and elements $\mathcal{F}$-*conjugate* to P and to g).

The following version of the definition of a saturated fusion system is the most convenient one to use here. (See Definitions I.2.2 and I.2.4 and Proposition I.2.5 in [5].)

Definition 1.1. Let $\mathcal{F}$ be a fusion system over a finite p-group S.

(a) A subgroup $P \leqslant S$ is *fully normalized (fully centralized)* in $\mathcal{F}$ if $|N_S(P)| \geqslant |N_S(Q)|$ ($|C_S(P)| \geqslant |C_S(Q)|$) for each $Q \in P^{\mathcal{F}}$.

(b) A subgroup $P \leqslant S$ is *fully automized* in $\mathcal{F}$ if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(c) A subgroup $P \leqslant S$ is *receptive* in $\mathcal{F}$ if each isomorphism $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q,P)$ in $\mathcal{F}$ extends to a morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$, where

$$N_\varphi = \{g \in N_S(Q) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P)\}.$$

(d) The fusion system $\mathcal{F}$ is *saturated* if it satisfies the following two conditions:
   (I) *(Sylow axiom)* each subgroup $P \leqslant S$ fully normalized in $\mathcal{F}$ is also fully automized and fully centralized; and
   (II) *(extension axiom)* each subgroup $P \leqslant S$ fully centralized in $\mathcal{F}$ is also receptive.

The above definition is motivated by fusion systems of finite groups. When G is a finite group and $S \in \mathrm{Syl}_p(G)$, the *p-fusion system* of G is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and where $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for each $P, Q \leqslant S$. For a proof that $\mathcal{F}_S(G)$ is saturated, see, for example, [5, Lemma I.1.2]. In general, a saturated fusion system $\mathcal{F}$ over a finite p-group S

will be called *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and will be called *exotic* otherwise.

The following lemma lists relations between some of these conditions that hold for *all* fusion systems, not just those that are saturated.

Lemma 1.2 [5, Lemma I.2.6]. *If $\mathcal{F}$ is a fusion system over a finite p-group S, then each receptive subgroup of S is fully centralized, and each subgroup that is fully automized and receptive is fully normalized.*

We next list some of the terminology used to describe certain subgroups in a fusion system.

Definition 1.3. Let $\mathcal{F}$ be a fusion system over a finite p-group S. For a subgroup $P \leqslant S$,

(a) P is *$\mathcal{F}$-centric* if $C_S(Q) \leqslant Q$ for each $Q \in P^{\mathcal{F}}$;
(b) P is *$\mathcal{F}$-radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$;
(c) P is *$\mathcal{F}$-quasi-centric* if for each $Q \in P^{\mathcal{F}}$ that is fully centralized in $\mathcal{F}$, the centralizer fusion system $C_{\mathcal{F}}(Q)$ (see Definition 1.5(b)) is the fusion system of the group $C_S(Q)$;
(d) P is *weakly closed* in $\mathcal{F}$ if $P^{\mathcal{F}} = \{P\}$;
(e) P is *strongly closed* in $\mathcal{F}$ if for each $x \in P$, $x^{\mathcal{F}} \subseteq P$;
(f) P is *normal* in $\mathcal{F}$ ($P \trianglelefteq \mathcal{F}$) if each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ (for $Q, R \leqslant S$) extends to a morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PR)$ such that $\bar{\varphi}(P) = P$; and
(g) P is *central* in $\mathcal{F}$ if each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ (for $Q, R \leqslant S$) extends to a morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PR)$ such that $\bar{\varphi}|_P = \mathrm{Id}_P$.

Let $\mathcal{F}^{cr} \subseteq \mathcal{F}^c \subseteq \mathcal{F}^q$ denote the sets of $\mathcal{F}$-centric $\mathcal{F}$-radical, $\mathcal{F}$-centric, and $\mathcal{F}$-quasi-centric subgroups of S, respectively, or (depending on the context) the full subcategories of $\mathcal{F}$ with those objects. Let $O_p(\mathcal{F}) \geqslant Z(\mathcal{F})$ denote the (unique) largest normal and central subgroups, respectively, in $\mathcal{F}$.

The following result is one of the versions of Alperin's fusion theorem for fusion systems.

Theorem 1.4. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms of subgroups that are $\mathcal{F}$-centric, $\mathcal{F}$-radical, and fully normalized in $\mathcal{F}$.*

Proof. This follows from [5, Theorem I.3.6] (the same statement but for $\mathcal{F}$-essential subgroups), together with [5, Proposition I.3.3(a)] (all $\mathcal{F}$-essential subgroups are $\mathcal{F}$-centric and $\mathcal{F}$-radical). Alternatively, the result as stated here is shown directly (without mention of essential subgroups) in [11, Theorem A.10]. □

Definition 1.5. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $Q \leqslant S$ be a subgroup.

(a) For each $K \leqslant \mathrm{Aut}(Q)$, set $N_S^K(Q) = \{x \in N_S(Q) \mid c_x \in K\}$, and let $N_{\mathcal{F}}^K(Q)$ be the fusion system over $N_S^K(Q)$ in which for $P, R \leqslant N_S^K(Q)$,

$$\mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P,R) = \big\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R) \,\big|\, \text{there is } \bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,RQ) \text{ such that } \bar{\varphi}|_P = \varphi,\ \bar{\varphi}(Q) = Q,\ \bar{\varphi}|_Q \in K\big\}.$$

(b) Set $N_{\mathcal{F}}(Q) = N_{\mathcal{F}}^{\mathrm{Aut}(Q)}(Q)$ and $C_{\mathcal{F}}(Q) = C_{\mathcal{F}}^{\{1\}}(Q)$.

If Q is fully normalized (fully centralized) in $\mathcal{F}$, then $N_{\mathcal{F}}(Q)$ ($C_{\mathcal{F}}(Q)$) is a saturated fusion system (see [11, Proposition A.6] or [5, Theorem I.5.5]). There is a similar condition (see [5, Theorem I.5.5]) that implies that $N_{\mathcal{F}}^{K}(Q)$ is saturated. Note that $Q \trianglelefteq \mathcal{F}$ if and only if $N_{\mathcal{F}}(Q) = \mathcal{F}$, and Q is central in $\mathcal{F}$ (i.e., $Q \leqslant Z(\mathcal{F})$) if and only if $C_{\mathcal{F}}(Q) = \mathcal{F}$.

Lemma 1.6. *Let G be a finite group with $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$.*

(a) *For each $Q \leqslant S$, Q is fully normalized (fully centralized) if and only if $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$ ($C_S(Q) \in \mathrm{Syl}_p(C_G(Q))$). If this holds, then $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_S(Q)}(N_G(Q))$ ($C_{\mathcal{F}}(Q) = \mathcal{F}_{C_S(Q)}(C_G(Q))$).*
(b) *In all cases, $O_p(G) \leqslant O_p(\mathcal{F})$ and $O_p(Z(G)) \leqslant Z(\mathcal{F})$.*

Proof. Point (a) is shown in [5, Proposition I.5.4]. In particular, when $Q = O_p(G)$, we have $N_{\mathcal{F}}(Q) = \mathcal{F}_S(G) = \mathcal{F}$ and hence $Q \leqslant O_p(\mathcal{F})$. Similarly, when $Q = O_p(Z(G))$, it says that $C_{\mathcal{F}}(Q) = \mathcal{F}_S(G) = \mathcal{F}$ and hence that $Q \leqslant Z(\mathcal{F})$. □

1.2 | Linking systems

Before recalling the definition of linking systems, we need to introduce more notation. If $P, Q \leqslant G$ are subgroups of a finite group G, the *transporter set* $T_G(P, Q)$ is defined by setting

$$T_G(P, Q) = \{g \in G \mid {}^gP \leqslant Q\}.$$

The *transporter category of G* is the category $\mathcal{T}(G)$ whose objects are the subgroups of G, and whose morphisms sets are the transporter sets:

$$\mathrm{Mor}_{\mathcal{T}(G)}(P, Q) = T_G(P, Q).$$

Composition in $\mathcal{T}(G)$ is given by multiplication in G. If $\mathcal{H}$ is a set of subgroups of G, then $\mathcal{T}_{\mathcal{H}}(G) \subseteq \mathcal{T}(G)$ denotes the full subcategory with object set $\mathcal{H}$.

The following definition of linking system taken from [5, Definition III.4.1].

Definition 1.7. Let $\mathcal{F}$ be a fusion system over a finite p-group S. A *linking system* associated to $\mathcal{F}$ is a triple $(\mathcal{L}, \delta, \pi)$ where $\mathcal{L}$ is a finite category, and δ and π are a pair of functors

$$\mathcal{T}_{\mathrm{Ob}(\mathcal{L})}(S) \xrightarrow{\ \delta\ } \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F},$$

which satisfy the following conditions.

(A1) $\mathrm{Ob}(\mathcal{L})$ is a set of subgroups of S closed under $\mathcal{F}$-conjugacy and overgroups, and contains $\mathcal{F}^{cr}$. Each object in $\mathcal{L}$ is isomorphic (in $\mathcal{L}$) to one that is fully centralized in $\mathcal{F}$.
(A2) δ is the identity on objects, and π is the inclusion on objects. For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ such that P is fully centralized in $\mathcal{F}$, $C_S(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ via δ_P and right composition,

and

$$\pi_{P,Q} : \mathrm{Mor}_{\mathcal{L}}(P,Q) \to \mathrm{Hom}_{\mathcal{F}}(P,Q)$$

is the orbit map for this action.

(B) For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ and each $g \in T_S(P,Q)$, $\pi_{P,Q}$ sends $\delta_{P,Q}(g) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to $c_g \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$.

(C) For all $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and all $g \in P$, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\psi} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(\psi)(g))} \\ P & \xrightarrow{\psi} & Q \end{array}$$

commutes in $\mathcal{L}$.

When the functors δ and π are understood, we refer directly to the category $\mathcal{L}$ as a linking system.

A *centric linking system* associated to $\mathcal{F}$ is a linking system $\mathcal{L}$ associated to $\mathcal{F}$ such that $\mathrm{Ob}(\mathcal{L}) = \mathcal{F}^c$.

Linking systems associated to a fusion system were originally motivated by centric linking systems of finite groups. For a finite group G, a p-subgroup $P \leqslant G$ is *p-centric in G* if $Z(P) \in \mathrm{Syl}_p(C_G(P))$; equivalently, if $C_G(P) = Z(P) \times O_{p'}(C_G(P))$. For $S \in \mathrm{Syl}_p(G)$, the *centric linking system of G* over S consists of the category $\mathcal{L}^c_S(G)$ whose objects are the subgroups of S that are p-centric in G and whose morphism sets are given by

$$\mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q) = T_G(P,Q)/O_{p'}(C_G(P)) \qquad (\text{all } P, Q \in \mathrm{Ob}(\mathcal{L}^c_S(G))),$$

together with functors $\mathcal{T}_{\mathrm{Ob}(\mathcal{L}^c_S(G))}(S) \xrightarrow{\delta} \mathcal{L}^c_S(G) \xrightarrow{\pi} \mathcal{F}_S(G)$ defined in the obvious way.

When G is a finite group and $S \in \mathrm{Syl}_p(G)$, then $P \leqslant S$ is $\mathcal{F}$-centric (see Definition 1.3) if and only if P is p-centric in G (see [10, Lemma A.5]). Moreover, $\mathcal{F}_S(G)$ is always saturated (see [5, Theorem I.2.3]), and $(\mathcal{L}^c_S(G), \delta, \pi)$ is a centric linking system associated to $\mathcal{F}_S(G)$.

Some of the basic properties of linking systems are listed in the next proposition.

Proposition 1.8. *Let $(\mathcal{L}, \delta, \pi)$ be a linking system associated to a saturated fusion system $\mathcal{F}$ over a finite p-group S. For each pair of subgroups $P \leqslant Q \leqslant S$ with $P, Q \in \mathrm{Ob}(\mathcal{L})$, set $\iota_{P,Q} = \delta_{P,Q}(1) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ (the* inclusion *in $\mathcal{L}$ of P into Q). Then*

(a) *δ is injective on all morphism sets; and*

(b) *all morphisms in $\mathcal{L}$ are monomorphisms and epimorphisms in the categorical sense.*

Conditions for the existence of restrictions and extensions of morphisms are as follows:

(c) *For every morphism $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, and every $P_0, Q_0 \in \mathrm{Ob}(\mathcal{L})$ such that $P_0 \leqslant P$, $Q_0 \leqslant Q$, and $\pi(\psi)(P_0) \leqslant Q_0$, there is a unique morphism $\psi|_{P_0,Q_0} \in \mathrm{Mor}_{\mathcal{L}}(P_0,Q_0)$ (the "restriction" of ψ) such that $\psi \circ \iota_{P_0,P} = \iota_{Q_0,Q} \circ \psi|_{P_0,Q_0}$.*

(d) *Let* $P, Q, \bar{P}, \bar{Q} \in \mathrm{Ob}(\mathcal{L})$ *and* $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ *be such that* $P \trianglelefteq \bar{P}$, $Q \leqslant \bar{Q}$, *and for each* $g \in \bar{P}$, *there is* $h \in \bar{Q}$ *such that* $\iota_{Q,\bar{Q}} \circ \psi \circ \delta_P(g) = \delta_{Q,\bar{Q}}(h) \circ \psi$. *Then there is a unique morphism* $\bar{\psi} \in \mathrm{Mor}_{\mathcal{L}}(\bar{P}, \bar{Q})$ *such that* $\bar{\psi}|_{P,Q} = \psi$.

Proof. See points (c), (f), (b), and (e), respectively, in [32, Proposition 4]. □

We note here the existence and uniqueness of linking systems shown by Chermak, Oliver, and Glauberman-Lynd. Two linking systems $(\mathcal{L}_1, \delta_1, \pi_1)$ and $(\mathcal{L}_2, \delta_2, \pi_2)$ associated to the same fusion system $\mathcal{F}$ are *isomorphic* if there is an isomorphism of categories $\rho : \mathcal{L}_1 \xrightarrow{\cong} \mathcal{L}_2$ such that $\rho \circ \delta_1 = \delta_2$ and $\pi_2 \circ \rho = \pi_1$.

Theorem 1.9 ([13, 18, 33]). *Let* $\mathcal{F}$ *be a saturated fusion system over a finite* p*-group* S, *and let* $\mathcal{H}$ *be a set of subgroups of* S *such that* $\mathcal{F}^{cr} \subseteq \mathcal{H} \subseteq \mathcal{F}^q$, *and such that* $\mathcal{H}$ *is closed under* $\mathcal{F}$*-conjugacy and overgroups. Then up to isomorphism, there is a unique linking system* $\mathcal{L}^{\mathcal{H}}$ *associated to* $\mathcal{F}$ *with object set* $\mathcal{H}$.

Proof. The existence and uniqueness of a centric linking system associated to $\mathcal{F}$ was shown by Chermak. See [13, Main theorem] and [33, Theorem A] for two versions of his original proof, and [18, Theorem 1.2] for the changes to the proof in [33] needed to make it independent of the classification of finite simple groups.

More generally, if $\mathcal{F}^{cr} \subseteq \mathcal{H} \subseteq \mathcal{F}^c$, the uniqueness of an $\mathcal{H}$-linking system follows by the same obstruction theory (shown to vanish in [33, Theorem 3.4] and [18, Theorem 1.1]) as that used in the centric case (by the same argument as in the proof of [11, Proposition 3.1]). For arbitrary $\mathcal{H} \subseteq \mathcal{F}^q$ containing $\mathcal{F}^{cr}$, the existence and uniqueness now follows from [5, Proposition III.4.8], applied with $\mathcal{H} \supseteq \mathcal{H} \cap \mathcal{F}^c$ in the role of $\widehat{\mathcal{H}} \supseteq \mathcal{H}$. □

1.3 | Normal fusion and linking subsystems

Let $\mathcal{F}$ be a fusion system over a finite p-group S. A *fusion subsystem* of $\mathcal{F}$ is a subcategory $\mathcal{E} \subseteq \mathcal{F}$ that is itself a fusion system over a subgroup $T \leqslant S$ (in particular, $\mathrm{Ob}(\mathcal{E})$ is the set of subgroups of T). We write $\mathcal{E} \leqslant \mathcal{F}$ when $\mathcal{E}$ is a fusion subsystem, and also sometimes say that $\mathcal{E} \leqslant \mathcal{F}$ is a pair of fusion systems over $T \leqslant S$.

Definition 1.10. Let $\mathcal{F}$ be a fusion system over a finite p-group S.

(a) Let R be another finite p-group and let $\alpha : S \longrightarrow R$ be an isomorphism. We denote by ${}^{\alpha}\mathcal{F}$ the fusion system over R with morphism sets

$$\mathrm{Hom}_{{}^{\alpha}\mathcal{F}}(P,Q) = \alpha \circ \mathrm{Hom}_{\mathcal{F}}(\alpha^{-1}(P), \alpha^{-1}(Q)) \circ \alpha^{-1}$$

for each pair of subgroups $P, Q \leqslant R$.

(b) Let $\mathcal{E}$ be another fusion system over a finite p-group T. We say that $\mathcal{E}$ and $\mathcal{F}$ are isomorphic fusion systems if there is an isomorphism $\alpha : S \longrightarrow T$ such that $\mathcal{E} = {}^{\alpha}\mathcal{F}$.

A more general concept of morphism between fusion systems is given in [5, Definition II.2.2]. Consider now the following definition from [5, Definition I.6.1].

Definition 1.11. Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S.

(a) A fusion subsystem $\mathcal{E} \leqslant \mathcal{F}$ over $T \trianglelefteq S$ is *weakly normal* if $\mathcal{E}$ is saturated, T is strongly closed in $\mathcal{F}$, and the following conditions hold:
- *(invariance condition)* ${}^{\alpha}\mathcal{E} = \mathcal{E}$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$, and
- *(Frattini condition)* for each $P \leqslant T$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,T)$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,T)$ such that $\varphi = \alpha\circ\varphi_0$.

(b) A fusion subsystem $\mathcal{E} \leqslant \mathcal{F}$ over $T \trianglelefteq S$ is *normal* ($\mathcal{E} \trianglelefteq \mathcal{F}$) if $\mathcal{E}$ is weakly normal in $\mathcal{F}$ and
- *(Extension condition)* each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[\bar{\alpha}, C_S(T)] \leqslant Z(T)$.

(c) A saturated fusion system $\mathcal{F}$ over a finite p-group S is *simple* if it contains no proper nontrivial normal fusion subsystem.

It will be convenient to say that "$\mathcal{E} \trianglelefteq \mathcal{F}$ is a normal pair of fusion systems over $T \trianglelefteq S$" to mean that $\mathcal{F}$ is a fusion system over S and $\mathcal{E} \trianglelefteq \mathcal{F}$ is a normal subsystem over T.

Note that if $\mathcal{F}$ is a saturated fusion system over a finite p-subgroup S, and $P \leqslant S$, then $P \trianglelefteq \mathcal{F}$ if and only if $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$ [4, (7.9)].

Note also that what are called "normal fusion subsystems" in [2, Definition 1.18] are what we are calling "weakly normal" subsystems here.

When $(\mathcal{L}, \delta, \pi)$ is a linking system associated to the fusion system $\mathcal{F}$ over S, and $\mathcal{F}_0 \leqslant \mathcal{F}$ is a fusion subsystem over $S_0 \leqslant S$, then a *linking subsystem* associated to $\mathcal{F}_0$ is a linking system $(\mathcal{L}_0, \delta_0, \pi_0)$ associated to $\mathcal{F}_0$, where $\mathcal{L}_0$ is a subcategory of $\mathcal{L}$ and

$$\mathcal{T}_{\mathrm{Ob}(\mathcal{L}_0)}(S_0) \xrightarrow{\ \delta_0\ } \mathcal{L}_0 \xrightarrow{\ \pi_0\ } \mathcal{F}_0$$

are the restrictions of δ and π. In this situation, we write $\mathcal{L}_0 \leqslant \mathcal{L}$, and sometimes say that $\mathcal{L}_0 \leqslant \mathcal{L}$ is a pair of linking systems. Note in particular the special case where $S_0 = S$ and $\mathcal{F}_0 = \mathcal{F}$ but $\mathrm{Ob}(\mathcal{L}_0) \subseteq \mathrm{Ob}(\mathcal{L})$: a pair of linking systems with possibly different object sets associated to the same fusion system.

Definition 1.12. Fix a pair of saturated fusion systems $\mathcal{E} \leqslant \mathcal{F}$ over finite p-groups $T \trianglelefteq S$ such that $\mathcal{E} \trianglelefteq \mathcal{F}$, and let $\mathcal{M} \leqslant \mathcal{L}$ be a pair of associated linking systems. Then, $\mathcal{M}$ *is normal in* $\mathcal{L}$ ($\mathcal{M} \trianglelefteq \mathcal{L}$) if:

(a) $\mathrm{Ob}(\mathcal{L}) = \{P \leqslant S \mid P \cap T \in \mathrm{Ob}(\mathcal{M})\}$, and
(b) for all $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$ and $\psi \in \mathrm{Mor}(\mathcal{M})$, $\gamma\psi\gamma^{-1} \in \mathrm{Mor}(\mathcal{M})$.

If $\mathcal{M} \trianglelefteq \mathcal{L}$, then we define $\mathcal{L}/\mathcal{M} = \mathrm{Aut}_{\mathcal{L}}(T)/\mathrm{Aut}_{\mathcal{M}}(T)$.

Notice that not every normal pair of fusion systems has an associated normal pair of linking systems.

Definition 1.12 differs from Definition 1.27 in [2] in that there is no "Frattini condition" in the definition we give here. We have omitted it because it follows from the Frattini condition for normal fusion subsystems, as shown in the next lemma.

Lemma 1.13. *If $\mathcal{M} \trianglelefteq \mathcal{L}$ is a normal pair of linking systems associated to fusion systems $\mathcal{E} \trianglelefteq \mathcal{F}$ over finite p-groups $T \trianglelefteq S$, then for all $P, Q \in \mathrm{Ob}(\mathcal{M})$ and all $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, there are morphisms $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$ and $\psi_0 \in \mathrm{Mor}_{\mathcal{M}}(\gamma(P), Q)$ such that $\psi = \psi_0\circ\gamma|_{P,\gamma(P)}$.*

Proof. Let $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ be as above, and assume first that P is fully centralized in $\mathcal{F}$. Then $\pi(\psi) \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, and by the Frattini condition on $\mathcal{E} \trianglelefteq \mathcal{F}$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi \in \mathrm{Hom}_{\mathcal{E}}(\alpha(P),Q)$ such that $\pi(\psi) = \varphi\circ\alpha|_{P,\alpha(P)}$. Choose $\widetilde{\varphi} \in \mathrm{Mor}_{\mathcal{M}}(\alpha(P),Q)$ and $\widetilde{\alpha} \in \mathrm{Aut}_{\mathcal{L}}(T)$ such that $\pi_{\mathcal{M}}(\widetilde{\varphi}) = \varphi$ and $\pi_{\mathcal{L}}(\widetilde{\alpha}) = \alpha$. By axiom (A2) for the linking system $\mathcal{L}$, there is $z \in C_S(P)$ such that $\widetilde{\varphi}\circ\widetilde{\alpha}|_{P,\alpha(P)} = \psi\circ\delta_P(z)|_{P,P}$. (Here, we take restrictions of morphisms in $\mathcal{M}$ in the sense of Proposition 1.8(c).) Set $\gamma = \widetilde{\alpha}\circ\delta_S(z)^{-1} \in \mathrm{Aut}_{\mathcal{L}}(T)$, and then $\psi = \widetilde{\varphi}\circ\gamma|_{P,\gamma(P)}$.

If P is not fully centralized in $\mathcal{F}$, then choose $P^* \in P^{\mathcal{F}}$ that is fully centralized, and fix $\omega \in \mathrm{Iso}_{\mathcal{L}}(P^*,P)$. Then $P^* \leqslant T$ since T is strongly closed in $\mathcal{F}$, and so, $P^* \in \mathrm{Ob}(\mathcal{M})$. We just showed that there are morphisms $\gamma_1,\gamma_2 \in \mathrm{Aut}_{\mathcal{L}}(T)$, $\psi_1 \in \mathrm{Mor}_{\mathcal{M}}(\gamma_1(P^*),Q)$, and $\psi_2 \in \mathrm{Mor}_{\mathcal{M}}(\gamma_2(P^*),P)$ such that

$$\psi\circ\omega = \psi_1\circ\gamma_1|_{P^*,\gamma_1(P^*)} \qquad \text{and} \qquad \omega = \psi_2\circ\gamma_2|_{P^*,\gamma_2(P^*)}.$$

Also, ψ_2 is an isomorphism since ω is an isomorphism. Set $\gamma = \gamma_1\gamma_2^{-1} \in \mathrm{Aut}_{\mathcal{L}}(T)$. Then

$$\psi = (\psi\circ\omega)\circ\omega^{-1} = \psi_1\circ\gamma|_{\gamma_2(P^*),\gamma_1(P^*)}\circ\psi_2^{-1} = \psi_1\circ(\gamma\psi_2\gamma^{-1})^{-1}\circ\gamma|_{P,\gamma(P)},$$

where $\gamma\psi_2\gamma^{-1} \in \mathrm{Iso}_{\mathcal{M}}(\gamma_1(P^*),\gamma(P))$ by Definition 1.12(b). □

1.4 | Fusion subsystems of p-power index and index prime to p

We recall some more definitions.

Definition 1.14. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.

(a) Set $\mathfrak{foc}(\mathcal{F}) = \{g^{-1}h \mid g,h \in S,\ h \in g^{\mathcal{F}}\} = \left\{g^{-1}\alpha(g) \mid g \in P \leqslant S,\ \alpha \in \mathrm{Aut}_{\mathcal{F}}(P)\right\}$ (the *focal subgroup* of $\mathcal{F}$).

(b) Set $\mathfrak{hyp}(\mathcal{F}) = \left\{g^{-1}\alpha(g) \mid g \in P \leqslant S,\ \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\right\}$ (the *hyperfocal subgroup* of $\mathcal{F}$).

(c) A saturated fusion subsystem $\mathcal{E} \leqslant \mathcal{F}$ over $T \leqslant S$ has *p-power index* if $T \geqslant \mathfrak{hyp}(\mathcal{F})$, and $\mathrm{Aut}_{\mathcal{E}}(P) \geqslant O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leqslant T$. The smallest normal subsystem of p-power index is denoted as $O^p(\mathcal{F})$.

(d) A saturated fusion subsystem $\mathcal{E} \leqslant \mathcal{F}$ over $T \leqslant S$ has *index prime to p* if $T = S$ and $\mathrm{Aut}_{\mathcal{E}}(P) \geqslant O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leqslant T$. The smallest normal subsystem of index prime to p is denoted as $O^{p'}(\mathcal{F})$.

For the existence of the minimal subsystems $O^p(\mathcal{F})$ and $O^{p'}(\mathcal{F})$, see, for example, Theorems I.7.4 and I.7.7 in [5].

Lemma 1.15.

(a) *If G is a finite group with $S \in \mathrm{Syl}_p(G)$, then $\mathfrak{foc}(\mathcal{F}_S(G)) = S \cap [G,G]$ and $\mathfrak{hyp}(\mathcal{F}_S(G)) = S \cap O^p(G)$.*

(b) *If $\mathcal{F}$ is a saturated fusion system over a finite p-group S, then $O^p(\mathcal{F})$ and $O^{p'}(\mathcal{F})$ are fusion subsystems over $\mathfrak{hyp}(\mathcal{F})$ and S, respectively, and are both normal in $\mathcal{F}$. Also,*

$$O^p(\mathcal{F}) = \mathcal{F} \iff \mathfrak{hyp}(\mathcal{F}) = S \iff \mathfrak{foc}(\mathcal{F}) = S.$$

Proof. The first statement in (a) is the focal subgroup theorem (see [20, Theorem 7.3.4]), and the second is Puig's hyperfocal subgroup theorem [40, §1.1].

Point (b) is due to Puig, and is also shown in Theorems I.7.4 and I.7.7 and Corollary I.7.5 in [5]. □

Lemma 1.16. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, fix $Q \leqslant S$, and let $K \leqslant \mathrm{Aut}(Q)$ be a subgroup of p-power order. Assume that Q is fully centralized in $\mathcal{F}$ and that $\mathrm{Aut}_S^K(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^K(Q))$. Then $N_{\mathcal{F}}^K(Q)$ is saturated, and $C_{\mathcal{F}}(Q)$ is normal of p-power index in $N_{\mathcal{F}}^K(Q)$.*

Proof. By Proposition I.5.2 and Theorem I.5.5 in [5], the fusion systems $C_{\mathcal{F}}(Q)$ and $N_{\mathcal{F}}^K(Q)$ are saturated. So, it suffices to prove the lemma when $\mathcal{F} = N_{\mathcal{F}}^K(Q)$; that is, when $Q \trianglelefteq \mathcal{F}$ and $K \geqslant \mathrm{Aut}_{\mathcal{F}}(Q)$. Then, $C_{\mathcal{F}}(Q) \trianglelefteq \mathcal{F}$ by [15, Proposition 8.8].

We claim that

$$P \leqslant S,\ \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \text{ of order prime to } p \ \Rightarrow\ \alpha \text{ extends to } \bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(PQ) \text{ with } \alpha|_Q = \mathrm{Id}_Q. \quad (1.1)$$

Since $Q \trianglelefteq \mathcal{F}$, α extends to $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(PQ)$, and we can arrange that $\bar{\alpha}$ also has order prime to p. But $\bar{\alpha}|_Q \in K$ by assumption, hence has p-power order, and so $\bar{\alpha}|_Q = \mathrm{Id}_Q$.

We next check that $C_S(Q) \geqslant \mathfrak{hyp}(\mathcal{F})$. Fix $P \leqslant S$, and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ of order prime to p. By (1.1), α extends to $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(PQ)$ such that $\bar{\alpha}|_Q = \mathrm{Id}_Q$. But then $[[\alpha, P], Q] \leqslant [[\bar{\alpha}, PQ], Q] = 1$ by the 3-subgroup lemma (see [20, Theorem 2.2.3] or [3, 8.7]) and since $[PQ, Q] \leqslant Q$ and $[\bar{\alpha}, Q] = 1$. So, $[\alpha, P] \leqslant C_S(Q)$. Since $\mathfrak{hyp}(\mathcal{F})$ is generated by such subgroups $[\alpha, P]$, this proves that $\mathfrak{hyp}(\mathcal{F}) \leqslant C_S(Q)$.

It remains to show, for all $P \leqslant C_S(Q)$, that $\mathrm{Aut}_{C_{\mathcal{F}}(Q)}(P)$ contains $O^p(\mathrm{Aut}_{\mathcal{F}}(P))$. But this follows directly from (1.1), which says that each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ of order prime to p lies in $\mathrm{Aut}_{C_{\mathcal{F}}(Q)}(P)$. Thus, $C_{\mathcal{F}}(Q)$ has p-power index in $\mathcal{F}$. □

1.5 | Quotient fusion systems

We begin with the basic definition and properties.

Definition 1.17. Let $\mathcal{F}$ be a fusion system over a finite p-group S, and assume $Q \trianglelefteq S$ is strongly closed in $\mathcal{F}$. Then $\mathcal{F}/Q$ is defined to be the fusion system over S/Q where

$$\mathrm{Hom}_{\mathcal{F}/Q}(P/Q, R/Q) = \big\{\varphi/Q \,\big|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)\big\}$$

for $P, R \leqslant S$ containing Q. Here, $\varphi/Q \in \mathrm{Hom}(P/Q, R/Q)$ sends gQ to $\varphi(g)Q$.

Note that by definition, $\mathcal{F}/Q = N_{\mathcal{F}}(Q)/Q$. If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and $H \trianglelefteq G$ is such that $Q = H \cap S$, then $\mathcal{F}/Q \cong \mathcal{F}_{SH/H}(G/H)$ (see [15, Theorem 5.20]).

Lemma 1.18. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $Q \trianglelefteq S$ is strongly closed in $\mathcal{F}$. Then $\mathcal{F}/Q$ is saturated. If $\mathcal{E} \trianglelefteq \mathcal{F}$ is a normal fusion subsystem over $T \trianglelefteq S$ such that $T \geqslant Q$, then $\mathcal{E}/Q \trianglelefteq \mathcal{F}/Q$.*

Proof. For a proof that $\mathcal{F}/Q$ is saturated, see [15, Proposition 5.11] or [5, Lemma II.5.4]. If $\mathcal{E} \trianglelefteq \mathcal{F}$ over $T \geqslant Q$, then $\mathcal{E}/Q$ is saturated since $\mathcal{E}$ is, T/Q is strongly closed in $\mathcal{F}/Q$, and the invariance and Frattini conditions for normality of $\mathcal{E}/Q \leqslant \mathcal{F}/Q$ follow immediately from those for $\mathcal{E} \trianglelefteq \mathcal{F}$ (see [15, Lemma 5.59] for details).

It remains to prove the extension condition for $\mathcal{E}/Q \leqslant \mathcal{F}/Q$. We must show, for each $\varphi \in \mathrm{Aut}_{\mathcal{E}/Q}(T/Q)$, that φ extends to some $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}/Q}((T/Q)C_{S/Q}(T/Q))$ such that $[\bar{\varphi}, C_{S/Q}(T/Q)] \leqslant Z(T/Q)$. This clearly holds when $\varphi \in \mathrm{Inn}(T/Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{E}/Q}(T/Q))$, so it will suffice to prove it when φ has order prime to p. Let $U \leqslant S$ be such that $Q \leqslant U$ and $C_{S/Q}(T/Q) = U/Q$.

By [4, Theorem 5] or [25, Theorem 1], there is a (unique) saturated fusion subsystem $\mathcal{E}S \leqslant \mathcal{F}$ over S such that $\mathcal{E}$ is normal of p-power index in $\mathcal{E}S$. Since $\mathcal{E}S$ is saturated, so is $\mathcal{E}S/Q$. So, by the extension axiom, φ extends to some $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{E}S/Q}(TU/Q)$, and upon replacing $\bar{\varphi}$ by $\bar{\varphi}^k$ for some k, we can arrange that $\bar{\varphi}$ have order prime to p. Then $\bar{\varphi} = \hat{\varphi}/T$ for some $\hat{\varphi} \in \mathrm{Aut}_{\mathcal{E}S}(TU)$, and we can again arrange that $\hat{\varphi}$ have order prime to p.

By definition of the hyperfocal subgroup, $[\hat{\varphi}, TU] \leqslant \mathfrak{hyp}(\mathcal{E}S) \leqslant T$, where the last inclusion holds since $O^p(\mathcal{E}S) \leqslant \mathcal{E}$. Thus, $[\bar{\varphi}, TU/Q] \leqslant T/Q$, and so, $[\bar{\varphi}, C_{S/Q}(T/Q)] = [\bar{\varphi}, U/Q] \leqslant (T \cap U)/Q = Z(T/Q)$. (We thank the referee for pointing out this short argument.) □

We refer to [5, § II.5] and [15, § 5.2] for some of the other properties of these quotient systems.

The next lemma involves normal fusion subsystems of index prime to p (see Definition 1.14).

Lemma 1.19. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $Z \leqslant Z(\mathcal{F})$ be a central subgroup. Then $O^{p'}(\mathcal{F}/Z) = O^{p'}(\mathcal{F})/Z$.*

Proof. Set $\mathcal{H} = \{P \in \mathcal{F}^c \mid P/Z \in (\mathcal{F}/Z)^c\}$. (Note that for $P \in \mathcal{F}^c$, $P \geqslant Z(S) \geqslant Z$.) For each $P \in \mathcal{F}^c \setminus \mathcal{H}$ that is fully normalized in $\mathcal{F}$, P/Z is fully normalized in $\mathcal{F}/Z$, and hence, $C_{S/Z}(P/Z) \not\leq P/Z$. Choose $x \in S \setminus P$ such that $xZ \in C_{S/Z}(P/Z)$, and consider the automorphism $c_x \in \mathrm{Aut}_S(P)$. Then $c_x \notin \mathrm{Inn}(P)$ since $P \in \mathcal{F}^c$, and c_x induces the identity on Z and on P/Z. Since $\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \mid [\alpha, P] \leqslant Z\}$ is a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ (see [20, Corollary 5.3.3]), this proves that $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$, and hence, that

$$\text{for each } P \in \mathcal{F}^c \setminus \mathcal{H}, \text{ there is } P^* \in P^{\mathcal{F}} \text{ with } \mathrm{Out}_S(P^*) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P^*)) \neq 1. \tag{1.2}$$

By [5, Theorem I.7.7], there is a finite group Γ of order prime to p, and a map $\theta\colon \mathrm{Mor}((\mathcal{F}/Z)^c) \longrightarrow \Gamma$ that sends composites to products and inclusions to the identity, and is such that $\theta(\mathrm{Aut}_{\mathcal{F}/Z}(S/Z)) = \Gamma$ and $O^{p'}(\mathcal{F}/Z) = \langle \theta^{-1}(1) \rangle$. Let $\mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}$ be the full subcategory with object set $\mathcal{H}$, and let Φ be the natural map from $\mathrm{Mor}(\mathcal{F}^{\mathcal{H}})$ to $\mathrm{Mor}((\mathcal{F}/Z)^c)$. Set $\mathcal{F}_0 = \langle (\theta\Phi)^{-1}(1) \rangle$: a fusion subsystem of $\mathcal{F}$ over S. By [34, Lemma 1.6] and (1.2), $\mathcal{F}_0 \geqslant O^{p'}(\mathcal{F})$ and is saturated. Thus, $O^{p'}(\mathcal{F})/Z \leqslant \mathcal{F}_0/Z \leqslant O^{p'}(\mathcal{F}/Z)$.

Conversely, $O^{p'}(\mathcal{F})/Z$ has index prime to p in $\mathcal{F}/Z$ since for each $P/Z \leqslant S/Z$, we have $\mathrm{Aut}_{O^{p'}(\mathcal{F})/Z}(P/Z) \geqslant O^{p'}(\mathrm{Aut}_{\mathcal{F}/Z}(P/Z))$. So, $O^{p'}(\mathcal{F})/Z \geqslant O^{p'}(\mathcal{F}/Z)$. □

The following construction is needed when we want to look at the image of $\mathcal{E} \trianglelefteq \mathcal{F}$ in $\mathcal{F}/Q$ but $\mathcal{E}$ does not contain Q.

Definition 1.20. Let $\mathcal{E} \leqslant \mathcal{F}$ be a pair of saturated fusion systems over $T \leqslant S$, and let $Z \leqslant Z(\mathcal{F})$ be a central subgroup. Define $Z\mathcal{E} \leqslant \mathcal{F}$ to be the fusion subsystem over ZT where for each $P, Q \leqslant ZT$,

$$\mathrm{Hom}_{Z\mathcal{E}}(P,Q) = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \mid \varphi|_{P\cap T} \in \mathrm{Hom}_{\mathcal{E}}(P\cap T, Q\cap T)\}.$$

If $\mathcal{E} \trianglelefteq \mathcal{F}$, then the above definition is a special case of a construction of Aschbacher [4, Theorem 5]. But the definition and arguments in this very restricted case are much more elementary.

Lemma 1.21. *Let $\mathcal{E} \leqslant \mathcal{F}$ be a pair of saturated fusion systems over finite p-groups $T \leqslant S$. Let $Z \leqslant Z(\mathcal{F})$ be a central subgroup. Then $Z\mathcal{E}$ is saturated, and $Z\mathcal{E} \trianglelefteq \mathcal{F}$ if $\mathcal{E} \trianglelefteq \mathcal{F}$.*

Proof. A subgroup $P \leqslant ZT$ is fully normalized or fully centralized in $Z\mathcal{E}$ if and only if $P \cap T$ is fully normalized or fully centralized in $\mathcal{E}$. The saturation axioms for $Z\mathcal{E}$ follow easily from those for $\mathcal{E}$: note, for example, that $\mathrm{Aut}_{Z\mathcal{E}}(P) \cong \mathrm{Aut}_{\mathcal{E}}(P\cap T)$ for $P \leqslant ZT$. So, $Z\mathcal{E}$ is saturated.

If $\mathcal{E} \trianglelefteq \mathcal{F}$, then the subgroup ZT is strongly closed in $\mathcal{F}$ since for each $x = zt$ (for $z \in Z$ and $t \in T$), each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, S)$ extends to $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(Z\langle t\rangle, S)$, and $\varphi(x) = z\bar\varphi(t) \in ZT$. The extension condition for $Z\mathcal{E}$ follows directly from that for $\mathcal{E} \trianglelefteq \mathcal{F}$, and the invariance and Frattini conditions for $Z\mathcal{E}$ follow from those conditions applied to $\mathcal{E} \trianglelefteq \mathcal{F}$ and the definition of a central subgroup. Thus, $Z\mathcal{E} \trianglelefteq \mathcal{F}$. □

The following lemma will also be useful.

Lemma 1.22. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{E} \trianglelefteq \mathcal{F}$ be a normal fusion subsystem over $T \trianglelefteq S$. Let $Q \trianglelefteq S$ be a subgroup strongly closed in $\mathcal{F}$, and assume that*

(i) *each morphism in $\mathcal{F}$ between subgroups of T lies in $\mathcal{E}$ (i.e., $\mathcal{E}$ is a full subcategory of $\mathcal{F}$), and*
(ii) *$S = QT$.*

Then, $Q \cap T$ is strongly closed in $\mathcal{E}$ and $\mathcal{F}/Q \cong \mathcal{E}/(Q\cap T)$.

Proof. Since an intersection of strongly closed subgroups is strongly closed, $Q \cap T$ is strongly closed in $\mathcal{F}$ and hence in $\mathcal{E}$.

By (ii), the inclusion of T into S induces an isomorphism $\alpha\colon T/(Q\cap T) \xrightarrow{\cong} S/Q$. Then ${}^{\alpha}(\mathcal{E}/(Q\cap T)) \leqslant \mathcal{F}/Q$ as fusion systems over S/Q, and we will show that they are equal.

Assume $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R)$ for some $P, R \leqslant S$ containing Q, and set $\psi = \varphi|_{P\cap T}$ as a morphism from $P\cap T$ to $R\cap T$. Then $\psi \in \mathrm{Hom}_{\mathcal{E}}(P\cap T, R\cap T)$ by (i), and ${}^{\alpha}(\psi/(Q\cap T)) = \varphi/Q$ as homomorphisms from P/Q to R/Q. Thus, $\varphi/Q \in \mathrm{Mor}({}^{\alpha}(\mathcal{E}/(Q\cap T)))$. Since $\varphi/Q \in \mathrm{Mor}(\mathcal{F}/Q)$ was arbitrary, this proves that $\mathcal{F}/Q \leqslant {}^{\alpha}(C_{\mathcal{F}}(Q)/Z(Q))$. □

2 | AUTOMORPHISM GROUPS AND TAMENESS

The main aim of this section is to introduce the concept of tameness for fusion systems. This was originally defined in [2] and it is one of the main subjects of this article.

2.1 | Automorphisms of fusion and linking systems

Before defining tameness, we must define automorphism and outer automorphism groups of fusion and linking systems.

Definition 2.1. Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. Then,

(a) $\mathrm{Aut}(\mathcal{F}) = \{\alpha \in \mathrm{Aut}(S) \mid {}^{\alpha}\mathcal{F} = \mathcal{F}\}$: the group of automorphisms of S that send $\mathcal{F}$ to itself;
(b) $\mathrm{Out}(\mathcal{F}) = \mathrm{Aut}(\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S)$ is the group of outer automorphisms of $\mathcal{F}$; and
(c) for each $\alpha \in \mathrm{Aut}(\mathcal{F})$, we let $c_\alpha : \mathcal{F} \longrightarrow \mathcal{F}$ denote the functor that sends an object P to $\alpha(P)$ and a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ to $\alpha\varphi\alpha^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\alpha(P), \alpha(Q))$.

Now that we have defined automorphisms, we can define characteristic subsystems.

Definition 2.2. Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. A fusion subsystem $\mathcal{E} \leqslant \mathcal{F}$ over $T \trianglelefteq S$ is *characteristic* if $\mathcal{E}$ is normal in $\mathcal{F}$ and $c_\alpha(\mathcal{E}) = \mathcal{E}$ for all $\alpha \in \mathrm{Aut}(\mathcal{F})$. Likewise, a subgroup P of S is characteristic in $\mathcal{F}$ if $P \trianglelefteq \mathcal{F}$ and $\alpha(P) = P$ for all $\alpha \in \mathrm{Aut}(\mathcal{F})$; equivalently, if $\mathcal{F}_P(P)$ is a characteristic subsystem of $\mathcal{F}$.

For example, when $\mathcal{F}$ is a saturated fusion system over a finite p-group S, then the subsystems $O^p(\mathcal{F})$ and $O^{p'}(\mathcal{F})$ (see Definition 1.14(b,c)) and the subgroups $O_p(\mathcal{F})$ and $Z(\mathcal{F})$ are all characteristic in $\mathcal{F}$.

Lemma 2.3. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be a normal pair of fusion systems over finite p-groups $T \trianglelefteq S$. Then*

(a) *if $\mathcal{D} \trianglelefteq \mathcal{E}$ is characteristic in $\mathcal{E}$, then $\mathcal{D} \trianglelefteq \mathcal{F}$; and*
(b) $O_p(\mathcal{E}) \leqslant O_p(\mathcal{F})$.

Proof. Point (a) is shown in [4, 7.4]. Since $O_p(\mathcal{E})$ is characteristic in $\mathcal{E}$, $O_p(\mathcal{E}) \trianglelefteq \mathcal{F}$ by (a), and so, $O_p(\mathcal{E}) \leqslant O_p(\mathcal{F})$, proving (b). □

The following condition for a subnormal fusion system to be normal is due to Aschbacher.

Lemma 2.4 ([4, 7.4]). *Let $\mathcal{D} \trianglelefteq \mathcal{E} \trianglelefteq \mathcal{F}$ be saturated fusion systems over finite p-groups $U \trianglelefteq T \trianglelefteq S$ such that $c_\alpha(\mathcal{D}) = \mathcal{D}$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$. Then $\mathcal{D} \trianglelefteq \mathcal{F}$.*

The following definitions of automorphism groups are taken from [2, Definition 1.13 & Lemma 1.14]. Recall that for each pair of objects $P \leqslant Q$ in a linking system $(\mathcal{L}, \delta, \pi)$, we write $\iota_{P,Q} = \delta_{P,Q}(1) \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, and regard it as the "inclusion" in $\mathcal{L}$ of P into Q.

Definition 2.5. Let $\mathcal{F}$ be a fusion system over a finite p-group S and let $(\mathcal{L}, \delta, \pi)$ be an associated linking system. For each P in $\mathcal{L}$, we call $\delta_P(P) \leqslant \mathrm{Aut}_{\mathcal{L}}(P)$ the *distinguished subgroup* of $\mathrm{Aut}_{\mathcal{L}}(P)$.

(a) Let $\mathrm{Aut}(\mathcal{L})$ be the group of automorphisms of the category $\mathcal{L}$ that send inclusions to inclusions and distinguished subgroups to distinguished subgroups.

(b) For $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$, let $c_\gamma \in \mathrm{Aut}(\mathcal{L})$ be the automorphism that sends an object P to $c_\gamma(P) \overset{\text{def}}{=} \pi(\gamma)(P)$, and sends $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to $c_\gamma(\psi) \overset{\text{def}}{=} \gamma|_{Q,c_\gamma(Q)} \circ \psi \circ (\gamma|_{P,c_\gamma(P)})^{-1}$. Set

$$\mathrm{Out}(\mathcal{L}) = \mathrm{Aut}(\mathcal{L}) \big/ \{c_\gamma \mid \gamma \in \mathrm{Aut}_{\mathcal{L}}(S)\}.$$

The notation in Definitions 2.1 and 2.5 is slightly different from that used in [2] and [5], as described in the following table:

Notation used here	**Aut($\mathcal{F}$)**	**Out($\mathcal{F}$)**	**Aut($\mathcal{L}$)**	**Out($\mathcal{L}$)**
Used in [2, 5]	$\mathrm{Aut}(S,\mathcal{F})$	$\mathrm{Out}(S,\mathcal{F})$	$\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$	$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$

By [2, Lemma 1.14], the above definition of $\mathrm{Out}(\mathcal{L})$ is equivalent to $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ in [11], and by [11, Lemma 8.2], both are equivalent to $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ in [10]. So, by [10, Theorem 4.5(a)], $\mathrm{Out}(\mathcal{L}^c_S(G)) \cong \mathrm{Out}(BG^\wedge_p)$: the group of homotopy classes of self-homotopy equivalences of the space $BG^\wedge_p$.

The next result shows how an automorphism of a linking system automatically preserves the structure functors. For use in the next section, we state this for certain full subcategories of a linking system that need not themselves be linking systems because their objects might not be closed under overgroups. (Compare with Proposition 6 in [32].)

For a group G, a set $\mathcal{H}$ of subgroups of G, and $\beta \in \mathrm{Aut}(G)$ that permutes the members of $\mathcal{H}$, let $\mathcal{T}(\beta)\colon \mathcal{T}_{\mathcal{H}}(G) \longrightarrow \mathcal{T}_{\mathcal{H}}(G)$ denote the functor that sends $H \in \mathcal{H}$ to $\beta(H)$, and sends $g \in T_G(H,K)$ to $\beta(g) \in T_G(\beta(H),\beta(K))$.

Proposition 2.6. *Let $(\mathcal{L},\delta,\pi)$ be a linking system associated to a fusion system $\mathcal{F}$ over a finite p-group S, let $\mathcal{L}_0 \subseteq \mathcal{L}$ be a full subcategory such that $\mathrm{Ob}(\mathcal{L}_0) \supseteq \mathcal{F}^{cr}$, and let $\mathrm{Aut}(\mathcal{L}_0)$ be the group of automorphisms of the category $\mathcal{L}_0$ that send inclusions to inclusions and distinguished subgroups to distinguished subgroups. Fix $\alpha \in \mathrm{Aut}(\mathcal{L}_0)$, and let $\beta \in \mathrm{Aut}(S)$ be the unique automorphism such that $\alpha(\delta_S(g)) = \delta_S(\beta(g))$ for all $g \in S$. Then $\beta \in \mathrm{Aut}(\mathcal{F})$, $\alpha(P) = \beta(P)$ for each $P \in \mathrm{Ob}(\mathcal{L}_0)$, and the following diagram of functors*

$$\begin{array}{ccccc} \mathcal{T}_{\mathrm{Ob}(\mathcal{L}_0)}(S) & \xrightarrow{\delta} & \mathcal{L}_0 & \xrightarrow{\pi} & \mathcal{F} \\ \downarrow{\scriptstyle \mathcal{T}(\beta)} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle c_\beta} \\ \mathcal{T}_{\mathrm{Ob}(\mathcal{L}_0)}(S) & \xrightarrow{\delta} & \mathcal{L}_0 & \xrightarrow{\pi} & \mathcal{F} \end{array} \tag{2.1}$$

commutes.

Proof. Clearly, $\alpha(S) = S$, and hence, α_S sends $\delta_S(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}}(S))$ to itself. Thus, β is well defined. Since α sends inclusions to inclusions, it commutes with restrictions. So for $P,Q \in \mathrm{Ob}(\mathcal{L}_0)$ and $g \in T_S(P,Q)$ (the transporter set), we have

$$\alpha(\delta_{P,Q}(g)) = \delta_{\alpha(P),\alpha(Q)}(\beta(g)) \tag{2.2}$$

since $\alpha(\delta_S(g)) = \delta_S(\beta(g))$. In particular, the left-hand square in (2.1) commutes.

When $Q = P$, (2.2) says that $\delta_{\alpha(P)}(\beta(P)) = \alpha_P(\delta_P(P))$, and $\alpha_P(\delta_P(P)) = \delta_{\alpha(P)}(\alpha(P))$ since α sends distinguished subgroups to distinguished subgroups. So, $\alpha(P) = \beta(P)$ since $\delta_{\alpha(P)}$ is a monomorphism (Proposition 1.8(a)).

Fix $P, Q \in \mathrm{Ob}(\mathcal{L})$ and $\psi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, and set $\varphi = \pi(\psi) \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. For each $g \in P$, consider the following three squares:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & Q \\ \downarrow{\scriptstyle \delta_P(g)} & & \downarrow{\scriptstyle \delta_Q(\varphi(g))} \\ P & \xrightarrow{\psi} & Q \end{array} \qquad \begin{array}{ccc} \alpha(P) & \xrightarrow{\alpha(\psi)} & \alpha(Q) \\ \downarrow{\scriptstyle \delta_{\alpha(P)}(\beta(g))} & & \downarrow{\scriptstyle \delta_{\alpha(Q)}(\beta(\varphi(g)))} \\ \alpha(P) & \xrightarrow{\alpha(\psi)} & \alpha(Q) \end{array} \qquad \begin{array}{ccc} \alpha(P) & \xrightarrow{\alpha(\psi)} & \alpha(Q) \\ \downarrow{\scriptstyle \delta_{\alpha(P)}(\beta(g))} & & \downarrow{\scriptstyle \delta_{\alpha(Q)}(\pi(\alpha(\psi))(\beta(g)))} \\ \alpha(P) & \xrightarrow{\alpha(\psi)} & \alpha(Q)\,. \end{array}$$

The first and third of these squares commute by axiom (C) in Definition 1.7, and the second commutes since it is the image under α of the first. Since morphisms in $\mathcal{L}$ are epimorphisms and $\delta_{\alpha(Q)}$ is injective (Proposition 1.8(a,b)), this implies $\beta(\varphi(g)) = \pi(\alpha(\psi))(\beta(g))$. Thus, $\pi(\alpha(\psi)) = \beta\varphi\beta^{-1} = c_\beta(\pi(\psi))$, proving that the right-hand square in (2.1) commutes.

In particular, since π is surjective on morphism sets (axiom (A2) in Definition 1.7), $\beta\varphi\beta^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\beta(P), \beta(Q))$ for each $P, Q \in \mathrm{Ob}(\mathcal{L}_0)$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Since $\mathrm{Ob}(\mathcal{L}_0)$ includes all subgroups that are $\mathcal{F}$-centric and $\mathcal{F}$-radical, all morphisms in $\mathcal{F}$ are composites of restrictions of morphisms between objects of $\mathcal{L}_0$ by Theorem 1.4. Hence, ${}^\beta\mathcal{F} \leqslant \mathcal{F}$ with equality since $\mathcal{F}$ is a finite category, and so, $\beta \in \mathrm{Aut}(\mathcal{F})$. □

Proposition 2.6 motivates the following definition.

Definition 2.7. Let $(\mathcal{L}, \delta, \pi)$ be a linking system associated to a fusion system $\mathcal{F}$ over a finite p-group S. Let $\widetilde{\mu}_{\mathcal{L}} : \mathrm{Aut}(\mathcal{L}) \longrightarrow \mathrm{Aut}(\mathcal{F})$ denote the homomorphism that sends $\alpha \in \mathrm{Aut}(\mathcal{L}_0)$ to $\beta \in \mathrm{Aut}(\mathcal{F})$ such that diagram (2.1) commutes. Let $\mu_{\mathcal{L}} : \mathrm{Out}(\mathcal{L}) \longrightarrow \mathrm{Out}(\mathcal{F})$ be the induced homomorphisms on the quotient groups.

That $\widetilde{\mu}_{\mathcal{L}}$ is a homomorphism follows easily from its definition via diagram (2.1). For $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$, we have $\widetilde{\mu}_{\mathcal{L}}(c_\gamma) = \pi(\gamma) \in \mathrm{Aut}_{\mathcal{F}}(S)$ since π is a functor, so $\mu_{\mathcal{L}}$ is well defined.

2.2 | Tameness of fusion systems

We next define a homomorphism κ_G that connects the automorphisms of a group to those of its linking system. We refer to [2, § 2.2] for more details about κ_G and the proof that it is well defined.

Definition 2.8. Let G be a finite group and choose $S \in \mathrm{Syl}_p(G)$. Let

$$\kappa_G : \mathrm{Out}(G) \longrightarrow \mathrm{Out}(\mathcal{L}_S^c(G))$$

denote the homomorphism that sends the class of $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(S) = S$ to the class of the automorphism of $\mathcal{L}_S^c(G)$ induced by α.

In these terms, tameness can be defined as follows.

Definition 2.9. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then

(a) $\mathcal{F}$ is *tamely realized by* a finite group G if $\mathcal{F} \cong \mathcal{F}_{S^*}(G)$ for some $S^* \in \mathrm{Syl}_p(G)$ and the homomorphism $\kappa_G : \mathrm{Out}(G) \longrightarrow \mathrm{Out}(\mathcal{L}^c_{S^*}(G))$ is split surjective; and
(b) $\mathcal{F}$ is *tame* if it is tamely realized by some finite group.

2.3 | Centric fusion and linking subsystems

Some of the results in later sections need the hypothesis that a certain fusion or linking subsystem be *centric*, which we now define.

Definition 2.10. Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be a normal pair of saturated fusion systems over finite p-groups $T \trianglelefteq S$.

(a) Let $C_S(\mathcal{E})$ denote the unique largest subgroup $X \leqslant C_S(T)$ such that $C_{\mathcal{F}}(X) \geqslant \mathcal{E}$. Such a largest subgroup exists by [4, 6.7], or (via a different proof) by [26, Theorem 1(a)].
(b) The subsystem $\mathcal{E}$ is *centric* in $\mathcal{F}$ if $C_S(\mathcal{E}) \leqslant T$; that is, if there is no $x \in C_S(T) \setminus T$ such that $C_{\mathcal{F}}(x) \geqslant \mathcal{E}$.
(c) If $\mathcal{M} \trianglelefteq \mathcal{L}$ are linking systems associated to $\mathcal{E} \trianglelefteq \mathcal{F}$, set

$$C_{\mathrm{Aut}_{\mathcal{L}}(T)}(\mathcal{M}) = \big\{\gamma \in \mathrm{Aut}_{\mathcal{L}}(T) \,\big|\, c_\gamma = \mathrm{Id}_{\mathcal{M}}\big\},$$

where c_γ is a well-defined element of $\mathrm{Aut}(\mathcal{M})$ by Definition 1.12(b).
(d) If $\mathcal{M} \trianglelefteq \mathcal{L}$ are linking systems associated to $\mathcal{E} \trianglelefteq \mathcal{F}$, then $\mathcal{M}$ is *centric* in $\mathcal{L}$ if $C_{\mathrm{Aut}_{\mathcal{L}}(T)}(\mathcal{M}) \leqslant \mathrm{Aut}_{\mathcal{M}}(T)$; equivalently, if $c_\psi \neq \mathrm{Id}_{\mathcal{M}}$ for each $\psi \in \mathrm{Aut}_{\mathcal{L}}(T) \setminus \mathrm{Aut}_{\mathcal{M}}(T)$.

For pairs of linking systems, this is the definition used in [2] (Definition 1.27). The term "centric fusion subsystem" was not used in [34], but the condition in Definition 2.10(b) appears in Proposition 2.1 and Theorem 2.3 of that paper (and the term is used in [35]).

In the next lemma, we look at the relation between normal centric fusion subsystems and normal centric linking subsystems.

Lemma 2.11. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be a normal pair of saturated fusion systems over finite p-groups $T \trianglelefteq S$ with associated linking systems $\mathcal{M} \trianglelefteq \mathcal{L}$, and set*

$$C_S(\mathcal{M}) = \big\{x \in S \,\big|\, c_{\delta_T(x)} = \mathrm{Id}_{\mathcal{M}}\big\} \trianglelefteq S.$$

Then,

(a) $C_{\mathrm{Aut}_{\mathcal{L}}(T)}(\mathcal{M}) = \delta_T(C_S(\mathcal{M}))$ *and* $Z(\mathcal{E})Z(\mathcal{F}) \leqslant C_S(\mathcal{M}) \leqslant C_S(\mathcal{E})$;
(b) $\mathcal{M}$ *is centric in* $\mathcal{L}$ *if and only if* $C_S(\mathcal{M}) \leqslant T$, *and this holds if* $\mathcal{E}$ *is centric in* $\mathcal{F}$; *and*
(c) *the conjugation action of* $\mathrm{Aut}_{\mathcal{L}}(T)$ *on* $C_S(\mathcal{M}) \cong \delta_T(C_S(\mathcal{M})) \trianglelefteq \mathrm{Aut}_{\mathcal{L}}(T)$ *induces an action of the quotient group* $\mathcal{L}/\mathcal{M} = \mathrm{Aut}_{\mathcal{L}}(T)/\mathrm{Aut}_{\mathcal{M}}(T)$ *on* $C_S(\mathcal{M})$, *and* $C_{C_S(\mathcal{M})}(\mathcal{L}/\mathcal{M}) = Z(\mathcal{F})$.

Proof. Throughout the proof, "axiom (–)" always refers to one of the axioms in the definition of a linking system (Definition 1.7).

We first claim that

$$\forall x \in S, \qquad \delta_T(x) \in \mathrm{Aut}_{\mathcal{M}}(T) \iff x \in T. \tag{2.3}$$

The implication "$\Longleftarrow$" is clear. To see the converse, fix $x \in S$ such that $\delta_T(x) \in \mathrm{Aut}_{\mathcal{M}}(T)$. Then $c_x \in \mathrm{Aut}_{\mathcal{E}}(T)$, and $c_x \in \mathrm{Inn}(T) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{E}}(T))$ since it has p-power order. Thus, there is $t \in T$ such that $xt^{-1} \in C_S(T)$, $\delta_T(xt^{-1}) = \delta_T(x)\delta_T(t)^{-1} \in \mathrm{Aut}_{\mathcal{M}}(T)$, and so, $xt^{-1} \in Z(T)$ by axiom (A2) applied to $\mathcal{M}$ and since δ_T is injective (Proposition 1.8(a)). Hence, $x \in T$.

We next claim that

$$C_S(\mathcal{E}) \cap T = Z(\mathcal{E}). \tag{2.4}$$

The inclusion $Z(\mathcal{E}) \leqslant C_S(\mathcal{E}) \cap T$ is immediate from the definitions. If $x \in C_S(\mathcal{E}) \cap T \leqslant Z(T)$, then since $\mathcal{E} \leqslant C_{\mathcal{F}}(x)$, each $\varphi \in \mathrm{Hom}_{\mathcal{E}}(\langle x\rangle, S)$ extends to a morphism in $\mathcal{F}$ that sends x to itself, and hence, $\varphi(x) = x$. Thus, $x^{\mathcal{E}} = \{x\}$, so $x \in Z(\mathcal{E})$ by [5, Lemma I.4.2], finishing the proof of (2.4).

(a) Fix $\alpha \in C_{\mathrm{Aut}_{\mathcal{L}}(T)}(\mathcal{M})$. By axiom (C) for the linking system $\mathcal{L}$, for all $g \in T$, $\alpha\delta_T(g)\alpha^{-1} = \delta_T(\pi(\alpha)(g))$, so $\delta_T(g) = \delta_T(\pi(\alpha)(g))$ since $c_\alpha = \mathrm{Id}_{\mathcal{M}}$, and $g = \pi(\alpha)(g)$ since δ_T is injective (see Proposition 1.8(a)). So, $\pi(\alpha) = \mathrm{Id}_T$, and $\alpha = \delta_T(x)$ for some $x \in C_S(T)$ by axiom (A2). Then $x \in C_S(\mathcal{M})$ by definition, and thus, $C_{\mathrm{Aut}_{\mathcal{L}}(T)}(\mathcal{M}) = \delta_T(C_S(\mathcal{M}))$.

If $x \in Z(\mathcal{E}) = C_S(\mathcal{E}) \cap T$, then for each $P, Q \leqslant T$ and $\psi \in \mathrm{Mor}_{\mathcal{M}}(P, Q)$, $\pi(\psi)$ extends to some $\bar\varphi \in \mathrm{Hom}_{\mathcal{E}}(P\langle x\rangle, Q\langle x\rangle)$, and $\bar\varphi = \pi(\widehat\psi)$ for some $\widehat\psi \in \mathrm{Mor}_{\mathcal{M}}(P\langle x\rangle, Q\langle x\rangle)$. Set $\psi' = \widehat\psi|_{P,Q} \in \mathrm{Mor}_{\mathcal{M}}(P, Q)$. Then $\psi'\delta_P(x) = \delta_Q(x)\psi'$ since $\widehat\psi\delta_{P\langle x\rangle}(x) = \delta_{Q\langle x\rangle}(x)\widehat\psi$ by axiom (C). Also, $\pi(\psi') = \pi(\psi)$, so if P is fully centralized in $\mathcal{E}$, then $\psi' = \psi\delta_P(y)$ for some $y \in C_T(P)$. Then $[x, y] = 1$ since $x \in Z(T)$, so $c_{\delta_T(x)}(\delta_P(y)) = \delta_P(y)$, and thus, $c_{\delta_T(x)}(\psi) = \psi$. This holds for all $\psi \in \mathrm{Mor}(\mathcal{M})$ whose domain is fully centralized, and hence for all morphisms in $\mathcal{M}$. So, $x \in C_T(\mathcal{M}) \leqslant C_S(\mathcal{M})$.

Thus, $Z(\mathcal{E}) \leqslant C_S(\mathcal{M})$. By a similar argument but working in $\mathcal{L}$ and $\mathcal{F}$ instead of $\mathcal{M}$ and $\mathcal{E}$, we also have $Z(\mathcal{F}) \leqslant C_S(\mathcal{L}) \leqslant C_S(\mathcal{M})$, and so $Z(\mathcal{E})Z(\mathcal{F}) \leqslant C_S(\mathcal{M})$.

It remains to show that $C_S(\mathcal{M}) \leqslant C_S(\mathcal{E})$. Fix $x \in C_S(\mathcal{M})$; we must show that $\mathcal{E} \leqslant C_{\mathcal{F}}(x)$. Fix $P, Q \leqslant T$ and $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P, Q)$, and choose $\psi \in \mathrm{Mor}_{\mathcal{M}}(P, Q)$ such that $\pi(\psi) = \varphi$. Set $\bar P = P\langle x\rangle$ and $\bar Q = Q\langle x\rangle$. Then $\psi\delta_P(x) = \delta_Q(x)\psi$ since $c_{\delta_T(x)} = \mathrm{Id}_{\mathcal{M}}$, and $\psi\delta_P(g) = \delta_Q(\varphi(g))\psi$ for all $g \in P$ by axiom (C) applied to $\mathcal{M}$. So, for each $g \in \bar P$, there is $h \in \bar Q$ such that $\psi\delta_P(g) = \delta_Q(h)\psi$ in $\mathrm{Mor}_{\mathcal{L}}(P, Q)$, and by Proposition 1.8(d), ψ extends to a unique morphism $\bar\psi \in \mathrm{Mor}_{\mathcal{L}}(\bar P, \bar Q)$. Set $\bar\varphi = \pi(\bar\psi) \in \mathrm{Hom}_{\mathcal{F}}(\bar P, \bar Q)$. By axiom (C) again (but applied to $\mathcal{L}$), we have $\bar\psi\delta_{\bar P}(x) = \delta_{\bar Q}(\bar\varphi(x))\bar\psi$, and after restriction to P and Q this gives $\psi\delta_P(x) = \delta_Q(\bar\varphi(x))\psi$. Hence, $\delta_Q(\bar\varphi(x))\psi = \delta_Q(x)\psi$, so $\delta_Q(\bar\varphi(x)) = \delta_Q(x)$ since ψ is an epimorphism in the categorical sense (see Proposition 1.8(b)), and $\bar\varphi(x) = x$ by the injectivity of δ_Q. Thus, each morphism in $\mathcal{E}$ extends to a morphism in $\mathcal{F}$ that sends x to itself, and hence, $C_{\mathcal{F}}(x) \geqslant \mathcal{E}$ and $x \in C_S(\mathcal{E})$.

(b) By (a) and Definition 2.10(c), $\mathcal{M}$ is centric in $\mathcal{L}$ if and only if $\delta_T(C_S(\mathcal{M})) \leqslant \mathrm{Aut}_{\mathcal{M}}(T)$, and this holds exactly when $C_S(\mathcal{M}) \leqslant T$ by (2.3). Since $C_S(\mathcal{M}) \leqslant C_S(\mathcal{E})$ by (a), this is the case whenever $\mathcal{E}$ is centric in $\mathcal{F}$.

(c) By (a), $\delta_T(C_S(\mathcal{M}))$ is the kernel of the homomorphism from $\mathrm{Aut}_{\mathcal{L}}(T)$ to $\mathrm{Aut}(\mathcal{M})$ induced by conjugation. So, $\delta_T(C_S(\mathcal{M})) \trianglelefteq \mathrm{Aut}_{\mathcal{L}}(T)$, and $\mathrm{Aut}_{\mathcal{L}}(T)$ acts by conjugation on $C_S(\mathcal{M}) \cong \delta_T(C_S(\mathcal{M}))$. Since this subgroup centralizes $\mathcal{M}$ by definition, $\mathrm{Aut}_{\mathcal{M}}(T)$ acts trivially, and hence, the action of $\mathrm{Aut}_{\mathcal{L}}(T)$ factors through an action of the quotient group $\mathcal{L}/\mathcal{M}$.

By Definitions 1.3(g) and 2.10(a), $Z(\mathcal{F}) = C_S(\mathcal{F})$. So, by (a) applied when $\mathcal{M} = \mathcal{L}$,

$$C_{\mathrm{Aut}_{\mathcal{L}}(S)}(\mathcal{L}) = \delta_S(Z(\mathcal{F})). \tag{2.5}$$

Thus, for each $x \in Z(\mathcal{F})$, $\delta_S(x)$ acts trivially on $\mathcal{L}$ and hence $\delta_T(x)$ acts trivially on $\mathrm{Aut}_{\mathcal{L}}(T)$. So, $Z(\mathcal{F}) \leqslant C_{C_S(\mathcal{M})}(\mathrm{Aut}_{\mathcal{L}}(T)) = C_{C_S(\mathcal{M})}(\mathcal{L}/\mathcal{M})$, and it remains to prove the opposite inclusion.

Fix $x \in C_S(\mathcal{M})$ such that $\delta_T(x) \in Z(\mathrm{Aut}_{\mathcal{L}}(T))$. In particular, $\delta_T(x) \in Z(\delta_T(S))$, so $x \in Z(S)$ by the injectivity of δ_T (Proposition 1.8(a)), and $c_{\delta_T(x)}(P) = {}^xP = P$ for each $P \in \mathrm{Ob}(\mathcal{L})$. We must show that $\delta_T(x)$ acts trivially on $\mathcal{L}$. Fix $P, Q \leqslant S$ and $\psi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, and set $P_0 = P \cap T$, $Q_0 = Q \cap T$, and $\psi_0 = \psi|_{P,Q}$ (see Proposition 1.8(c)). By the Frattini condition on a normal linking subsystem (Lemma 1.13), ψ_0 is the composite of the restriction of a morphism $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$ followed by some $\chi \in \mathrm{Mor}(\mathcal{M})$. Since $\delta_T(x)$ commutes with γ by assumption and commutes with χ by (a) ($Z(\mathcal{F}) \leqslant C_S(\mathcal{M})$), we have

$$(\psi\delta_P(x))|_{P_0,Q_0} = \psi_0\delta_{P_0}(x) = \delta_{Q_0}(x)\psi_0 = (\delta_Q(x)\psi)|_{P_0,Q_0}.$$

Then $\psi\delta_P(x) = \delta_Q(x)\psi$ by the uniqueness of extensions in a linking system (Proposition 1.8(d)), and hence $c_{\delta_T(x)}(\psi) = \psi$.

Thus, $c_{\delta_S(x)}$ is the identity on all objects and morphisms in $\mathcal{L}$. So, $x \in Z(\mathcal{F})$ by (2.5) and the injectivity of δ_S (Proposition 1.8(a)). □

The following consequence of Lemma 2.11 will be needed in Section 4.

Lemma 2.12. *Let $H \trianglelefteq G$ be finite groups, choose $S \in \mathrm{Syl}_p(G)$, and set $T = S \cap H$. Set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{E} = \mathcal{F}_T(H) \trianglelefteq \mathcal{F}$. If $\mathrm{Ker}(\varkappa_H)$ has order prime to p, then $Z(\mathcal{F}) \leqslant Z(\mathcal{E})C_S(H)$.*

Proof. Set $G_0 = SH$ and $\mathcal{F}_0 = \mathcal{F}_S(G_0)$, and set $\mathcal{H} = \{P \leqslant S \mid P \cap T \in \mathcal{E}^c\}$. For each $P \in \mathcal{H}$, $C_H(P \cap H) = Z(P \cap H) \times O_{p'}(C_H(P \cap H))$ since $P \cap H \in \mathcal{E}^c$, so $C_H(P) = (Z(P) \cap H) \times O_{p'}(C_H(P))$, and this has p-power index in $C_{G_0}(P)$. Thus, $O^p(C_{G_0}(P))$ has order prime to p, so P is $\mathcal{F}_0$-quasi-centric by [5, Lemma III.4.6(e)].

Thus, $\mathcal{H} \subseteq \mathcal{F}_0^q$. Set $\mathcal{L}_0 = \mathcal{L}_S^{\mathcal{H}}(G_0)$ (see [5, p. 146]), and $\mathcal{M} = \mathcal{L}_T^c(H)$. Then $\mathcal{M} \trianglelefteq \mathcal{L}_0$ is a normal pair of linking systems associated to $\mathcal{E} \trianglelefteq \mathcal{F}_0$, so by Lemma 2.11(a),

$$Z(\mathcal{F}_0) \leqslant C_S(\mathcal{M}) = \{x \in S \mid c_{\delta_T(x)} = \mathrm{Id}_{\mathcal{M}}\}. \tag{2.6}$$

Fix $x \in Z(\mathcal{F}) \leqslant Z(\mathcal{F}_0)$. By (2.6), $[c_x] \in \mathrm{Ker}(\varkappa_H)$, and so $c_x \in \mathrm{Inn}(H)$ since $\mathrm{Ker}(\varkappa_H)$ has order prime to p. Thus, c_x is conjugation by some element $y \in C_H(T) = Z(T) \times O_{p'}(C_H(T))$, and since c_x has p-power order, we can assume $y \in Z(T)$. Then c_y induces the identity on $\mathcal{M}$ since c_x does by (2.6), so $y \in Z(\mathcal{E})$ by the exact sequence in [2, Lemma 1.14(a)]. Also, $y^{-1}x \in C_S(H)$, and so $x \in Z(\mathcal{E})C_S(H)$. □

3 | PRODUCTS OF FUSION AND LINKING SYSTEMS

In this section, we first define centric linking systems associated to products of two or more fusion systems (Lemma 3.5). This is followed by a description of the group of automorphisms of such a

product linking systems that leave the factors invariant up to permutation, as well as conditions on the fusion systems that guarantee that these are the only automorphisms of the linking system (Proposition 3.7(a,b)). As a consequence, we show that a product of tame fusion systems that satisfy these same conditions is always tame (Proposition 3.7(c)).

Recall that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over finite p-groups S_1 and S_2, then $\mathcal{F}_1 \times \mathcal{F}_2$ is the fusion system over $S_1 \times S_2$ generated by all morphisms $\varphi_1 \times \varphi_2 \in \mathrm{Hom}(P_1 \times P_2, Q_1 \times Q_2)$ for $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i)$ $(i = 1, 2)$. See [5, Definition I.6.5] for more details. When $\mathcal{F}_1$ and $\mathcal{F}_2$ are both saturated, then so is $\mathcal{F}_1 \times \mathcal{F}_2$ [5, Theorem I.6.6].

The following notation and hypotheses will be used throughout the section.

Hypothesis 3.1. Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be saturated fusion systems over finite p-groups $S_1, \dots, S_k$ (some $k \geqslant 2$), and set $S = S_1 \times \cdots \times S_k$ and $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$. For each i, let $\mathrm{pr}_i : S \longrightarrow S_i$ be the projection. For each $P \leqslant S$, we write $P_i = \mathrm{pr}_i(P)$ (for $1 \leqslant i \leqslant k$) and $\widehat{P} = P_1 \times \cdots \times P_k \leqslant S$. Thus, $\widehat{P} \geqslant P$ for each P.

We first check which subgroups are centric in a product of fusion systems.

Lemma 3.2. *Assume Hypothesis 3.1. Then a subgroup $P \leqslant S$ is $\mathcal{F}$-centric if and only if P_i is $\mathcal{F}$-centric for all i and $Z(P_1) \times \cdots \times Z(P_k) \leqslant P$.*

Proof. For each $P \leqslant S$, we have $C_S(P) = C_{S_1}(P_1) \times \cdots \times C_{S_k}(P_k) = C_S(\widehat{P})$. Hence,

- P is fully centralized in $\mathcal{F}$ if and only if P_i is fully centralized in $\mathcal{F}_i$ for each i, and
- $C_S(P) \leqslant P$ if and only if $C_{S_i}(P_i) \leqslant P_i$ for each i and $Z(\widehat{P}) \leqslant P$.

The result now follows since P is $\mathcal{F}$-centric if and only if P is fully centralized in $\mathcal{F}$ and $C_S(P) \leqslant P$. □

Note also that under Hypothesis 3.1, if $P \leqslant S$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then $P = \widehat{P} = P_1 \times \cdots \times P_k$ (see, e.g., [2, Lemma 3.1]). But that will not be needed here.

The following easy consequence of the Krull–Remak–Schmidt theorem will be needed. Recall that a group is indecomposable if it is not the direct product of two of its proper subgroups.

Proposition 3.3. *Assume $G_1, \dots, G_k$ are finite, indecomposable groups, and set $G = G_1 \times \cdots \times G_k$. Then, the following hold for each $\alpha \in \mathrm{Aut}(G)$.*

(a) *There is $\sigma \in \Sigma_k$ such that $\alpha(G_i Z(G)) = G_{\sigma(i)} Z(G)$ for each $1 \leqslant i \leqslant k$.*
(b) *If $\big(|Z(G)|, |G/[G,G]|\big) = 1$, then there is $\sigma \in \Sigma_k$ such that $\alpha(G_i) = G_{\sigma(i)}$ for each i.*

Proof. An automorphism $\beta \in \mathrm{Aut}(G)$ is *normal* if β commutes with all inner automorphisms of G; equivalently, if $[\beta, G] \leqslant Z(G)$. For such β, one can define a homomorphism $\delta : G \longrightarrow Z(G)$ by setting $\delta(g) = \beta(g)g^{-1}$, and δ factors through $G/[G,G]$. In particular, if $\big(|Z(G)|, |G/[G,G]|\big) = 1$, then the only normal automorphism of G is the identity.

By the Krull–Remak–Schmidt theorem in the form stated in [42, Theorem 2.4.8] (and applied with $\Omega = 1$ or $\Omega = \mathrm{Inn}(G)$), any two direct product decompositions of G with indecomposable factors have the same number of factors, and there is always a normal automorphism of G that sends one to the other up to a permutation of the factors . Points (a) and (b) follow immediately from this, applied to the decompositions of G as the product of the G_i and of the $\alpha(G_i)$. □

One immediate consequence of Proposition 3.3 is the following description of $\mathrm{Out}(G)$ when G is a product of simple groups.

Proposition 3.4. *Assume $G = G_1 \times \cdots \times G_k$, where $G_1, \dots, G_k$ are finite indecomposable groups and $\big(|Z(G)|, |G/[G,G]|\big) = 1$. Let Γ be the group of all $\gamma \in \Sigma_k$ such that $G_{\gamma(i)} \cong G_i$ for each i. Then there is an isomorphism*

$$\Phi_G : \big(\mathrm{Aut}(G_1) \times \cdots \times \mathrm{Aut}(G_k)\big) \rtimes \Gamma \xrightarrow{\ \cong\ } \mathrm{Aut}(G)$$

with the property that $\Phi_G(\beta_1, \dots, \beta_k) = \beta_1 \times \cdots \times \beta_k$ for each k-tuple of automorphisms $\beta_i \in \mathrm{Aut}(G_i)$. Also, Φ_G sends $\prod_{i=1}^k \mathrm{Inn}(G_i)$ isomorphically to $\mathrm{Inn}(G)$, and hence induces an isomorphism

$$\bar{\Phi}_G : \big(\mathrm{Out}(G_1) \times \cdots \times \mathrm{Out}(G_k)\big) \rtimes \Gamma \xrightarrow{\ \cong\ } \mathrm{Out}(G).$$

To define Φ_G more precisely, fix isomorphisms $\lambda_{ij} : G_i \longrightarrow G_j$ for each $1 \leqslant i < j \leqslant k$ such that $G_i \cong G_j$, chosen so that $\lambda_{i\ell} = \lambda_{j\ell}\lambda_{ij}$ whenever $G_i \cong G_j \cong G_\ell$, and set $\lambda_{ji} = \lambda_{ij}^{-1}$. Also, set $\lambda_{ii} = \mathrm{Id}_{G_i}$ for each i. Then Φ_G can be chosen so that for each $\gamma \in \Gamma$,

$$\Phi_G(\gamma)(g_1, \dots, g_k) = \big(\lambda_{\gamma^{-1}(1),1}(g_{\gamma^{-1}(1)}), \dots, \lambda_{\gamma^{-1}(k),k}(g_{\gamma^{-1}(k)})\big).$$

Proof. Without loss of generality, we can assume that $G_i = G_j$ and $\lambda_{ij} = \mathrm{Id}_{G_i}$ for each i, j such that $G_i \cong G_j$. Thus, $\Phi_G(\gamma)(g_1, \dots, g_k) = (g_{\gamma^{-1}(1)}, \dots, g_{\gamma^{-1}(k)})$ for each $\gamma \in \Gamma$. Then Φ_G is clearly an injective homomorphism, and it factors through a homomorphism $\bar{\Phi}_G$ as above since $\mathrm{Inn}(G) = \Phi_G\big(\prod_{i=1}^k \mathrm{Inn}(G_i)\big)$. Each automorphism of G permutes the factors by Proposition 3.3(b), and hence, Φ_G is surjective. □

In the next proposition, we describe one way to construct linking systems associated to products of fusion systems.

Proposition 3.5. *Assume Hypothesis 3.1. For each $1 \leqslant i \leqslant k$, let $\mathcal{L}_i$ be a centric linking system associated to $\mathcal{F}_i$, with structure functors δ_i and π_i. Let $\mathcal{L}$ be the category whose objects are the $\mathcal{F}$-centric subgroups of S, and where for each $P, Q \in \mathrm{Ob}(\mathcal{L})$,*

$$\mathrm{Mor}_{\mathcal{L}}(P,Q) = \Big\{(\varphi_1, \dots, \varphi_k) \in \prod_{i=1}^k \mathrm{Mor}_{\mathcal{L}_i}(P_i, Q_i) \,\Big|\, (\pi_1(\varphi_1), \dots, \pi_k(\varphi_k))(P) \leqslant Q\Big\}.$$

Define

$$\mathcal{T}_{\mathrm{Ob}(\mathcal{L})}(S) \xrightarrow{\ \delta\ } \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F}$$

by setting, for all $P, Q \in \mathcal{F}^c = \mathrm{Ob}(\mathcal{L})$,

$$\delta_{P,Q}(g) = \big((\delta_1)_{P_1,Q_1}(g_1), \dots, (\delta_k)_{P_k,Q_k}(g_k)\big) \qquad \text{all } g = (g_1, \dots, g_k) \in T_S(P,Q)$$

$$\pi_{P,Q}(\varphi) = \big(\pi_1(\varphi_1), \dots, \pi_k(\varphi_k)\big) \qquad \text{all } \varphi = (\varphi_1, \dots, \varphi_k) \in \mathrm{Mor}_{\mathcal{L}}(P,Q).$$

Then the following hold.

(a) *The functors δ and π make $\mathcal{L}$ into a centric linking system associated to $\mathcal{F}$.*

(b) *Let $\mathcal{L}_1 \times \cdots \times \mathcal{L}_k$ be the product of the* categories *$\mathcal{L}_i$. Define $\xi_{\mathcal{L}} : \mathcal{L}_1 \times \cdots \times \mathcal{L}_k \longrightarrow \mathcal{L}$ by setting*

$$\xi_{\mathcal{L}}(P_1, \dots, P_k) = P_1 \times \cdots \times P_k \qquad \text{and} \qquad \xi_{\mathcal{L}}(\varphi_1, \dots, \varphi_k) = (\varphi_1, \dots, \varphi_k).$$

Then $\xi_{\mathcal{L}}$ is an isomorphism of categories from $\mathcal{L}_1 \times \cdots \times \mathcal{L}_k$ to the full subcategory $\widehat{\mathcal{L}} \subseteq \mathcal{L}$ whose objects are those $P \in \mathcal{F}^c$ such that $P = \widehat{P}$; equivalently, the products $P_1 \times \cdots \times P_k$ for $P_i \in \mathcal{F}_i^c$. Also, the following square commutes:

$$\begin{array}{ccc} \mathcal{T}_{\mathrm{Ob}(\mathcal{L}_1)}(S_1) \times \cdots \times \mathcal{T}_{\mathrm{Ob}(\mathcal{L}_k)}(S_k) & \xrightarrow[\cong]{\eta} & \mathcal{T}_{\mathrm{Ob}(\widehat{\mathcal{L}})}(S) \\ \downarrow{\scriptstyle \delta_1 \times \cdots \times \delta_k} & & \downarrow{\scriptstyle \delta} \\ \mathcal{L}_1 \times \cdots \times \mathcal{L}_k & \xrightarrow{\xi_{\mathcal{L}}} & \mathcal{L} \end{array}$$

where η is the natural isomorphism that sends $(P_1, \dots, P_k)$ to $\prod_{i=1}^k P_i$.

(c) *Let $\rho_i : \mathcal{L}_i \longrightarrow \mathcal{L}$ be the functor that sends $P_i \in \mathrm{Ob}(\mathcal{L}_i)$ to its product with the S_j for $j \neq i$, and sends $\varphi_i \in \mathrm{Mor}(\mathcal{L}_i)$ to its product with Id_{S_j} for $j \neq i$. Then ρ_i is injective on objects and on morphism sets. If $\alpha \in \mathrm{Aut}(\mathcal{L})$ is such that $\alpha_S(\delta_S(S_i)) = \delta_S(S_i)$ for each i, then $\alpha(\rho_i(\mathcal{L}_i)) = \rho_i(\mathcal{L}_i)$ for each i.*

Proof. By Lemma 3.2, for each $P \in \mathrm{Ob}(\mathcal{L}) = \mathcal{F}^c$, $\mathrm{pr}_i(P)$ is $\mathcal{F}_i$-centric for each i. So, the definitions of $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ and δ make sense.

(a) Axiom (A1) is clear. Fix $P, Q \in \mathrm{Ob}(\mathcal{L})$ and set $P_i = \mathrm{pr}_i(P)$ and $Q_i = \mathrm{pr}_i(Q)$; then $C_S(P) = Z(P) = Z(P_1) \times \cdots \times Z(P_k)$. So, by axiom (A2) for the $\mathcal{L}_i$, for $\varphi, \varphi' \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, $\pi(\varphi) = \pi(\varphi')$ if and only if $\varphi' = \varphi \circ \delta_P(z)$ for some $z \in Z(P)$. For each $P, Q \in \mathrm{Ob}(\mathcal{L})$, each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the restriction of some morphism $\prod_{i=1}^k \varphi_i \in \mathrm{Hom}_{\mathcal{F}}(\widehat{P}, \widehat{Q})$ (see [5, Theorem I.6.6]), and hence, the surjectivity of π on morphism sets follows from that of the π_i. The rest of (A2) (the effect of φ and π on objects) is clear. Likewise, axioms (B) and (C) for $\mathcal{L}$ follow immediately from the corresponding axioms for the $\mathcal{L}_i$. Thus, $\mathcal{L}$ is a centric linking system with structure functors δ and π.

(b) Both statements ($\xi_{\mathcal{L}}$ is an isomorphism of categories and the diagram commutes) are immediate from the definitions and since $P_1 \times \cdots \times P_k$ is $\mathcal{F}$-centric if P_i is $\mathcal{F}_i$-centric for each i (Lemma 3.2).

(c) Let $\alpha \in \mathrm{Aut}(\mathcal{L})$ be such that $\alpha(\delta_S(S_i)) = \delta_S(S_i) \leqslant \mathrm{Aut}_{\mathcal{L}}(S)$ for each i. We must show that $\alpha(\rho_i(\mathcal{L}_i)) = \rho_i(\mathcal{L}_i)$ for each i. Fix some i, let $\bar{S}_i \leqslant S$ be the product of the S_j for $j \neq i$, and identify $S_i \times \bar{S}_i$ with S. Thus, $\rho_i(P_i) = P_i \times \bar{S}_i$ and $\rho_i(\varphi_i) = (\varphi_i \times \mathrm{Id}_{\bar{S}_i})$ for $P_i \in \mathrm{Ob}(\mathcal{L}_i)$ and $\varphi_i \in \mathrm{Mor}(\mathcal{L}_i)$.

Set $\beta = \widetilde{\mu}_{\mathcal{L}}(\alpha) \in \mathrm{Aut}(\mathcal{F})$ (see Definition 2.7). By Proposition 2.6, $\alpha(P) = \beta(P)$ for $P \in \mathrm{Ob}(\mathcal{L})$, and $\pi \circ \alpha = c_\beta \circ \pi$ as functors from $\mathcal{L}$ to $\mathcal{F}$. By assumption, $\beta(S_i) = S_i$ and $\beta(\bar{S}_i) = \bar{S}_i$. So, for each $P_i \in \mathrm{Ob}(\mathcal{L}_i)$, we have $\alpha(\rho_i(P_i)) = \beta(P_i \times \bar{S}_i) = P_i^* \times \bar{S}_i$ for some $P_i^* \leqslant S_i$. Thus, α permutes the objects in $\rho_i(\mathcal{L}_i)$.

Now fix a morphism $\varphi_i \in \mathrm{Mor}_{\mathcal{L}_i}(P_i, Q_i)$. Since $\pi\circ\alpha = c_\beta\circ\pi$, we have

$$\pi(\alpha(\rho_i(\varphi_i))) = c_\beta(\pi(\varphi_i \times \mathrm{Id}_{\bar{S}_i})) = c_\beta(\pi(\varphi_i)\times \mathrm{Id}_{\bar{S}_i}) = \pi(\psi)\times \mathrm{Id}_{\bar{S}_i} = \pi(\rho_i(\psi))$$

for some $\psi \in \mathrm{Mor}(\mathcal{L}_i)$. So, by axiom (A2) in Definition 1.7, $\alpha(\rho_i(\varphi_i)) = \rho_i(\psi)\delta_S(z, z')$ for some $z \in Z(P_i)$ and $z' \in Z(\bar{S}_i)$. Since z' has p-power order, this shows that $z' = 1$ and $\alpha(\rho_i(\varphi_i)) \in \mathrm{Mor}(\rho_i(\mathcal{L}_i))$ if φ_i is an automorphism of order prime to p. Also,

$$\alpha(\rho_i(\delta_{S_i}(g_i))) = \alpha(\delta_S(g_i)) = \delta_S(\beta(g_i)) \in \mathrm{Mor}(\rho_i(\mathcal{L}_i))$$

for all $g \in S_i$ since $\beta(g_i) \in \beta(S_i) = S_i$.

By [2, Theorem 1.12], each morphism in $\mathcal{L}_i$ is a composite of restrictions of elements of $\mathrm{Aut}_{\mathcal{L}_i}(P)$ for fully normalized subgroups $P \in \mathcal{F}_i^{cr}$. Also, when P is fully normalized, $\delta_P(N_S(P)) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}_i}(P))$ (see [5, Proposition III.4.2(c)]), and so, $\mathrm{Aut}_{\mathcal{L}_i}(P)$ is generated by $\delta_P(N_S(P))$ and elements of order prime to P. Thus, each morphism in $\mathcal{L}_i$ is a composite of restrictions of automorphisms of order prime to p and elements of $\delta_i(S_i)$, and so, $\alpha(\rho_i(\mathcal{L}_i)) = \rho_i(\mathcal{L}_i)$. □

As one example, if $G_1, \dots, G_k$ are finite groups, $S_i \in \mathrm{Syl}_p(G_i)$, and $\mathcal{L}_i = \mathcal{L}^c_{S_i}(G_i)$, then it is an easy exercise to show that the linking system $\mathcal{L}$ defined in Proposition 3.5 is the centric linking system of $G_1 \times \cdots \times G_k$.

The subcategory $\widehat{\mathcal{L}} \subseteq \mathcal{L}$ defined in Proposition 3.5(b) is not a linking system, since $\mathrm{Ob}(\widehat{\mathcal{L}})$ is not closed under overgroups. However,

$$\mathrm{Ob}(\widehat{\mathcal{L}}) = \left\{P = P_1 \times \cdots \times P_k \,\middle|\, P_i \leqslant S_i,\ P \in \mathcal{F}^c\right\}$$

does include all subgroups of S that are $\mathcal{F}$-centric and $\mathcal{F}$-radical: this is shown in [2, Lemma 3.1] when $k = 2$ and follows in the general case by iteration. So, Proposition 2.6 applies to the automorphism group

$$Aut(\hat{\mathcal{L}}) = \left\{\alpha \in Aut_{\mathrm{cat}}(\hat{\mathcal{L}}) | \alpha(\delta_{P,S}(1)) = \delta_{\alpha(P),S}(1),\ \ \alpha(\delta_P(P)) = \delta_{\alpha(P)}(\alpha(P)) \forall P \in Ob(\hat{\mathcal{L}})\right\}.$$

By analogy with finite groups, a saturated fusion system is *indecomposable* if it is not the direct product of two proper fusion subsystems.

Lemma 3.6. *Assume Hypothesis 3.1. Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be centric linking systems associated to $\mathcal{F}_1, \dots, \mathcal{F}_k$, respectively, and let $\mathcal{L}$ be the centric linking system associated to $\mathcal{F}$ defined as in Proposition 3.5. Let $\widehat{\mathcal{L}} \subseteq \mathcal{L}$ be the full subcategory with objects the subgroups $P_1 \times \cdots \times P_k \leqslant S$ for $P_i \in \mathcal{F}_i^c = \mathrm{Ob}(\mathcal{L}_i)$, and let $\xi_{\mathcal{L}} : \prod_{i=1}^k \mathcal{L}_i \xrightarrow{\cong} \widehat{\mathcal{L}} \leqslant \mathcal{L}$ be as in Proposition 3.5(b). Then*

(a) *$\xi_{\mathcal{L}}$ induces a homomorphism*

$$c_\xi : \mathrm{Aut}(\mathcal{L}_1) \times \cdots \times \mathrm{Aut}(\mathcal{L}_k) \longrightarrow \mathrm{Aut}(\widehat{\mathcal{L}}),$$

which sends $(\alpha_1, \dots, \alpha_k)$ to $\xi_{\mathcal{L}}(\alpha_1 \times \cdots \times \alpha_k)\xi_{\mathcal{L}}^{-1}$;

(b) *each $\widehat{\alpha} \in \mathrm{Aut}(\widehat{\mathcal{L}})$ has a unique extension $E_{\mathcal{L}}(\widehat{\alpha})$ to an automorphism of $\mathcal{L}$, in this way defining an injective homomorphism $E_{\mathcal{L}} : \mathrm{Aut}(\widehat{\mathcal{L}}) \longrightarrow \mathrm{Aut}(\mathcal{L})$; and*

(c) *if $Z(\mathcal{F}_i) = 1$ and $\mathcal{F}_i$ is indecomposable for each i, then $E_{\mathcal{L}}$ is an isomorphism.*

Proof.

(a) This formula clearly defines a homomorphism to the group $\mathrm{Aut}_{\mathrm{cat}}(\widehat{\mathcal{L}})$ of all automorphisms of $\widehat{\mathcal{L}}$ as a category. That $\xi_{\mathcal{L}}(\alpha_1 \times \cdots \times \alpha_k)\xi_{\mathcal{L}}^{-1}$ (for $\alpha_i \in \mathrm{Aut}(\mathcal{L}_i)$) sends inclusions in $\widehat{\mathcal{L}}$ to inclusions and sends distinguished subgroups to distinguished subgroups follows from the commutativity of the square in Proposition 3.5(b).

(b) We will show that each $\widehat{\alpha} \in \mathrm{Aut}(\widehat{\mathcal{L}})$ extends to some $\alpha \in \mathrm{Aut}(\mathcal{L})$. By the definition in Proposition 3.5, each morphism in $\mathcal{L}$ is a restriction of a morphism in $\widehat{\mathcal{L}}$, and hence, there is at most one such extension. So, upon setting $E_{\mathcal{L}}(\widehat{\alpha}) = \alpha$, we get a well-defined injective homomorphism from $\mathrm{Aut}(\widehat{\mathcal{L}})$ to $\mathrm{Aut}(\mathcal{L})$.

Fix $\widehat{\alpha} \in \mathrm{Aut}(\widehat{\mathcal{L}})$, and let $\beta = \widetilde{\mu}_{\mathcal{L}}(\widehat{\alpha}) \in \mathrm{Aut}(\mathcal{F})$ be the automorphism of Proposition 2.6 and Definition 2.7. Thus, $\pi \circ \widehat{\alpha} = c_\beta \circ \pi$, and $\widehat{\alpha}(P) = \beta(P)$ for all $P \in \mathrm{Ob}(\widehat{\mathcal{L}})$. In terms of $\mathcal{L}$ and $\widehat{\mathcal{L}}$, the definition of morphisms in Proposition 3.5 takes the form

$$\mathrm{Mor}_{\mathcal{L}}(P,Q) = \left\{\psi \in \mathrm{Mor}_{\widehat{\mathcal{L}}}(\widehat{P},\widehat{Q}) \,\middle|\, \pi(\psi)(P) \leqslant Q\right\} \tag{3.1}$$

for all $P, Q \in \mathrm{Ob}(\mathcal{L})$, where $\widehat{P}$ and $\widehat{Q}$ are as in Hypothesis 3.1. Also, $\beta(\widehat{P}) = \widehat{\beta(P)}$ for each $P \in \mathrm{Ob}(\mathcal{L})$, since $\widehat{P}$ and $\widehat{\beta(P)}$ are the unique minimal objects of $\widehat{\mathcal{L}}$ containing P and $\beta(P)$, and since β permutes the objects of $\widehat{\mathcal{L}}$ and of $\mathcal{L}$. (We are *not* assuming here that β permutes the factors S_i.)

For all $P, Q \in \mathrm{Ob}(\mathcal{L})$ and $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q) \subseteq \mathrm{Mor}_{\widehat{\mathcal{L}}}(\widehat{P},\widehat{Q})$, we have

$$\pi(\widehat{\alpha}(\psi))(\beta(P)) = c_\beta(\pi(\psi))(\beta(P)) = \beta(\pi(\psi)(P)) \leqslant \beta(Q)\ :$$

the first equality since $\pi \circ \widehat{\alpha} = c_\beta \circ \pi$ and the inequality since $\pi(\psi)(P) \leqslant Q$ by (3.1). So, $\widehat{\alpha}(\psi) \in \mathrm{Mor}_{\mathcal{L}}(\beta(P), \beta(Q))$ by (3.1) again. We can thus define $\alpha \in \mathrm{Aut}(\mathcal{L})$ extending $\widehat{\alpha}$ by setting $\alpha(P) = \beta(P)$ for all P, and letting $\alpha_{P,Q}$ be the restriction of $\widehat{\alpha}_{\widehat{P},\widehat{Q}}$ for $P, Q \in \mathrm{Ob}(\mathcal{L})$.

(c) Now assume that $Z(\mathcal{F}_i) = 1$ and $\mathcal{F}_i$ is indecomposable for each i. By [37, Corollary 5.3], $\mathcal{F}$ has a unique factorization as a product of indecomposable fusion systems.

Fix $\alpha \in \mathrm{Aut}(\mathcal{L})$, and set $\beta = \widetilde{\mu}_{\mathcal{L}}(\alpha) \in \mathrm{Aut}(\mathcal{F})$ (Definition 2.7). By the uniqueness of the factorization, there is $\gamma \in \Sigma_k$ such that $c_\beta(\mathcal{F}_i) = \mathcal{F}_{\gamma(i)}$ for each $1 \leqslant i \leqslant k$. In particular, $\beta(S_i) = S_{\gamma(i)}$ for each i. So, for each object $P = P_1 \times \cdots \times P_k$ in $\widehat{\mathcal{L}}$, $\beta(P) = \prod_{i=1}^k \beta(P_i)$ is also an object in $\widehat{\mathcal{L}}$. Hence, $\alpha(\widehat{\mathcal{L}}) = \widehat{\mathcal{L}}$ and $\alpha = E_{\mathcal{L}}(\alpha|_{\widehat{\mathcal{L}}})$. Since $\alpha \in \mathrm{Aut}(\mathcal{L})$ was arbitrary, $E_{\mathcal{L}}$ is onto. □

We are now ready to prove our main results concerning the automorphism group of a product of linking systems.

Proposition 3.7. *Assume Hypothesis 3.1. Let $\mathcal{L}_i$ be a centric linking system associated to $\mathcal{F}_i$ for each $1 \leqslant i \leqslant k$, and let $\mathcal{L}$ be the centric linking system associated to $\mathcal{F}$ defined as in Proposition 3.5. Set*

$$\Gamma = \left\{\sigma \in \Sigma_k \,\middle|\, \mathcal{F}_{\sigma(i)} \cong \mathcal{F}_i \text{ for each } 1 \leqslant i \leqslant k\right\}.$$

(a) *There is an injective homomorphism*

$$\Phi_{\mathcal{L}} : \left(\mathrm{Aut}(\mathcal{L}_1) \times \cdots \times \mathrm{Aut}(\mathcal{L}_k)\right) \rtimes \Gamma \longrightarrow \mathrm{Aut}(\mathcal{L})$$

with the property that for each $(\alpha_1,\dots,\alpha_k)\in\prod_{i=1}^k \mathrm{Aut}(\mathcal{L}_i)$, $(P_1,\dots,P_k)\in\prod_{i=1}^k \mathrm{Ob}(\mathcal{L}_i)$, *and* $(\varphi_1,\dots,\varphi_k)\in\prod_{i=1}^k \mathrm{Mor}(\mathcal{L}_i)$, *we have*

$$\begin{aligned}\Phi_{\mathcal{L}}\big(\alpha_1,\dots,\alpha_k\big)\big(P_1\times\cdots\times P_k\big) &= \alpha_1(P_1)\times\cdots\times\alpha_k(P_k)\\ \Phi_{\mathcal{L}}\big(\alpha_1,\dots,\alpha_k\big)\big(\varphi_1,\dots,\varphi_k\big) &= \big(\alpha_1(\varphi_1),\dots,\alpha_k(\varphi_k)\big)\end{aligned}\tag{3.2}$$

for $\alpha_i\in\mathrm{Aut}(\mathcal{L}_i)$. *Furthermore,*

$$\Phi_{\mathcal{L}}\big(\textstyle\prod_{i=1}^k \mathrm{Aut}(\mathcal{L}_i)\big) = \mathrm{Aut}^0(\mathcal{L}) \overset{\text{def}}{=} \big\{\alpha\in\mathrm{Aut}(\mathcal{L})\,\big|\,\alpha_S(\delta_S(S_i))=\delta_S(S_i)\ \textit{for each}\ 1\leqslant i\leqslant k\big\}.$$

(b) *If* $Z(\mathcal{F})=1$, *and* $\mathcal{F}_i$ *is indecomposable for each* i, *then* $\Phi_{\mathcal{L}}$ *is an isomorphism, and induces an isomorphism*

$$\bar{\Phi}_{\mathcal{L}}:\ \big(\mathrm{Out}(\mathcal{L}_1)\times\cdots\times\mathrm{Out}(\mathcal{L}_k)\big)\rtimes\Gamma \xrightarrow{\ \cong\ } \mathrm{Out}(\mathcal{L}).$$

(c) *Assume that* $Z(\mathcal{F})=1$, *and that* $\mathcal{F}_i$ *is indecomposable and tame for each* i. *Then* $\mathcal{F}$ *is tame. If* $G_1,\dots,G_k$ *are finite groups such that* $O_{p'}(G_i)=1$ *and* $\mathcal{F}_i$ *is tamely realized by* G_i *for each* i, *and such that* $\mathcal{F}_i\cong\mathcal{F}_j$ *implies* $G_i\cong G_j$, *then* $\mathcal{F}$ *is tamely realized by the product* $G_1\times\cdots\times G_k$.

Proof. Let $\widehat{\mathcal{L}}\subseteq\mathcal{L}$ be the full subcategory defined in Proposition 3.5(b). Thus, $\mathrm{Ob}(\widehat{\mathcal{L}})$ is the set of all $P\in\mathrm{Ob}(\mathcal{L})=\mathcal{F}^c$ such that $P=\widehat{P}$.

Without loss of generality, for each pair of indices i,j such that $\mathcal{F}_i\cong\mathcal{F}_j$, we can assume that $\mathcal{F}_i=\mathcal{F}_j$ and $S_i=S_j$. Then, $\mathcal{L}_i\cong\mathcal{L}_j$ by Theorem 1.9, and so, we can also assume that $\mathcal{L}_i=\mathcal{L}_j$.

(a) Define $\Phi_{\mathcal{L}}$ to be the composite

$$\Phi_{\mathcal{L}}:\ \big(\mathrm{Aut}(\mathcal{L}_1)\times\cdots\times\mathrm{Aut}(\mathcal{L}_k)\big)\rtimes\Gamma \xrightarrow{\ \widehat{c}_\xi\ } \mathrm{Aut}(\widehat{\mathcal{L}}) \xrightarrow{\ E_{\mathcal{L}}\ } \mathrm{Aut}(\mathcal{L}),$$

where $E_{\mathcal{L}}$ is the homomorphism of Lemma 3.6(b), where the restriction of $\widehat{c}_\xi$ to $\prod_{i=1}^k\mathrm{Aut}(\mathcal{L}_i)$ is the homomorphism c_ξ of Lemma 3.6(a), and where

$$\widehat{c}_\xi(\gamma)(\varphi_1,\dots,\varphi_k)=(\varphi_{\gamma^{-1}(1)},\dots,\varphi_{\gamma^{-1}(k)})\tag{3.3}$$

for $\gamma\in\Gamma$ and $\varphi_i\in\mathrm{Mor}(\mathcal{L}_i)$. Then (3.2) holds by the definition of c_ξ. One easily checks using (3.2) and (3.3) that $\Phi_{\mathcal{L}}$ is an injective homomorphism.

It remains to check that $\Phi_{\mathcal{L}}\big(\prod_{i=1}^k\mathrm{Aut}(\mathcal{L}_i)\big)=\mathrm{Aut}^0(\mathcal{L})$: the subgroup of those $\alpha\in\mathrm{Aut}(\mathcal{L})$ such that $\alpha_S(\delta_S(S_i))=\delta_S(S_i)$ for each i. The inclusion of the first group in $\mathrm{Aut}^0(\mathcal{L})$ is clear. By Proposition 3.5(c), there are embeddings of categories $\rho_i:\ \mathcal{L}_i\longrightarrow\mathcal{L}$ sending $P_i\leqslant S_i$ to its product with the S_j for all $j\neq i$, and $\alpha(\rho_i(\mathcal{L}_i))=\rho_i(\mathcal{L}_i)$ for each $\alpha\in\mathrm{Aut}^0(\mathcal{L})$. We can thus define $\Psi_{\mathcal{L},i}:\ \mathrm{Aut}^0(\mathcal{L})\longrightarrow\mathrm{Aut}(\mathcal{L}_i)$ by sending α to $\rho_i^{-1}\alpha\rho_i$. Set $\Psi^0_{\mathcal{L}}=(\Psi_{\mathcal{L},1},\dots,\Psi_{\mathcal{L},k})$; then $\Phi_{\mathcal{L}}\circ\Psi^0_{\mathcal{L}}$ is the inclusion of $\mathrm{Aut}^0(\mathcal{L})$ into $\mathrm{Aut}(\mathcal{L})$, and so, $\mathrm{Aut}^0(\mathcal{L})\leqslant\mathrm{Im}(\Phi_{\mathcal{L}})$.

(b) Fix $\alpha\in\mathrm{Aut}(\mathcal{L})$, and set $\beta=\widetilde{\mu}_{\mathcal{L}}(\alpha)\in\mathrm{Aut}(\mathcal{F})$ (see Definition 2.7). By [2, Proposition 3.6] and since $Z(\mathcal{F})=1$ and the $\mathcal{F}_i$ are indecomposable, c_β permutes the factors $\mathcal{F}_i$.

Let $\gamma \in \Sigma_k$ be such that $c_\beta(\mathcal{F}_i) = \mathcal{F}_{\gamma(i)}$ for each i, and hence also $\beta(S_i) = S_{\gamma(i)}$ and $\alpha_S(\delta_S(S_i)) = \delta_S(S_{\gamma(i)})$. In particular, $\gamma \in \Gamma$. Then $\Phi_{\mathcal{L}}(\gamma)^{-1} \circ \alpha \in \mathrm{Aut}^0(\mathcal{L})$, and since $\mathrm{Aut}^0(\mathcal{L}) \leqslant \mathrm{Im}(\Phi_{\mathcal{L}})$ by (a), this shows that $\mathrm{Aut}(\mathcal{L}) \leqslant \mathrm{Im}(\Phi_{\mathcal{L}})$ and hence that $\Phi_{\mathcal{L}}$ is onto.

Since $\mathrm{Aut}_{\mathcal{L}}(S) = \mathrm{Aut}_{\mathcal{L}_1}(S_1) \times \cdots \times \mathrm{Aut}_{\mathcal{L}_k}(S_k)$, $\Phi_{\mathcal{L}}$ induces an isomorphism of quotient groups

$$\bar{\Phi}_{\mathcal{L}} : \big(\mathrm{Out}(\mathcal{L}_1) \times \cdots \times \mathrm{Out}(\mathcal{L}_k)\big) \rtimes \Gamma \xrightarrow{\cong} \mathrm{Out}(\mathcal{L}).$$

(c) Assume now that $Z(\mathcal{F}) = 1$, and that $\mathcal{F}_i$ is indecomposable and tame for each i. Let $G_1, \dots, G_k$ be such that $O_{p'}(G_i) = 1$ and $\mathcal{F}_i$ is tamely realized by G_i for each i, and such that $\mathcal{F}_i \cong \mathcal{F}_j$ implies $G_i \cong G_j$. For each i, $Z(G_i)$ is a p-group and hence $Z(G_i) \leqslant Z(\mathcal{F}_i) = 1$ by Lemma 1.6(b). Without loss of generality, we can assume that $G_i = G_j$ whenever $G_i \cong G_j$ and also (by the uniqueness of linking systems again) that $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$ and $\mathcal{L}_i = \mathcal{L}^c_{S_i}(G_i)$ for each i. Note that each G_i is indecomposable: since $O_{p'}(G_i) = 1$, a nontrivial factorization of G_i would induce a nontrivial factorization of $\mathcal{F}_i$.

Set $\kappa_i = \kappa_{G_i} : \mathrm{Out}(G_i) \longrightarrow \mathrm{Out}(\mathcal{L}_i)$ for short. Fix splittings $s_i : \mathrm{Out}(\mathcal{L}_i) \longrightarrow \mathrm{Out}(G_i)$ for all i, chosen so that $s_i = s_j$ if $G_i = G_j$.

Consider the following diagram:

$$\begin{array}{ccc}
\big(\mathrm{Out}(\mathcal{L}_1) \times \cdots \times \mathrm{Out}(\mathcal{L}_k)\big) \rtimes \Gamma & \xrightarrow[\cong]{\bar{\Phi}_{\mathcal{L}}} & \mathrm{Out}(\mathcal{L}) \\
\downarrow{\scriptstyle (s_1,\dots,s_k)\rtimes \mathrm{Id}_\Gamma} & & \downarrow{\scriptstyle s} \\
\big(\mathrm{Out}(G_1) \times \cdots \times \mathrm{Out}(G_k)\big) \rtimes \Gamma & \xrightarrow[\cong]{\bar{\Phi}_{G}} & \mathrm{Out}(G) \\
\downarrow{\scriptstyle (\kappa_1,\dots,\kappa_k)\rtimes \mathrm{Id}_\Gamma} & & \downarrow{\scriptstyle \kappa_G} \\
\big(\mathrm{Out}(\mathcal{L}_1) \times \cdots \times \mathrm{Out}(\mathcal{L}_k)\big) \rtimes \Gamma & \xrightarrow[\cong]{\bar{\Phi}_{\mathcal{L}}} & \mathrm{Out}(\mathcal{L})
\end{array}
\qquad \mathrm{Id}$$

where $\bar{\Phi}_G$ is the isomorphism of Proposition 3.4 (as defined when taking $\lambda_{ij} = \mathrm{Id}_{G_i}$ for each $i < j$ such that $G_i = G_j$), where s is defined to make the top square commute, and where the commutativity of the bottom square is immediate from the definitions. Thus, $\kappa_G \circ s = \mathrm{Id}_{\mathrm{Out}(\mathcal{L})}$, so κ_G is split surjective, and $\mathcal{F}$ is tamely realized by G. □

4 | COMPONENTS OF GROUPS AND OF FUSION SYSTEMS

In this section, we set up some tools that will be used later when proving inductively that all realizable fusion systems are tame. The starting point for the inductive procedure is Theorem 4.5, which summarizes the main results in [38]. Note that if $\mathcal{F}$ is a saturated fusion system such that $O^{p'}(\mathcal{F})$ is simple, then for each finite group G with $O_{p'}(G) = 1$ that realizes $\mathcal{F}$, $O^{p'}(G)$ is simple (so G is almost simple), and $O^{p'}(G)$ realizes $\mathcal{F}$ if $\mathcal{F}$ is simple.

Let $\mathrm{Comp}(G)$ denote the set of components of a finite group G; that is, the set of subnormal subgroups of G that are quasi-simple. (Recall that a subgroup H of G is subnormal, denoted as $H \trianglelefteq\trianglelefteq G$, if there is a sequence $H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_k = G$ with each subgroup normal in the following one, and H is quasi-simple if H is perfect and $H/Z(H)$ is simple.) The components

of G commute with each other pairwise (see [3, § 31] or [5, Lemma A.12]). In particular, when $O_q(G) = 1$ for all primes q, they are all simple groups, and the subgroup $E(G) \overset{\text{def}}{=} \langle \mathrm{Comp}(G) \rangle$ is their direct product.

Similarly, the components of a saturated fusion system $\mathcal{F}$ over a finite p-group S are its subnormal fusion subsystems $\mathcal{C} \trianglelefteq\trianglelefteq \mathcal{F}$ that are quasi-simple (i.e., $O^p(\mathcal{C}) = \mathcal{C}$ and $\mathcal{C}/Z(\mathcal{C})$ is simple). The set of components of $\mathcal{F}$ will be denoted as $\mathrm{Comp}(\mathcal{F})$.

By analogy with the case for groups, a *central product* of fusion systems $\mathcal{E}_1, \dots, \mathcal{E}_k$ is a fusion system $\mathcal{E} \cong (\mathcal{E}_1 \times \cdots \times \mathcal{E}_k)/Z$, for some central subgroup $Z \leqslant \prod_{i=1}^k Z(\mathcal{E}_i)$ that intersects trivially with each factor $Z(\mathcal{E}_i)$. More precisely, if $\mathcal{F}$ is a fusion system over S and $\mathcal{E}_1, \dots, \mathcal{E}_k \leqslant \mathcal{F}$ are fusion subsystems over $T_1, \dots, T_k \leqslant S$, then the subsystems commute in $\mathcal{F}$ if the T_i commute pairwise, and for each k-tuple of morphisms $(\varphi_1, \dots, \varphi_k)$, where $\varphi_i \in \mathrm{Hom}_{\mathcal{E}_i}(P_i, Q_i)$, there is a morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P_1 \cdots P_k, Q_1 \cdots Q_k)$ that extends each of the φ_i. Note, in particular, that

$$\mathcal{E}_1, \dots, \mathcal{E}_k \text{ commute } \Rightarrow \mathcal{E}_i \leqslant C_{\mathcal{F}}(T_j) \text{ for each } i \neq j.$$

In this situation, the (internal) central product of the $\mathcal{E}_i$ is the fusion subsystem

$$\mathcal{E}_1 \cdots \mathcal{E}_k = \Big\langle \bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P_1 \cdots P_k, Q_1 \cdots Q_k) \,\Big|\, P_i, Q_i \leqslant T_i, \ \bar{\varphi}|_{P_i} \in \mathrm{Hom}_{\mathcal{E}_i}(P_i, Q_i)\ \forall\, 1 \leqslant i \leqslant k \Big\rangle \leqslant \mathcal{F}$$

over $T_1 \cdots T_k \leqslant S$. See Definition 2.4 and Lemma 2.8 in [37] for some more details, and see [26, Proposition 3.3] for a slightly different approach to defining central products of fusion subsystems.

By [4, 9.8–9.9], the components of a saturated fusion system $\mathcal{F}$ commute, and also commute with $O_p(\mathcal{F})$. So, by analogy with finite groups, when $\mathcal{F}$ is a saturated fusion system over a finite p-group and $\mathrm{Comp}(\mathcal{F}) = \{\mathcal{C}_1, \dots, \mathcal{C}_k\}$, one defines

$$E(\mathcal{F}) = \mathcal{C}_1 \cdots \mathcal{C}_k \qquad \text{and} \qquad F^*(\mathcal{F}) = E(\mathcal{F}) O_p(\mathcal{F})$$

(central products). In particular, $F^*(\mathcal{F})$ is the *generalized Fitting subsystem* of $\mathcal{F}$.

Note that when $O_p(\mathcal{F}) = 1$, the components of $\mathcal{F}$ are all simple, and $F^*(\mathcal{F}) = E(\mathcal{F})$ is their direct product.

Lemma 4.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then*

(a) *$E(\mathcal{F})$ is characteristic in $\mathcal{F}$;*
(b) *$F^*(\mathcal{F})$ is characteristic and centric in $\mathcal{F}$; and*
(c) *if $\mathcal{E} \trianglelefteq \mathcal{F}$ and $\mathcal{C} \in \mathrm{Comp}(\mathcal{F}) \setminus \mathrm{Comp}(\mathcal{E})$, then $\mathcal{F}$ contains a central product of $\mathcal{C}$ and $\mathcal{E}$.*

Proof. These are all shown in Chapter 9 of [4]: point (a) in 9.8.1 and 9.8.2, point (b) in 9.9 and 9.11, and (c) in 9.13. More precisely, $F^*(\mathcal{F})$ is centric in $\mathcal{F}$ since $C_S(F^*(\mathcal{F})) = Z(F^*(\mathcal{F}))$ by [4, 9.11]. □

Lemma 4.2. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be a normal pair of saturated fusion systems over finite p-groups. Then*

(a) *$\mathrm{Comp}(\mathcal{E})$ is equal to the set of all $\mathcal{C} \in \mathrm{Comp}(\mathcal{F})$ such that $\mathcal{C} \leqslant \mathcal{E}$;*
(b) *if $\mathcal{E}$ is centric in $\mathcal{F}$ or has p-power index in $\mathcal{F}$, then $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$; and*
(c) *if $O_p(\mathcal{F}) = 1$ and $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$, then $\mathcal{E}$ is centric in $\mathcal{F}$.*

Proof.

(a,b) In all cases, each fusion subsystem subnormal in $\mathcal{E}$ is subnormal in $\mathcal{F}$, and hence $\mathrm{Comp}(\mathcal{E}) \subseteq \mathrm{Comp}(\mathcal{F})$. If $\mathcal{C} \in \mathrm{Comp}(\mathcal{F}) \setminus \mathrm{Comp}(\mathcal{E})$, then by Lemma 4.1(c), $\mathcal{F}$ contains a central product of $\mathcal{C}$ and $\mathcal{E}$, and, in particular, $\mathcal{C} \not\leq \mathcal{E}$. This proves (a), and also shows that $\mathcal{E}$ is not centric in $\mathcal{F}$ in this case, proving the first part of (b). If $\mathcal{E}$ has p-power index in $\mathcal{F}$, then $\mathcal{F}$ cannot be a central product of $\mathcal{E}$ with a quasi-simple system, so $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$ also in this case.

(c) If $O_p(\mathcal{F}) = 1$ and $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$, then $\mathcal{E} \geqslant E(\mathcal{F}) = F^*(\mathcal{F})$ is the generalized Fitting subsystem of $\mathcal{F}$. Since $F^*(\mathcal{F})$ is centric in $\mathcal{F}$ by Lemma 4.1(b), so is $\mathcal{E}$. □

In the proof of the next lemma, we need to work with the centralizer fusion subsystem $C_{\mathcal{F}}(\mathcal{E})$ of a normal fusion subsystem $\mathcal{E} \trianglelefteq \mathcal{F}$ over $T \trianglelefteq S$. This was defined by Henke [26] to be the unique fusion subsystem over $C_S(\mathcal{E})$ of p-power index in $C_{\mathcal{F}}(T)$. (There is such a subsystem by [5, Theorem I.7.4] and since $C_S(\mathcal{E}) \geqslant \mathfrak{foc}(C_{\mathcal{F}}(T))$ by [26, Proposition 1].) By [26, Proposition 6.3], it is equal to the subsystem $C_{\mathcal{F}}(\mathcal{E})$ defined by Aschbacher in [4, Chapter 6].

Lemma 4.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Set $\mathrm{Comp}(\mathcal{F}) = \{C_1, \dots, C_k\}$ where C_i is a fusion subsystem over U_i for each $1 \leqslant i \leqslant k$. Let $Z \leqslant Z(\mathcal{F})$ be a central subgroup. Then $\mathrm{Comp}(\mathcal{F}/Z) = \{ZC_1/Z, \dots, ZC_k/Z\}$.*

Proof. Set $\mathrm{Comp}_0(\mathcal{F}/Z) = \{ZC_1/Z, \dots, ZC_k/Z\}$ for short. For each i, $ZC_i/Z \trianglelefteq\trianglelefteq \mathcal{F}/Z$ by Lemmas 1.21 and 1.18 and since $C_i \trianglelefteq\trianglelefteq \mathcal{F}$. Also, $ZC_i/Z \cong C_i/(Z \cap U_i)$ by Lemma 1.22 and hence is quasi-simple. Thus, $ZC_i/Z \in \mathrm{Comp}(\mathcal{F}/Z)$ for each i, and $\mathrm{Comp}(\mathcal{F}/Z) \supseteq \mathrm{Comp}_0(\mathcal{F}/Z)$.

It remains to prove the opposite inclusion. Set $\mathcal{E} = E(\mathcal{F})$: the central product of the C_i. It is a saturated fusion system over $U = U_1 \cdots U_k$, and is normal in $\mathcal{F}$ by Lemma 4.1(a). Set $K = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(ZU) \mid [\alpha, U] \leqslant Z\}$: a p-group of automorphisms by [20, Corollary 5.3.3]. Each $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}/Z}(ZU/Z)}(P/Z, Q/Z)$ (for $Z \leqslant P, Q \leqslant N_S^K(ZU)$) extends to $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}/Z}(PU/Z, QU/Z)$ such that $\bar{\varphi}|_{ZU/Z} = \mathrm{Id}$, and this, in turn, lifts to $\psi \in \mathrm{Hom}_{\mathcal{F}}(PU, QU)$ with $\psi|_{ZU} \in K$. Thus, $N_{\mathcal{F}}^K(ZU)/Z = C_{\mathcal{F}/Z}(ZU/Z)$.

Recall that K is a p-group. Hence, by Lemma 1.16, the centralizer $C_{\mathcal{F}}(ZU)$ is normal of p-power index in $N_{\mathcal{F}}^K(ZU)$. Also, $C_{\mathcal{F}}(\mathcal{E})$ has p-power index in $C_{\mathcal{F}}(U) = C_{\mathcal{F}}(ZU)$ by Henke's definition in [26], and so, we have inclusions

$$C_{\mathcal{F}}(\mathcal{E})/Z \leqslant C_{\mathcal{F}}(ZU)/Z \leqslant N_{\mathcal{F}}^K(ZU)/Z = C_{\mathcal{F}/Z}(ZU/Z),$$

each of p-power index in the next. Each component of $\mathcal{F}/Z$ not in $\mathrm{Comp}_0(\mathcal{F}/Z)$ commutes with the ZC_i/Z, hence is contained in $C_{\mathcal{F}/Z}(ZU/Z)$, hence is contained in $C_{\mathcal{F}}(\mathcal{E})/Z$ by Lemma 4.2(b), and hence lies in $\mathrm{Comp}(C_{\mathcal{F}}(\mathcal{E})/Z)$ by Lemma 4.2(a) and since $C_{\mathcal{F}}(\mathcal{E})/Z \trianglelefteq \mathcal{F}/Z$ by Lemma 1.18.

By [4, 9.12.3] and since $\mathcal{E} = E(\mathcal{F})$, the centralizer subsystem $C_{\mathcal{F}}(\mathcal{E})$ is constrained. Hence, $C_{\mathcal{F}}(\mathcal{E})/Z$ is also constrained by [27, Lemma 2.10], so $\mathrm{Comp}(C_{\mathcal{F}}(\mathcal{E})/Z) = \emptyset$, finishing the proof that $\mathrm{Comp}(\mathcal{F}/Z) = \mathrm{Comp}_0(\mathcal{F}/Z)$. □

In the next lemma, we use the following notation to describe certain automorphism groups. For a fixed prime p and integers $k \mid m \mid (p-1)$ and $n, \ell \geqslant 1$, let $G(m,k,n) \leqslant GL_n(\mathbb{Z}/p^\ell)$ be the subgroup

$$G(m,k,n) = \left\{\mathrm{diag}(u_1, \dots, u_n) \in GL_n(\mathbb{Z}/p^\ell) \,\middle|\, u_i^m = 1\ \forall i,\ (u_1 \cdots u_n)^{m/k} = 1\right\} \cdot \mathfrak{Perm}(n),$$

where $\mathfrak{Perm}(n) \simeq \Sigma_n$ is the group of all permutation matrices. Thus, $G(m,1,n) \cong C_m \wr \Sigma_n$, the group of all monomial matrices whose nonzero entries are mth roots of unity in $\mathbb{Z}/p^\ell$, and $G(m,k,n)$ has index k in $G(m,1,n)$.

The following is a version of [38, Lemma 4.7] that has been reformulated so as to not depend on the classification of finite simple groups.

Lemma 4.4. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, for some prime $p \geqslant 5$, and assume $A \trianglelefteq S$ is abelian and $\mathcal{F}$-centric. Assume also, for some $\ell \geqslant 1$, $\kappa \geqslant p$, and $2 < m \mid (p-1)$, that A is homocyclic of rank κ and exponent p^ℓ, and that with respect to some basis $\{a_1, \dots, a_\kappa\}$ for A as a $\mathbb{Z}/p^\ell$-module, $\mathrm{Aut}_{\mathcal{F}}(A)$ contains $G(m,m,\kappa)$ with index prime to p, and*

$$\mathrm{Aut}_{\mathcal{F}}(A) \cap G(m,1,\kappa) = G(m,r,\kappa) \leqslant GL_\kappa(\mathbb{Z}/p^\ell) \qquad \text{for some } 2 < r \mid m.$$

Then either $A \trianglelefteq \mathcal{F}$, or $O^{p'}(\mathcal{F})$ is simple and $\mathcal{F}$ is not realized by any known finite almost simple group.

Proof. Since $\mathrm{Aut}_{\mathcal{F}}(A)$ contains $G(m,m,\kappa)$ with index prime to p, some subgroup conjugate to $\mathrm{Aut}_S(A)$ is contained in $G(1,1,\kappa) \cong \Sigma_\kappa$, and hence, $\mathrm{Aut}_S(A)$ permutes some basis of $\Omega_1(A)$. Also, $G(m,m,\kappa)$ acts faithfully on $\Omega_1(A)$, as does each subgroup of $\mathrm{Aut}_{\mathcal{F}}(A)$ of order prime to p (see [20, Theorem 5.2.4]). So, by the assumptions on $\mathrm{Aut}_{\mathcal{F}}(A)$,

$$\begin{aligned} &\mathrm{Aut}_{\mathcal{F}}(A) \text{ acts faithfully on } \Omega_1(A), \ \mathrm{Aut}_S(A) \text{ permutes} \\ &\text{a basis of } \Omega_1(A), \text{ and } C_S(\Omega_1(A)) = A. \end{aligned} \tag{4.1}$$

We next claim that

$$\Omega_1(A) \text{ is the only elementary abelian subgroup of } S \text{ of rank } \kappa. \tag{4.2}$$

This is well known, but the proof is simple enough that we give it here. Set $V = \Omega_1(A)$ for short, let $W \leqslant S$ be another elementary abelian subgroup, and set $\overline{W} = \mathrm{Aut}_W(V)$ and $r = \mathrm{rk}(\overline{W})$. Then $r = \mathrm{rk}(W/C_W(V))$ where $C_W(V) = W \cap A = W \cap V$ by (4.1). Let $\mathcal{B}$ be a basis for V permuted by $\overline{W}$, and assume that $\overline{W}$ acts on $\mathcal{B}$ with s orbits (including fixed orbits) of lengths $p^{m_1}, \dots, p^{m_s}$. Then $p^r = |\overline{W}| \leqslant p^{m_1} \cdots p^{m_s}$, and hence $m_1 + \cdots + m_s \geqslant r$. So,

$$\mathrm{rk}(W) = r + \mathrm{rk}(W \cap V) \leqslant r + \mathrm{rk}(C_V(\overline{W})) = r + s \leqslant \sum_{i=1}^{s}(m_i+1) < \sum_{i=1}^{s} p^{m_i} = \mathrm{rk}(V),$$

proving (4.2). In particular, $\Omega_1(A)$ and $A = C_S(\Omega_1(A))$ are weakly closed in $\mathcal{F}$.

Set $G_0(m,m,\kappa) = O^{p'}(G(m,m,\kappa)) \cong (C_m)^{\kappa-1} \rtimes A_\kappa$: the unique subgroup of index 2 in $G(m,m,\kappa)$. There are exactly κ one-dimensional subspaces of $\Omega_1(A)$ invariant under the action of $O_{p'}(G_0(m,m,\kappa)) \cong (C_m)^{\kappa-1}$, and they are permuted transitively by the alternating group A_κ. Hence,

$$\Omega_1(A) \text{ is a simple } \mathbb{F}_p[G_0(m,m,\kappa)]\text{-module.} \tag{4.3}$$

Set $\mathcal{F}_0 = O^{p'}(\mathcal{F})$. Set $\Gamma = \mathrm{Aut}_{\mathcal{F}}(A)$ and $\Gamma_0 = \mathrm{Aut}_{\mathcal{F}_0}(A)$. Thus, $\Gamma_0 \geqslant G_0(m,m,\kappa)$ since $\Gamma \geqslant G(m,m,\kappa)$.

Step 1: Assume that $\mathcal{F}_0$ is not simple, and let $\mathcal{E} \trianglelefteq \mathcal{F}_0$ be a proper nontrivial normal subsystem over $1 \neq T \trianglelefteq S$. Then T is strongly closed in $\mathcal{F}_0$, so $T \cap A$ is normalized by the action of Γ_0 on A, and $T \cap A = \Omega_k(A)$ for some $1 \leqslant k \leqslant \ell$ since $\Omega_1(A)$ is simple by (4.3). Also, $T/\Omega_k(A)$ is normal in $S/\Omega_k(A)$, so if $k < \ell$ and $T > \Omega_k(A)$, then $T/\Omega_k(A) \cap Z(S/\Omega_k(A)) \neq 1$. Since $Z(S/\Omega_k(A)) \leqslant A/\Omega_k(A)$, this implies that $T \cap A > \Omega_k(A)$, contradicting the choice of k. Thus, either $T = \Omega_k(A)$ for some $1 \leqslant k \leqslant \ell$, or $T > A$.

If $T = \Omega_k(A)$ for some $k \leqslant \ell$, then since T is abelian and strongly closed, $T = \Omega_k(A) \trianglelefteq \mathcal{F}$ by [5, Corollary I.4.7(a)]. Hence, for each $a \in A$ and each $x \in a^{\mathcal{F}}$, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\Omega_k(A)\langle a\rangle, S)$ such that $\varphi(a) = x$ and $\varphi(\Omega_k(A)) = \Omega_k(A)$. Then $x \in C_S(\Omega_1(A)) = A$ by (4.1), so A is strongly closed in this case, and $A \trianglelefteq \mathcal{F}$ by [5, Corollary I.4.7(a)] again.

Thus, if $\mathcal{F}_0$ is not simple and $A \notin \mathcal{F}$, then there is a proper normal subsystem $\mathcal{E} \trianglelefteq \mathcal{F}_0$ over $T \trianglelefteq S$ such that $T > A$. Set $\Delta = \mathrm{Aut}_{\mathcal{E}}(A) \trianglelefteq \Gamma_0$. Then $\Delta \geqslant \mathrm{Aut}_T(A) \neq 1$ since $T > A$, so $p \mid |\Delta|$. Since Γ_0 contains $G_0(m, m, \kappa)$ with index prime to p, we have

$$\Delta \cap G_0(m, m, \kappa) \trianglelefteq G_0(m, m, \kappa) \cong (C_m)^{\kappa-1} \rtimes A_\kappa$$

where $\kappa \geqslant p \geqslant 5$ and $p \mid |\Delta \cap G_0(m, m, \kappa)|$. Since A_κ and hence $G_0(m, m, \kappa)$ have no proper normal subgroups of order a multiple of p, it follows that $\Delta \geqslant G_0(m, m, \kappa)$, and hence that Δ has index prime to p in $\mathrm{Aut}_{\mathcal{F}}(A)$. But then $\mathrm{Aut}_T(A) = \mathrm{Aut}_S(A)$, so $T = S$ since A is $\mathcal{F}$-centric, and $\mathcal{E}$ has index prime to p in $\mathcal{F}_0$ and $\mathcal{F}$ by [2, Lemma 1.26]. Thus, $\mathcal{E} = \mathcal{F}_0 = O^{p'}(\mathcal{F})$, contradicting our assumption that $\mathcal{E}$ is proper.

Step 2: It remains to show, when $A \notin \mathcal{F}$, that $\mathcal{F}$ is not realized by any known finite almost simple group. Assume otherwise: assume $\mathcal{F} = \mathcal{F}_S(G)$ where G is almost simple, and set $G_0 = O^{p'}(G)$. Since $O^{p'}(\mathcal{F})$ is simple, G_0 is a known simple group.

We claim that this is impossible. Note that A is a radical p-subgroup of G_0, since $O_p(\mathrm{Aut}_{G_0}(A)) = 1$ and $p \nmid |C_{G_0}(A)/A|$ (i.e., A is $\mathcal{F}_0$-centric). Although we do not know $\mathrm{Aut}_{G_0}(A)$ precisely, we know that it is contained in $\mathrm{Aut}_{\mathcal{F}}(A)$ and contains $G_0(m, m, \kappa) \cong (C_m)^{\kappa-1} \rtimes A_\kappa$.

Since $p \geqslant 5$ and $\mathrm{rk}_p(G_0) \geqslant p$, G_0 cannot be a sporadic group by [22, Table 5.6.1].

By [1, § 2], for each abelian radical p-subgroup $B \leqslant \Sigma_\kappa$, $\mathrm{Aut}_{\Sigma_\kappa}(B)$ is a product of wreath products of the form $GL_c(p) \wr \Sigma_\kappa$ for $c \geqslant 1$ and $\kappa \geqslant 1$. Thus, $\mathrm{Aut}_{A_\kappa}(B)$ can have index 2 in $C_{p-1} \wr \Sigma_\kappa$ for some κ, but not index larger than 2. So, G_0 cannot be an alternating group.

If $G_0 \in \mathfrak{Lie}(p)$, then $N_{G_0}(A)$ is a parabolic subgroup by the Borel–Tits theorem [22, Corollary 3.1.5] and since A is centric and radical. So, in the notation of [22, § 2.6], $A = U_J$ and $N_{G_0}(A) = P_J$ (up to conjugacy) for some set J of primitive roots for G_0. Hence, by [22, Theorem 2.6.5(f,g)], $O^{p'}(N_{G_0}(A)/A) \cong O^{p'}(L_J)$ is a central product of groups in $\mathfrak{Lie}(p)$, contradicting the assumption that $O^{p'}(\mathrm{Aut}_G(A)) \cong G_0(m, m, \kappa)$.

Now assume that $G_0 \in \mathfrak{Lie}(q_0)$ for some prime $q_0 \neq p$. By [21, 10-2] (and since $p \geqslant 5$), $S \in \mathrm{Syl}_p(G_0)$ contains a unique elementary abelian p-subgroup of maximal rank, and by (4.2), it must be equal to $\Omega_1(A)$. Hence, $\mathrm{Aut}_{\mathcal{F}}(A)$ must be as in one of the entries in Table 4.2 or 4.3 in [38].

- If G_0 is a classical group and hence $\mathrm{Aut}_{\mathcal{F}}(A) \cong G(\hat{m}, \hat{r}, \kappa)$ for $\hat{m} = \mu$ or 2μ and $\hat{r} \leqslant 2$ (see the next-to-last column in [38, Table 4.2] and recall that $G(\hat{m}, 1, \kappa) \cong C_{\hat{m}} \wr \Sigma_\kappa$), then the identifications $\mathrm{Aut}_{\mathcal{F}}(A) \cong G(\hat{m}, \hat{r}, \kappa)$ and $\mathrm{Aut}_{\mathcal{F}}(A) \geqslant G(m, r, \kappa)$ are based on the same decompositions of A as a direct sum of cyclic subgroups, and hence, we have $m \mid \hat{m}$ and $r \leqslant 2$, contradicting our original assumption.

- If G_0 is an exceptional group, then by [38, Table 4.3], either $\kappa = \mathrm{rk}(A) < p$, or $p = 3$, or (in case (b)) $m^{\kappa-1} \cdot \kappa!$ does not divide $|\mathrm{Aut}_{\mathcal{F}}(A)|$ for any $m > 2$, and hence, $\mathrm{Aut}_{\mathcal{F}}(A)$ cannot contain any such $G(m, r, \kappa)$. □

As noted at the beginning of the section, if $\mathcal{F}$ is a realizable fusion system such that $O^{p'}(\mathcal{F})$ is simple, then $\mathcal{F}$ is realized by a finite almost simple group, and by a simple group if $\mathcal{F}$ is simple. We now consider the converse, by determining which fusion systems of known finite simple groups are simple or almost simple.

Theorem 4.5. *Fix a prime p and a known finite quasi-simple group L such that $Z(L)$ is a p-group and $p \mid |L|$. Fix $T \in \mathrm{Syl}_p(L)$, and set $\mathcal{E} = \mathcal{F}_T(L)$ and $\mathcal{C} = O^{p'}(\mathcal{E})$. Then $T > Z(L)$, and either*

(a) *$T \trianglelefteq \mathcal{E}$ and hence $\mathcal{C}$ is not quasi-simple; or*
(b) *$p = 3$ and $L \cong G_2(q)$ for some $q \equiv \pm 1$ (mod 9), in which case $|O_3(\mathcal{E})| = 3$, $Z(\mathcal{E}) = 1$, and $\mathcal{C} < \mathcal{E}$ is quasi-simple and is realized by $SL_3^{\pm}(q)$; or*
(c) *$p \geqslant 5$, L is one of the simple classical groups $PSL_n^{\pm}(q)$, $PSp_{2n}(q)$, $\Omega_{2n+1}(q)$, or $P\Omega_{2n+2}^{\pm}(q)$ where $n \geqslant 2$ and $q \not\equiv 0, \pm 1$ (mod p), in which case $\mathcal{C}$ is simple and is not realized by any known finite simple group; or*
(d) *$\mathcal{C}$ is quasi-simple, $Z(\mathcal{C}) = Z(L)$, and $\mathcal{C}$ is realized by a known finite quasi-simple group with center $Z(\mathcal{C})$.*

In cases (b), (c), and (d), there is a normal fusion subsystem $\mathcal{C}^* \trianglelefteq \mathcal{E}$ over T containing $\mathcal{C}$ that is realized by a known finite quasi-simple group, and is such that for each saturated fusion system $\mathcal{E}'$ over T such that $O^{p'}(\mathcal{E}') = \mathcal{C}$, $\mathcal{E}'$ is realized by a known finite quasi-simple group only if it contains $\mathcal{C}^*$. Thus, $\mathcal{C}^* = \mathcal{C}$ in cases (b) and (d), while $\mathcal{C}^* > \mathcal{C}$ in case (c).

Proof. In all cases, $T \neq 1$ since $p \mid |L|$ by assumption, and $T > Z(L)$ since otherwise $p \nmid |L/Z(L)|$ while $p \mid |Z(L)|$, contradicting the assumption that L is perfect.

When L is simple, this is essentially [38, Theorem 4.8], but restated to make its proof independent of the classification of finite simple groups. The only difference between the proof of this version and that of Theorem 4.8 in [38] is that we replace [38, Lemma 4.7] by the above Lemma 4.4. Note in case (a) that $\mathcal{C} = \mathcal{F}_T(T)$ is not quasi-simple since $O^p(\mathcal{C}) = 1$, and in case (d) that $\mathcal{C}$ is simple and is realized by a known finite simple group (thus with center $Z(L) = L(\mathcal{C}) = 1$). In case (b), $Z(\mathcal{E}) = 1$ since $\mathrm{Aut}_L(T)$ acts nontrivially on $O_3(\mathcal{E}) = O_3(\mathcal{F}_T(L)) \cong C_3$.

Now assume that $Z(L) \neq 1$. Then $Z(L) \leqslant T$ since $Z(L)$ is a p-group by assumption, and $\mathcal{F}_{T/Z(L)}(L/Z(L)) \cong \mathcal{E}/Z(L)$. Also, $L/Z(L)$ is not isomorphic to $G_2(q)$ for any prime power q and is not one of the groups in case (c) (see [21, 6-1], or Theorem 6.1.4 and Table 6.1.2 in [22], for the description of Schur multipliers of groups of Lie type in cross characteristic). So, either $T/Z(L) \trianglelefteq \mathcal{E}/Z(L)$ (case (a)), in which case $T \trianglelefteq \mathcal{E}$; or else $O^{p'}(\mathcal{E}/Z(L)) = \mathcal{C}/Z(L)$ is simple and is realized by a known simple group (case (d)).

If $\mathcal{C}/Z(L)$ is realized by a known simple group H, then $\mathcal{C}$ is realized by a central extension $\widehat{H}$ of $Z(L)$ by H. This follows from the proof of [9, Corollary 6.14] (the statement itself only says that $\mathcal{C}$ is realizable). We have $\mathfrak{foc}(\mathcal{C}/Z(L)) = T/Z(L)$ since H is simple, and hence $T = Z(L)\mathfrak{foc}(\mathcal{C})$. Also,

$$T = \mathfrak{foc}(\mathcal{E}) = \mathfrak{foc}(\mathcal{C})[\mathrm{Aut}_{\mathcal{E}}(T), T] = \mathfrak{foc}(\mathcal{C}),$$

where the first equality holds since $\mathcal{E} = \mathcal{F}_T(L)$ where L is perfect, the second holds since $\mathcal{E}$ is generated by $\mathcal{C} = O^{p'}(\mathcal{E})$ and $\mathrm{Aut}_{\mathcal{E}}(T)$ by the Frattini condition for $O^{p'}(\mathcal{E}) \trianglelefteq \mathcal{E}$ (see [5, Theorem I.7.7]),

and the last equality holds since $T = Z(L)\mathfrak{foc}(\mathcal{C})$ where $\mathrm{Aut}_{\mathcal{E}}(T)$ acts trivially on $Z(L)$ and sends $\mathfrak{foc}(\mathcal{C})$ to itself. Thus, $\mathcal{C}$ and $\widehat{H}$ are quasi-simple by the focal subgroup theorems (Lemma 1.15(a,b)), and $\widehat{H}$ is a known quasi-simple group. So, the last statement in the theorem holds in this case with $\mathcal{C}^* = \mathcal{C}$. □

The following terminology will be useful when stating many of the results throughout the rest of the paper.

Definition 4.6. Let G be a finite group. We say that

(a) G is *p'-reduced* if $O_{p'}(G) = 1$; and
(b) G is a *$\mathcal{KC}$-group* if all of its components are known quasi-simple groups.

Most of our statements from now on about groups will be formulated in terms of "finite p'-reduced $\mathcal{KC}$-groups," a restriction that allows us to avoid assuming the classification of finite simple groups in our proofs. Note that when working with the p-local structure of a finite group G, there is not much point in assuming that all components of G are known without also assuming that $O_{p'}(G) = 1$.

Lemma 4.7. *Let G be a finite p'-reduced $\mathcal{KC}$-group. Then for $U \in \mathrm{Syl}_p(F^*(G))$, the centralizer $C_G(U)$ is p-solvable.*

Proof. Let $L_1, \dots, L_k$ be the components of G, and fix $T_i \in \mathrm{Syl}_p(L_i)$. Set $L = L_1 \cdots L_k$ and $T = T_1 \cdots T_k$, and set $Q = O_p(G)$. Then $F^*(G) = QL$ and $QT \in \mathrm{Syl}_p(QL)$, so we can assume $U = QT$.

Conjugation by each element of G permutes the subgroups L_i. Since $O_{p'}(G) = 1$, and since the Schur multiplier of a group of order prime to p has order prime to p, we have $T_i > Z(L_i)$ for each i. So, each element of $C_G(U)$ normalizes each of the L_i.

Consider the homomorphism $\gamma : C_G(U) \longrightarrow \mathrm{Out}(L)$ that sends $g \in C_G(U)$ to the class of $c_g|_L$. We just showed that $\mathrm{Im}(\gamma) \leqslant \prod_{i=1}^k \mathrm{Out}(L_i)$, when identified with a subgroup of $\mathrm{Out}(L)$ in the obvious way. Moreover, $\mathrm{Out}(L_i) \leqslant \mathrm{Out}(L_i/Z(L_i))$ is solvable for each i by the Schreier conjecture and since $L_i/Z(L_i)$ is a known simple group (see [22, Theorem 7.1.1]), so $\mathrm{Im}(\gamma)$ is also solvable.

If $g \in \mathrm{Ker}(\gamma)$, then $c_g|_L = c_h|_L$ for some $h \in L$, so $gh^{-1} \in C_G(L)$. Then $gh^{-1} \in C_G(QL)$ since $h \in L \leqslant C_G(Q)$, and $h \in C_L(T)$ since g and gh^{-1} both commute with T. Thus, $\mathrm{Ker}(\gamma) \leqslant C_G(QL)C_L(T)$ (and the opposite inclusion is obvious). Also, $C_G(QL) \trianglelefteq G$ since $QL \trianglelefteq G$, and so $C_G(QL) \trianglelefteq \mathrm{Ker}(\gamma)$. By [3, 31.13] or [5, Theorem A.13(c)], $C_G(QL) \leqslant QL$, and hence, $C_G(QL) = Z(QL) = Z(Q)$: an abelian p-group. Also, $C_L(T)/Z(T)$ has order prime to p since $T \in \mathrm{Syl}_p(L)$. So $\mathrm{Ker}(\gamma)$, and hence $C_G(U)$, are p-solvable. □

The following notation will be used in several of the results in the rest of the section.

Notation 4.8. Let G be a finite p'-reduced $\mathcal{KC}$-group, fix $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Set $\mathrm{Comp}(G) = \{L_1, \dots, L_k\}$; thus, each L_i is a known quasi-simple group. For each $1 \leqslant i \leqslant k$, set

$$T_i = S \cap L_i, \quad \mathcal{E}_i = \mathcal{F}_{T_i}(L_i) \leqslant \mathcal{F}, \quad \text{and} \quad C_i = O^{p'}(\mathcal{E}_i).$$

Assume that the L_i were ordered so that for some $0 \leqslant m \leqslant k$, C_i is quasi-simple if and only if $i \leqslant m$.

Proposition 4.9. *Let G be a finite p'-reduced $\mathscr{K}\mathscr{C}$-group, and assume Notation 4.8. Then*

$$\mathrm{Comp}(\mathcal{F}) = \{C_1, \dots, C_m\} \qquad \textit{and} \qquad O_p(\mathcal{F}) \geqslant O_p(G)T_{m+1} \cdots T_k.$$

Proof. Set $L = L_1 \cdots L_k$ and $T = T_1 \cdots T_k$. For each $1 \leqslant i \leqslant k$, we have $T_i \in \mathrm{Syl}_p(L_i)$ since $L_i \trianglelefteq L \trianglelefteq G$, and also

$$C_i = O^{p'}(\mathcal{F}_{T_i}(L_i)) \trianglelefteq \mathcal{F}_{T_i}(L_i) \trianglelefteq \mathcal{F}_T(L) \trianglelefteq \mathcal{F}_S(G) = \mathcal{F} :$$

the first normality relation by [5, Theorem I.7.7] and the other two by [5, Proposition I.6.2] and since $L_i \trianglelefteq L \trianglelefteq G$. Thus, $C_i \trianglelefteq\trianglelefteq \mathcal{F}$ for each i. If $i \leqslant m$, then C_i is quasi-simple and hence is a component of $\mathcal{F}$. If $i \geqslant m+1$, then $T_i \trianglelefteq \mathcal{E}_i$ by Theorem 4.5, and hence $T_i \leqslant O_p(\mathcal{F}_T(L)) \leqslant O_p(\mathcal{F})$ by Lemma 2.3(b).

Set $Q = O_p(G)$ for short. Thus, $QT_{m+1} \cdots T_k \leqslant O_p(\mathcal{F})$, and $T_1 \cdots T_m$ is the Sylow of the central product $C_1 \cdots C_m$.

Assume $\mathcal{D} \trianglelefteq\trianglelefteq \mathcal{F}$ is another component. By [4, 9.8–9.9], the components of $\mathcal{F}$ commute with each other and with $O_p(\mathcal{F})$. Hence, $\mathcal{D} \leqslant C_{\mathcal{F}}(QT)$, where $C_{\mathcal{F}}(QT)$ is the fusion system of $C_G(QT)$ by Lemma 1.6. (Note that QT is fully centralized in $\mathcal{F}$ since it is normal in S.)

By Lemma 4.7, the centralizer $C_G(QT)$ is p-solvable. Hence, $C_{\mathcal{F}}(QT) = \mathcal{F}_{C_S(QT)}(C_G(QT))$ is solvable in the sense of [5, Definition II.12.1], and its saturated fusion subsystems are all solvable by [5, Lemma II.12.8]. So, $\mathcal{D}$ is solvable, and hence is constrained by [5, Lemma II.12.5(b)] (see also [5, Definition I.4.8]), which is impossible since $\mathcal{D}$ was assumed to be quasi-simple. We conclude that $C_1, \dots, C_m$ are the only components of $\mathcal{F}$. □

In the next section, we will be working mostly with fusion systems that are (tamely) realized by finite p'-reduced $\mathscr{K}\mathscr{C}$-groups. It will be important to know that in such situations, the fusion system and the group always have the same center. The following technical lemma is needed to prove that.

Lemma 4.10. *Fix an odd prime p. Let G be a central product of known p'-reduced quasi-simple groups, choose $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then $Z(\mathcal{F}) = Z(G)$, and* $\mathrm{Ker}(\varkappa_G)$ *(see Definition 2.8) has order prime to p.*

Proof. Assume that the lemma holds for $G/Z(G)$. Then $Z(\mathcal{F}/Z(G)) = Z(G/Z(G)) = 1$, so $Z(\mathcal{F}) \leqslant Z(G)$, while the opposite inclusion holds since G is p'-reduced. Also, $\mathrm{Ker}(\varkappa_G)$ has order prime to p by [2, Lemma 2.17].

It thus suffices to prove the lemma when G is a product of known simple groups.

$p \nmid |\mathrm{Ker}(\varkappa_G)|$: We first claim that if G is a finite group such that $Z(G) = 1$, and $U \in \mathrm{Syl}_p(\mathrm{Aut}(G))$ and $S = U \cap \mathrm{Inn}(G) \in \mathrm{Syl}_p(\mathrm{Inn}(G))$, then

$$C_U(S) \leqslant S \quad \Rightarrow \quad p \nmid |\mathrm{Ker}(\varkappa_G)|. \tag{4.4}$$

To simplify notation, we identify G with $\mathrm{Inn}(G)$ (recall $Z(G) = 1$), and thus, identify S with a Sylow p-subgroup of G. Assume (4.4) does not hold: thus, $C_U(S) \leqslant S$ and $p \mid |\mathrm{Ker}(\varkappa_G)|$. Let $\alpha \in \mathrm{Aut}(G)$ be such that $[\alpha]$ has order p in $\mathrm{Out}(G)$ and $\varkappa_G([\alpha]) = 1$. Since $\mathrm{Aut}(G) = \mathrm{Inn}(G)N_{\mathrm{Aut}(G)}(S)$ by the Frattini argument, we can assume that $\alpha(S) = S$ without changing the class $[\alpha]$. Then $\varkappa_G([\alpha]) = 1$

TABLE 1 Pairs (G, p) where G is simple, $S \in \mathrm{Syl}_p(G)$ is nonabelian, and $S \trianglelefteq \mathcal{F}_S(G)$

p	G	S	$N_G(S)/S$
p	$PSU_3(q)$ $(q = p^k)$	p^{k+2k}	$C_{(q^2-1)/(q+1,3)}$
3	${}^2G_2(q)$ $(q = 3^k)$	3^{k+k+k}	C_{q-1}
3	$G_2(q)$ $(q \equiv \pm 2, \pm 4 \pmod 9)$	3^{1+2}_+	SD_{16}
3	J_2	3^{1+2}_+	C_8
3	J_3	order 3^5	C_8
5	McL	5^{1+2}_+	$C_3 \rtimes C_8$
5	HS	5^{1+2}_+	order 16
5	Co_2	5^{1+2}_+	$4S_4$
5	Co_3	5^{1+2}_+	order 48
11	J_4	11^{1+2}_+	$5 \times 2S_4$

implies, in particular, that $\alpha|_S \in \mathrm{Aut}_G(S)$, and thus, $\alpha|_S = c_x|_S$ for some $x \in N_G(S)$. So, upon replacing α by $c_x^{-1}\alpha$, we can assume that α centralizes S, and upon replacing α by α^k for some appropriate k, we can also arrange that α have p-power order in $\mathrm{Aut}(G)$. Also, $\alpha \in N_{\mathrm{Aut}(G)}(S)$ and $U \in \mathrm{Syl}_p(N_{\mathrm{Aut}(G)}(S))$ imply that $\beta\alpha\beta^{-1} \in C_U(S) \setminus S$ for some $\beta \in N_{\mathrm{Aut}(G)}(S)$. This contradicts our original assumption, and finishes the proof of (4.4).

By [23, Theorem B], $C_U(S) \leqslant S$ whenever G is a known simple group. The corresponding relation for products of known simple groups then follows from the description in Proposition 3.4 of the automorphism group of a product. So, $p \nmid |\mathrm{Ker}(\varkappa_G)|$ for all such G by (4.4).

$Z(\mathcal{F}) = Z(G)$: If $S \notin \mathcal{F}$, then by Theorem 4.5, either G is as in case (b) and $Z(\mathcal{F}) = Z(G) = 1$; or it is as in case (c) or (d) and $Z(O^{p'}(\mathcal{F})) = Z(G) = 1$. Since $Z(\mathcal{F}) \leqslant Z(O^{p'}(\mathcal{F}))$ for every saturated fusion system over a finite p-group, this proves that $Z(\mathcal{F}) = Z(G) = 1$.

Now assume that $S \trianglelefteq \mathcal{F}$; that is, that G is p-Goldschmidt in the terminology of Aschbacher. Then, $Z(\mathcal{F}) = C_{Z(S)}(N_G(S)/S)$, and we must show this is trivial in all cases (recall that G is simple). If S is abelian, then since $\mathrm{Aut}_G(S) \cong N_G(S)/S$ has order prime to p, we have $S = C_S(\mathrm{Aut}_G(S)) \times [\mathrm{Aut}_G(S), S]$ (see [20, Theorem 5.2.3]), where the second factor is the focal subgroup $\mathfrak{foc}(\mathcal{F})$. Also, $\mathfrak{foc}(\mathcal{F}) = S$ since G is simple, and thus, $Z(\mathcal{F}) = C_S(\mathrm{Aut}_G(S)) = 1$.

If $S \trianglelefteq \mathcal{F}$ and S is nonabelian, then by [4, Theorem 15.6], (G, p) is one of pairs listed in Table 1.

In all cases, $C_S(N_G(S)) \leqslant [S, S]$ by an argument similar to that used when S is abelian. If $G \cong PSU_3(q)$ or ${}^2G_2(q)$, then the explicit description of S and $N_G(S)/S$ in [28, II.10.12(b)] or [29, §XI.3], respectively, shows that $C_S(\mathrm{Aut}_G(S)) = 1$. When $p = 3$ and $G \cong G_2(q)$ for $q \equiv \pm 2, \pm 4 \pmod 9$, the action of the Weyl group $W \cong D_{12}$ on $Z(S) \cong C_3$ is nontrivial. In all other cases where $S \cong p^{1+2}_+$, the generators of $Z(S)$ are conjugate (by [21, §5] or [22, Tables 5.3]), and so, $N(S)/S$ acts nontrivially on $Z(S) = [S, S]$.

This leaves the case $G \cong J_3$ and $p = 3$. By [30, Lemma 5.4] (where S is denoted as W_1), $Z(S) \cong E_9$, and there is a subgroup $W \trianglelefteq N_G(S)$ such that $Z(S) \leqslant W \cong E_{27}$ and $S \trianglelefteq N_G(W)$. Also, $N_G(W)/S \cong C_8$; and all elements in $Z(S)^{\#}$ and all those in $W \setminus Z(S)$ are conjugate in $N_G(W)$. Thus, $W \geqslant [S, S]$, and $C_S(N_G(S)) = 1$ in this case. □

By [19, Theorem 5.1] and Glauberman's Z^*-theorem, Lemma 4.10 also holds when $p = 2$. More generally, in the same theorem, Glauberman and Lynd showed that for each prime p for which the Z^*-theorem holds for all almost simple groups, one also has that $\mathrm{Ker}(\varkappa_G)$ has order prime to p for all simple groups.

We are now ready to prove the Z^*-theorem at odd primes for all finite p'-reduced $\mathscr{KC}$-groups.

Proposition 4.11. *Fix a prime p, let G be a p'-reduced $\mathscr{KC}$-group, and choose $S \in \mathrm{Syl}_p(G)$. Then $Z(\mathcal{F}_S(G)) = Z(G)$.*

Proof. When $p = 2$, this is just Glauberman's Z^*-theorem [17, Corollary 1], and holds for all finite $2'$-reduced groups. So, it remains to prove the proposition when p is odd.

Set $H = E(G)$: the central product of the components of G. Set $T = S \cap H$, and set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{E} = \mathcal{F}_T(H)$. By Lemma 4.10, $\mathrm{Ker}(\kappa_H)$ has order prime to p and $Z(\mathcal{E}) = Z(H)$. Also, $C_G(O_p(G)H) = Z(O_p(G)H) = Z(O_p(G))$ (see [3, 31.13]), so $O_p(G) \trianglelefteq C_G(H)$ is a centric subgroup. In particular, $Z(C_G(H)) \leqslant Z(O_p(G))$.

Assume $x \in Z(\mathcal{F})$. By Lemma 2.12, there are $y \in C_S(H)$ and $z \in Z(\mathcal{E}) = Z(H)$ such that $x = yz$. If $g \in C_G(H)$ is such that ${}^g y \in C_S(H)$, then ${}^g x = ({}^g y)z \in S$, so ${}^g x = x$ since $x \in Z(\mathcal{F})$ and hence ${}^g y = y$. Thus,

$$y \in Z(C_G(H)) \leqslant Z(O_p(G)),$$

and since $z \in Z(H) \leqslant Z(O_p(G))$, we have $x \in Z(O_p(G))$. Since $x \in Z(\mathcal{F})$, it must be invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(Z(O_p(G)))$, and so, $x \in C_{Z(O_p(G))}(G) = Z(G)$.

This proves that $Z(\mathcal{F}) \leqslant Z(G)$, and the opposite inclusion holds since G is p'-reduced. □

A very similar proof of the Z^*-theorem at odd primes was given by Guralnick and Robinson [24, Theorem 4.1]. If we give our own proof here, it is mostly to make our assumptions completely clear: we assume only that all components of G are known quasi-simple groups. Other proofs of the Z^*-theorem, where the analogous assumptions are less restrictive or less clear, are given in [44, Theorem 1] and in [22, Remark 7.8.3].

The next proposition is somewhat more technical.

Proposition 4.12. *Let G be a finite p'-reduced $\mathscr{KC}$-group, assume Notation 4.8, and set*

$$L = L_1 \cdots L_m, \quad T = T_1 \cdots T_m, \quad \mathcal{E} = \mathcal{F}_T(L), \quad \text{and} \quad \mathcal{C} = O^{p'}(\mathcal{E}).$$

Then $\mathcal{C} \trianglelefteq \mathcal{E}$ are the central products of the subsystems $\mathcal{C}_i \trianglelefteq \mathcal{E}_i$ for $1 \leqslant i \leqslant m$.

(a) *There is a unique minimal normal fusion subsystem $\mathcal{C}^* \trianglelefteq \mathcal{F}$ among those subsystems containing $\mathcal{C} = E(\mathcal{F})$ that are realized by finite p'-reduced $\mathscr{KC}$-groups, and $\mathcal{C}^*$ itself is realized by a central product of known finite quasi-simple groups. More precisely, $\mathcal{C}^* = \mathcal{C}_1^* \cdots \mathcal{C}_m^*$, where for each $1 \leqslant i \leqslant m$, $\mathcal{C}_i^* \leqslant \mathcal{E}_i$ is a fusion system over T_i such that $O^{p'}(\mathcal{C}_i^*) = \mathcal{C}_i$, and $\mathcal{C}_i^*$ is the fusion system of a known finite quasi-simple group.*

(b) *We have $\mathcal{C} = O^{p'}(\mathcal{C}^*)$ and $\mathrm{Aut}(\mathcal{C}^*) = \mathrm{Aut}(\mathcal{C})$, and $\mathcal{C}$ and $\mathcal{C}^*$ are both characteristic in $\mathcal{F}$.*

Proof. Let $\varphi\colon L_1 \times \cdots \times L_m \longrightarrow G$ be the homomorphism induced by the inclusions $L_i \leqslant G$, and let

$$\widehat{\varphi}\colon \mathcal{E}_1 \times \cdots \times \mathcal{E}_m = \mathcal{F}_{T_1 \times \cdots \times T_m}(L_1 \times \cdots \times L_m) \longrightarrow \mathcal{F}_S(G) = \mathcal{F}$$

be the induced functor between the fusion systems. Then $\mathrm{Im}(\widehat{\varphi}) = \mathcal{E}_1 \cdots \mathcal{E}_m$, the central product of the $\mathcal{E}_i$, and is equal to $\mathcal{F}_{T_1 \cdots T_m}(L_1 \cdots L_m)$. Set $Z = \mathrm{Ker}(\varphi) \leqslant \prod_{i=1}^m Z(L_i)$. Then $\mathcal{E}_1 \cdots \mathcal{E}_m \cong (\prod_{i=1}^m \mathcal{E}_i)/Z$, and hence,

$$O^{p'}(\mathcal{E}_1 \cdots \mathcal{E}_m) \cong O^{p'}\left(\prod_{i=1}^m \mathcal{E}_i\right)/Z \cong \left(\prod_{i=1}^m O^{p'}(\mathcal{E}_i)\right)/Z \cong O^{p'}(\mathcal{E}_1) \cdots O^{p'}(\mathcal{E}_m) :$$

the first isomorphism by Lemma 1.19 and the second by [2, Proposition 3.4]. Since these are finite categories and $O^{p'}(\mathcal{E}_1 \cdots \mathcal{E}_m) \leqslant O^{p'}(\mathcal{E}_1) \cdots O^{p'}(\mathcal{E}_m)$, the two are equal.

(a) For each $1 \leqslant i \leqslant m$, let $\mathcal{C}_i^* \leqslant \mathcal{E}_i$ be the minimal realizable fusion subsystem containing $\mathcal{C}_i$ of Theorem 4.5. More precisely, $\mathcal{C}_i^*$ is realized by a known finite quasi-simple group, and each saturated fusion subsystem of $\mathcal{E}_i$ that is realized by a known finite quasi-simple group and contains $\mathcal{C}_i$ also contains $\mathcal{C}_i^*$. Since $\mathcal{C}_i^* \leqslant \mathcal{E}_i$ for each i and the $\mathcal{E}_i$ commute, the $\mathcal{C}_i^*$ also commute.

Set $\mathcal{C}^* = \mathcal{C}_1^* \cdots \mathcal{C}_m^* \leqslant \mathcal{F}$, the central product of these subsystems. Thus, $\mathcal{C} \leqslant \mathcal{C}^* \leqslant \mathcal{E}$ where all three fusion systems are over $T = T_1 \cdots T_m$, and where $\mathcal{C} = E(\mathcal{F}) \trianglelefteq \mathcal{F}$, and $\mathcal{E} \trianglelefteq \mathcal{F}$ since $L_1 \cdots L_m \trianglelefteq G$.

It remains to show that each normal fusion subsystem of $\mathcal{F}$ containing $\mathcal{C}$ and realized by a finite p'-reduced $\mathscr{K}\mathscr{C}$-group also contains $\mathcal{C}^*$. Assume that H is such a group. In particular, there is $R \in \mathrm{Syl}_p(H)$ such that $T \leqslant R \trianglelefteq S$ and $\mathcal{C} \leqslant \mathcal{F}_R(H) \trianglelefteq \mathcal{F}$. Let $H_1, \dots, H_\ell \trianglelefteq\trianglelefteq H$ be the components of H (by assumption, known quasi-simple groups), and set $R_i = R \cap H_i$. By Proposition 4.9, the components of $\mathcal{F}_R(H)$ are those subsystems $O^{p'}(\mathcal{F}_{R_i}(H_i))$ for $1 \leqslant i \leqslant \ell$ that are quasi-simple. Each component of $\mathcal{F}_R(H)$ is subnormal in $\mathcal{F}$ and hence a component of $\mathcal{F}$ (recall $\mathcal{F}_R(H) \trianglelefteq \mathcal{F}$), and each component of $\mathcal{F}$ is contained in and hence a component of $\mathcal{F}_R(H)$ by assumption. Thus, $m \leqslant \ell$, and we can assume that the indices are chosen so that $T_i = R_i$ and $O^{p'}(\mathcal{F}_{T_i}(H_i)) = \mathcal{C}_i$ for each $i \leqslant m$.

For each i, $\mathcal{F}_{T_i}(H_i)$ and $\mathcal{C}_i^*$ are both realizable fusion systems over T_i where $O^{p'}(\mathcal{F}_{T_i}(H_i)) = O^{p'}(\mathcal{C}_i^*) = \mathcal{C}_i$, and so $\mathcal{F}_{T_i}(H_i) \geqslant \mathcal{C}_i^*$ by Theorem 4.5 (the last statement) and the minimality assumption. So, $\mathcal{F}_R(H)$ contains $\mathcal{C}^*$, and thus, $\mathcal{C}^*$ is minimal among normal subsystems of $\mathcal{F}$ containing $\mathcal{C}$ and realized by finite p'-reduced $\mathscr{K}\mathscr{C}$-groups.

(b) Since $\mathcal{C} \leqslant \mathcal{C}^* \leqslant \mathcal{E}$ and $\mathcal{C} = O^{p'}(\mathcal{E})$, we also have $\mathcal{C} = O^{p'}(\mathcal{C}^*)$. It remains to prove that $\mathrm{Aut}(\mathcal{C}^*) = \mathrm{Aut}(\mathcal{C})$, and that $\mathcal{C}$ and $\mathcal{C}^*$ are characteristic in $\mathcal{F}$.

Assume that the central factors $\mathcal{C}_i$ are ordered so that for some $\ell \leqslant m$, we have $\mathcal{C}_i < \mathcal{C}_i^*$ if and only if $1 \leqslant i \leqslant \ell$. For each $1 \leqslant i \leqslant \ell$, L_i falls under case (c) in Theorem 4.5, so $\mathcal{C}_i$ is simple, and $Z(\mathcal{C}_i^*) = 1$. So, if we set $T_0 = T_{\ell+1} \cdots T_m$ and $\mathcal{C}_0 = \mathcal{C}_{\ell+1} \cdots \mathcal{C}_m$, the central products of the remaining factors, we get direct product decompositions

$$T = T_0 \times \cdots \times T_\ell, \qquad \mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_\ell \qquad \text{and} \qquad \mathcal{C}^* = \mathcal{C}_0^* \times \cdots \times \mathcal{C}_\ell^*.$$

Since $\mathcal{C} = O^{p'}(\mathcal{C}^*)$, we have $\mathrm{Aut}(\mathcal{C}^*) \leqslant \mathrm{Aut}(\mathcal{C})$, and it remains to prove the opposite inclusion. Fix $\alpha \in \mathrm{Aut}(\mathcal{C}) \leqslant \mathrm{Aut}(T)$. Then, $\alpha(T_0) = T_0$ and ${}^\alpha\mathcal{C}_0 = \mathcal{C}_0$, and α permutes the factors T_i and $\mathcal{C}_i$ for $1 \leqslant i \leqslant \ell$. Thus, there is $\sigma \in \Sigma_\ell$ such that ${}^\alpha\mathcal{C}_i^* \geqslant {}^\alpha\mathcal{C}_i = \mathcal{C}_{\sigma(i)}$ for each $1 \leqslant i \leqslant \ell$. For each i, $\mathcal{C}_i^*$ and $\mathcal{C}_{\sigma(i)}^*$ were chosen to be the unique smallest saturated fusion systems over T_i and $T_{\sigma(i)}$ containing $\mathcal{C}_i$ and $\mathcal{C}_{\sigma(i)}$, respectively, that are realized by known finite quasi-simple groups. Since ${}^\alpha\mathcal{C}_i^*$ is also realized by a known quasi-simple group, we have ${}^\alpha\mathcal{C}_i^* = \mathcal{C}_{\sigma(i)}^*$. Upon

taking the direct product of these systems, it now follows that ${}^{\alpha}\mathcal{C}^* = \mathcal{C}^*$. Hence, $\alpha \in \mathrm{Aut}(\mathcal{C}^*)$, finishing the proof that $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{C}^*)$.

Now, $\mathcal{C} = E(\mathcal{F})$ since $\mathrm{Comp}(\mathcal{F}) = \{\mathcal{C}_1, \dots, \mathcal{C}_m\}$ by Proposition 4.9, and hence, $\mathcal{C}$ is characteristic in $\mathcal{F}$ by Lemma 4.1(a). The Frattini condition for $\mathcal{C}^* \leqslant \mathcal{F}$ holds since it holds for $\mathcal{C} \trianglelefteq \mathcal{F}$, and the extension condition holds since it holds for $\mathcal{E} \trianglelefteq \mathcal{F}$. For each $\alpha \in \mathrm{Aut}(\mathcal{F})$, $\alpha|_T \in \mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{C}^*)$ since $\mathcal{C}$ is characteristic in $\mathcal{F}$, and hence ${}^{\alpha}\mathcal{C}^* = \mathcal{C}^*$. Thus, the invariance condition holds (so $\mathcal{C}^* \trianglelefteq \mathcal{F}$), and $\mathcal{C}^*$ is characteristic in $\mathcal{F}$. □

We need to understand the role played by the components of G and of $\mathcal{F}_S(G)$ when determining automorphisms of the linking system $\mathcal{L}^c_S(G)$ and tameness. The next proposition is a first step toward that.

Proposition 4.13. *Let G be a finite p'-reduced $\mathscr{K}\mathscr{C}$-group, and assume Notation 4.8. Assume also that $O_p(G) = 1 = O_p(\mathcal{F})$. Thus, for each $1 \leqslant i \leqslant k$, L_i is a known simple group and $\mathcal{C}_i$ is a simple fusion system. Set*

$$L = L_1 \times \cdots \times L_k \trianglelefteq G \qquad \text{and} \qquad T = T_1 \times \cdots \times T_k = S \cap L \in \mathrm{Syl}_p(L);$$

and also $\mathcal{E} = \mathcal{F}_T(L)$ and $\mathcal{C} = O^{p'}(\mathcal{E})$.

(a) *There is a unique minimal normal fusion subsystem $\mathcal{C}^* \leqslant \mathcal{E}$ containing $\mathcal{C}$ that is realized by a product of known finite simple groups. Also, $\mathcal{C}^* = \mathcal{C}_1^* \times \cdots \times \mathcal{C}_k^*$, where for each $1 \leqslant i \leqslant k$, the subsystem $\mathcal{C}_i^*$ is realized by a known finite simple group and $\mathcal{C}_i \leqslant \mathcal{C}_i^* \leqslant \mathcal{E}_i$.*
(b) *The fusion subsystems $\mathcal{C}^*$ and $\mathcal{C}$ are both centric and characteristic in $\mathcal{F}$.*

Proof.

(a) This is the special case of Proposition 4.12 when $O_p(G) = 1 = O_p(\mathcal{F})$.
(b) The subsystems $\mathcal{C}$ and $\mathcal{C}^*$ are characteristic in $\mathcal{F}$ by Proposition 4.12.

Since $O_p(\mathcal{F}) = 1 = O_p(G)$, each of the subsystems $\mathcal{C}_i$ is simple, and hence, $\mathrm{Comp}(\mathcal{F}) = \{\mathcal{C}_1, \dots, \mathcal{C}_k\}$ by Proposition 4.9. So, $F^*(\mathcal{F}) = E(\mathcal{F}) = \mathcal{C}$, and $\mathcal{C}$ is centric in $\mathcal{F}$ by Lemma 4.1(b). So, $\mathcal{C}^* \geqslant \mathcal{C}$ is also centric in $\mathcal{F}$. □

We finish the section with two more specialized results. The first shows that in a saturated fusion system $\mathcal{F}$ where $E(\mathcal{F})$ is "almost realizable" in the sense that there is a minimal realizable fusion subsystem $\mathcal{C}^* \leqslant \mathcal{F}$ with $O^{p'}(\mathcal{C}^*) = E(\mathcal{F})$, the subsystem $\mathcal{C}^*$ is always contained and normal in $C_{\mathcal{F}}(O_p(\mathcal{F}))$.

Lemma 4.14. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be its components, where each $\mathcal{C}_i \trianglelefteq\trianglelefteq \mathcal{F}$ is a fusion system over $T_i \trianglelefteq\trianglelefteq S$. Set $T = T_1 \cdots T_m$ and $\mathcal{C} = \mathcal{C}_1 \cdots \mathcal{C}_m$. Set $Q = O_p(\mathcal{F})$.*

Assume, for each $i = 1, \dots, m$, that $\mathcal{C}_i^* \leqslant \mathcal{F}$ is a saturated fusion subsystem over T_i containing $\mathcal{C}_i$ such that $\mathcal{C}_i = O^{p'}(\mathcal{C}_i^*)$, such that $\mathcal{C}_i^*$ is realized by a known finite quasi-simple group, and such that $\mathcal{C}_i^* = \mathcal{C}_i$ for each i such that $\mathcal{C}_i$ is realized by a known finite quasi-simple group. Assume also that the subsystems $\mathcal{C}_1^*, \dots, \mathcal{C}_m^*$ commute in $\mathcal{F}$, and that their central product $\mathcal{C}^* = \mathcal{C}_1^* \cdots \mathcal{C}_m^*$ is normal in $\mathcal{F}$. Then, $\mathcal{C}^*$ is contained in $C_{\mathcal{F}}(Q)$, and is normal in $C_{\mathcal{F}}(Q)$ and in $N_{\mathcal{F}}^{\mathrm{Inn}(Q)}(Q)$.

Proof. Assume that the indices are chosen so that for some $0 \leqslant \ell \leqslant m$, $\mathcal{C}_i^* = \mathcal{C}_i$ for each $1 \leqslant i \leqslant \ell$ and $\mathcal{C}_i^* > \mathcal{C}_i$ for each $\ell + 1 \leqslant i \leqslant m$. For each $1 \leqslant i \leqslant m$, let L_i be a known finite quasi-simple group with $T_i \in \mathrm{Syl}_p(L_i)$ such that $\mathcal{C}_i^* = \mathcal{F}_{T_i}(L_i)$ (there is such a group by assumption). By Theorem 4.5 applied to L_i, we are in case (d) of the theorem whenever $i \leqslant \ell$ and in case (c) whenever $i \geqslant \ell + 1$. (Case (b) cannot occur since we assume $\mathcal{C}_i = O^{p'}(\mathcal{C}_i^*) = \mathcal{C}_i^*$ whenever $\mathcal{C}_i$ is realizable by a known finite quasi-simple group.) In particular, the $\mathcal{C}_i$ and L_i are all simple for $\ell + 1 \leqslant i \leqslant m$ by Theorem 4.5(c).

Set

$$\mathcal{C}_I = \mathcal{C}_1 \cdots \mathcal{C}_\ell = \mathcal{C}_1^* \cdots \mathcal{C}_\ell^*, \quad \mathcal{C}_{II} = \mathcal{C}_{\ell+1} \cdots \mathcal{C}_m, \quad \text{and} \quad \mathcal{C}_{II}^* = \mathcal{C}_{\ell+1}^* \cdots \mathcal{C}_m^*,$$

and also $T_I = T_1 \cdots T_\ell$ and $T_{II} = T_{\ell+1} \cdots T_m$. Since $\mathcal{C}_i$ is simple for $i \geqslant \ell + 1$, we have

$$Z(\mathcal{C}_{II}) = 1, \qquad \mathcal{C} = \mathcal{C}_I \times \mathcal{C}_{II}, \qquad \mathcal{C}^* = \mathcal{C}_I \times \mathcal{C}_{II}^*, \qquad \text{and} \qquad T = T_I \times T_{II}.$$

By [4, 9.9] or [14, Theorem 7.10(e)], we have $\mathcal{C} \trianglelefteq \mathcal{C}_{\mathcal{F}}(Q)$. By the Frattini condition for $\mathcal{C} \trianglelefteq \mathcal{C}^*$, the fusion system $\mathcal{C}^*$ is generated by $\mathcal{C}$ and automorphisms $\alpha \in \mathrm{Aut}_{\mathcal{C}^*}(T)$ of order prime to p that are the identity on T_I. By the extension condition for $\mathcal{C}^* \trianglelefteq \mathcal{F}$, each such α extends to an element $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[\bar{\alpha}, C_S(T)] \leqslant Z(T)$, and since $Q \trianglelefteq \mathcal{F}$, this implies that $[\bar{\alpha}, Q] \leqslant Q \cap Z(T) = Z(\mathcal{C}) \leqslant T_I$. Upon replacing $\bar{\alpha}$ by $\bar{\alpha}^k$ for some appropriate k, we can arrange that $\bar{\alpha}$ have order prime to p. Then

$$Q = C_Q(\bar{\alpha})[\bar{\alpha}, Q] \leqslant C_Q(\bar{\alpha})T_I \leqslant C_{QT}(\bar{\alpha})$$

(see [20, Theorem 5.3.5] for the first equality), and so $\bar{\alpha}|_Q = \mathrm{Id}_Q$. Thus, α is a morphism in $\mathcal{C}_{\mathcal{F}}(Q)$, finishing the proof that $\mathcal{C}^* \leqslant \mathcal{C}_{\mathcal{F}}(Q)$.

It remains to prove that $\mathcal{C}^*$ is normal in $\mathcal{C}_{\mathcal{F}}(Q)$ and in $N_{\mathcal{F}}^{\mathrm{Inn}(Q)}(Q)$. The Frattini condition holds since it holds for $\mathcal{C} \trianglelefteq \mathcal{C}_{\mathcal{F}}(Q)$ and $\mathcal{C} \trianglelefteq N_{\mathcal{F}}^{\mathrm{Inn}(Q)}(Q)$, and the invariance condition holds for both inclusions because it holds for $\mathcal{C}^* \trianglelefteq \mathcal{F}$. We just showed that $\mathcal{C}^*$ is generated by morphisms in $\mathcal{C}$ and morphisms $\alpha \in \mathrm{Aut}_{\mathcal{C}^*}(T)$ that extend to $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $\bar{\alpha}|_Q = \mathrm{Id}_Q$ and $[\bar{\alpha}, C_S(T)] \leqslant Z(T)$, and hence, the extension condition holds for both inclusions since $\mathcal{C} \trianglelefteq \mathcal{C}_{\mathcal{F}}(Q)$ and $\mathcal{C} \trianglelefteq N_{\mathcal{F}}^{\mathrm{Inn}(Q)}(Q)$. □

In the following proposition, we show that each saturated fusion system $\mathcal{F}$ has a maximal characteristic subsystem that normalizes all components of $\mathcal{F}$.

Proposition 4.15. *Let $\mathcal{F}$ be a saturated fusion system over a finite group S. Let $\mathcal{C}_1, \dots, \mathcal{C}_k$ be the components of $\mathcal{F}$, where $\mathcal{C}_i$ is a fusion system over U_i. Assume, for each $1 \leqslant i \leqslant k$, that $Z(\mathcal{C}_i) = 1$ (i.e., that $\mathcal{C}_i$ is simple). Set*

$$U = U_1 \times \cdots \times U_k \leqslant S, \qquad N = \bigcap\nolimits_{i=1}^{k} N_S(U_i),$$
$$\mathcal{H} = \{P \leqslant N \mid P \cap U_i \neq 1 \text{ for each } 1 \leqslant i \leqslant k\}, \qquad \text{and}$$
$$\mathcal{N} = \left\langle \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q) \,\middle|\, P, Q \in \mathcal{H},\ \varphi(P \cap U_i) \leqslant Q \cap U_i \text{ for each } 1 \leqslant i \leqslant k \right\rangle_N.$$

Here, $\langle - \rangle_N$ denotes the fusion system over N generated by the morphisms between the brackets. Thus, $E(\mathcal{F})$ is the direct product of the C_i, and is a fusion subsystem over U by Lemma 4.1(a). Then the fusion subsystem $\mathcal{N}$ is saturated and characteristic in $\mathcal{F}$, $E(\mathcal{F}) \trianglelefteq \mathcal{N}$, and $C_i \trianglelefteq \mathcal{N}$ for each $1 \leqslant i \leqslant k$.

Proof. By Lemma 4.1(a) and since $Z(C_i) = 1$ for each i, $E(\mathcal{F})$ is the direct product of the C_i and is characteristic in $\mathcal{F}$. In particular, $E(\mathcal{F}) \trianglelefteq \mathcal{F}$, and U is strongly closed in $\mathcal{F}$.

Set

$$\Delta = \{\delta \in \Sigma_k \mid \exists\, \alpha \in \mathrm{Aut}_{\mathcal{F}}(U) \text{ such that } \alpha(U_i) = U_{\delta(i)}\ \forall\, 1 \leqslant i \leqslant k\}.$$

We first claim that

$$\begin{aligned} &\text{for each } P \in \mathcal{H} \text{ and each } \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S), \text{ there is } \delta \in \Delta \text{ such that} \\ &\varphi(P \cap U_i) \leqslant U_{\delta(i)} \text{ for all } 1 \leqslant i \leqslant k. \end{aligned} \tag{4.5}$$

It suffices to prove this when $P \leqslant U$ (hence $\varphi(P) \leqslant U$). Since $E(\mathcal{F}) \trianglelefteq \mathcal{F}$ by Lemma 4.1(a), the Frattini condition implies that $\varphi = \alpha\varphi'$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(U)$ and some $\varphi' \in \mathrm{Hom}_{E(\mathcal{F})}(P, U)$. Since α permutes the components of $\mathcal{F}$, there is $\delta \in \Delta$ such that $\alpha(U_i) = U_{\delta(i)}$ for each i. Since φ' is in $E(\mathcal{F})$, we have $\varphi(P \cap U_i) \leqslant \alpha(U_i) = U_{\delta(i)}$ for each $1 \leqslant i \leqslant k$.

We next claim that

$$\text{for each } \delta \in \Delta, \text{ there is } \alpha \in \mathrm{Aut}_{\mathcal{F}}(N) \text{ such that } \alpha(U_i) = U_{\delta(i)} \text{ for each } 1 \leqslant i \leqslant k. \tag{4.6}$$

To see this, fix $\delta \in \Delta$, and choose $\beta \in \mathrm{Aut}_{\mathcal{F}}(U)$ such that $\beta(U_i) = U_{\delta(i)}$ for each $1 \leqslant i \leqslant k$. Since $\mathrm{Aut}_S(U) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(U))$ and $\mathrm{Aut}_{\mathcal{N}}(U) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(U)$, we have that $\mathrm{Aut}_N(U)$ and ${}^{\beta}\mathrm{Aut}_N(U)$ are both Sylow p-subgroups of $\mathrm{Aut}_{\mathcal{N}}(U)$. So, there is $\gamma \in \mathrm{Aut}_{\mathcal{N}}(U)$ such that $\gamma\beta$ normalizes $\mathrm{Aut}_N(U)$, and hence, by the extension axiom extends to $\alpha \in \mathrm{Aut}_{\mathcal{F}}(N)$. By construction, $\alpha(U_i) = U_{\delta(i)}$ for each $1 \leqslant i \leqslant k$.

Fix $\alpha \in \mathrm{Aut}_{\mathcal{F}}(N)$, and let $\delta \in \Sigma_k$ be such that $\alpha(U_i) = U_{\delta(i)}$ for all $1 \leqslant i \leqslant k$. For each $P, Q \in \mathcal{H}$ and $\varphi \in \mathrm{Hom}_{\mathcal{N}}(P, Q)$, $\varphi(P \cap U_i) \leqslant Q \cap U_i$ for each $1 \leqslant i \leqslant k$, and hence $\alpha\varphi\alpha^{-1}(\alpha(P) \cap U_i) \leqslant \alpha(Q) \cap U_i$ for each $1 \leqslant i \leqslant k$. Thus, ${}^{\alpha}\varphi \in \mathrm{Mor}(\mathcal{N})$. So, α normalizes the subsystem $\mathcal{N}$, and we have shown

$$\text{for all } \alpha \in \mathrm{Aut}_{\mathcal{F}}(N), \quad {}^{\alpha}\mathcal{N} = \mathcal{N}. \tag{4.7}$$

We show in Step 1 that $\mathcal{N}$ is saturated, in Step 2 that $\mathcal{N}$ is characteristic, and in Step 3 that $E(\mathcal{F}) \trianglelefteq \mathcal{N}$ and $C_i \trianglelefteq \mathcal{N}$ for $1 \leqslant i \leqslant k$.

Step 1: For each $1 \leqslant i \leqslant k$, since $U_i \trianglelefteq N$, we have $Z(N) \cap U_i \neq 1$. So, for each $P \in \mathcal{N}^c$, we have $P \cap U_i \geqslant Z(N) \cap U_i \neq 1$ for each $1 \leqslant i \leqslant k$, and hence, $P \in \mathcal{H}$. Thus, $\mathcal{N}^c \subseteq \mathcal{H}$.

By definition, $\mathcal{N}$ is $\mathcal{H}$-generated. So, by [5, Theorem I.3.10], to prove that $\mathcal{N}$ is saturated, it suffices to prove that it is $\mathcal{H}$-saturated; that is, that each $P \in \mathcal{H}$ is $\mathcal{N}$-conjugate to a subgroup that is fully automized and receptive in $\mathcal{N}$ (see [5, Definition I.3.9]).

If $P \in \mathcal{H}$ is receptive in $\mathcal{F}$ and $\varphi \in \mathrm{Iso}_{\mathcal{N}}(Q, P)$ for some $Q \leqslant N$, then φ extends to some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi^{\mathcal{F}}, S)$, and $\bar{\varphi}(N_\varphi^{\mathcal{F}} \cap U_i) \leqslant U_i$ for each $1 \leqslant i \leqslant k$ by (4.5). Hence, $\bar{\varphi}$ restricts to an element of $\mathrm{Hom}_{\mathcal{N}}(N_\varphi^{\mathcal{N}}, N)$. Thus, P is receptive in $\mathcal{N}$.

Assume $P \in \mathcal{H}$ is fully automized in $\mathcal{F}$. By (4.5), each $\beta \in \mathrm{Aut}_{\mathcal{F}}(P)$ permutes the subgroups $P \cap U_i$ for $1 \leqslant i \leqslant k$, while $\beta \in \mathrm{Aut}_{\mathcal{N}}(P)$ if and only if it sends each $P \cap U_i$ to itself. So, $\mathrm{Aut}_{\mathcal{N}}(P)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(P)$. Also, $\mathrm{Aut}_N(P) = \mathrm{Aut}_S(P) \cap \mathrm{Aut}_{\mathcal{N}}(P)$: if $c_x \in \mathrm{Aut}_{\mathcal{N}}(P)$ for $x \in N_S(P)$, then ${}^x(P \cap U_i) \leqslant U_i$ for each $1 \leqslant i \leqslant k$ and hence $x \in N$. So, $\mathrm{Aut}_N(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{N}}(P))$ since $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, and we conclude that P is fully automized in $\mathcal{N}$.

Now fix $P \in \mathcal{H}$, and let $\chi \in \mathrm{Hom}_{\mathcal{F}}(P, N)$ be such that $\chi(P)$ is fully normalized in $\mathcal{F}$. Then $\chi(P) \in \mathcal{H}$, and we just showed that $\chi(P)$ is fully automized and receptive in $\mathcal{N}$. By (4.5) and (4.6), there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(N)$ such that $\alpha\chi \in \mathrm{Hom}_{\mathcal{N}}(P, N)$. Since ${}^{\alpha}\mathcal{N} = \mathcal{N}$ by (4.7), the subgroup $\alpha\chi(P)$ is also fully automized and receptive in $\mathcal{N}$, and is $\mathcal{N}$-conjugate to P. Since $P \in \mathcal{H}$ was arbitrary, this proves that $\mathcal{N}$ is $\mathcal{H}$-saturated, and finishes the proof that it is saturated.

Step 2: We first check that N is strongly closed in $\mathcal{F}$. Set

$$K = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(U) \mid \alpha(U_i) = U_i, \text{ all } 1 \leqslant i \leqslant k\} \leqslant \mathrm{Aut}_{\mathcal{F}}(U).$$

Thus, $N = N_S^K(U)$. Let $x, y \in S$ be such that $x \in N$ and $y \in x^{\mathcal{F}}$; we claim that $y \in N$.

Let $P, Q \leqslant S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ be such that $x \in P$, $y \in Q$, and $\varphi(x) = y$. Then $\varphi(P \cap U) \leqslant Q \cap U$ since U is strongly closed in $\mathcal{F}$ as noted above, and so, φ induces a homomorphism $\bar{\varphi}$ from $PU/U \cong P/(P \cap U)$ to $QU/U \cong Q/(Q \cap U)$. By a theorem of Puig (see [15, Theorem 5.14]), $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}/U}(PU/U, QU/U)$. In other words, there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(PU, QU)$ such that for each $g \in P$, $\psi(g) \in \varphi(g)U$.

Now, $c_{\psi(x)} = (\psi|_U)c_x(\psi|_U)^{-1}$ where $\psi|_U \in \mathrm{Aut}_{\mathcal{F}}(U)$. Also, $c_x \in K$ since $x \in N = N_S^K(U)$, and K is normal in $\mathrm{Aut}_{\mathcal{F}}(U)$ since each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(U)$ permutes the U_i. So, $c_{\psi(x)} \in K$, and hence, $\psi(x) \in N$. Hence, $y \in \psi(x)U \subseteq N$, finishing the proof that N is strongly closed.

If $x \in C_S(N) \leqslant C_S(U)$, then ${}^xU_i = U_i$ for each $1 \leqslant i \leqslant k$ (hence for each $1 \leqslant i \leqslant k$), and so, $x \in N$. Thus, $C_S(N) \leqslant N$, so the extension condition holds for $\mathcal{N} \leqslant \mathcal{F}$.

By (4.5) and (4.6), for each $P, Q \leqslant N$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, there is $\delta \in \Delta$ such that $\varphi(P \cap U_i) \leqslant Q \cap U_{\delta(i)}$ for all $1 \leqslant i \leqslant k$, and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(N)$ such that $\alpha(U_i) = U_{\delta(i)}$ for each $1 \leqslant i \leqslant k$. So, $\alpha^{-1}\varphi \in \mathrm{Hom}_{\mathcal{N}}(P, \alpha^{-1}(Q))$, and the Frattini condition for normality holds. The invariance condition holds by (4.7), and thus, $\mathcal{N} \trianglelefteq \mathcal{F}$.

For each $\beta \in \mathrm{Aut}(\mathcal{F})$, β permutes the components of $\mathcal{F}$, and hence permutes the subgroups U_i and the members of the set $\mathcal{H}$. So, $c_\beta(\mathcal{N}) = \mathcal{N}$ by the above definition of $\mathcal{N}$, and $\mathcal{N}$ is characteristic in $\mathcal{F}$.

Step 3: By [4, 9.8.3] and since $\mathcal{N} \trianglelefteq \mathcal{F}$, we have $E(\mathcal{F}) = E(\mathcal{N}) \trianglelefteq \mathcal{N}$. For each $1 \leqslant i \leqslant k$, we have $C_i \trianglelefteq E(\mathcal{F})$ by [4, 9.8.2], and $c_\alpha(C_i) = C_i$ for each $\alpha \in \mathrm{Aut}_{\mathcal{N}}(U)$ by definition of $\mathcal{N}$. So, $C_i \trianglelefteq \mathcal{N}$ by Lemma 2.4. □

By construction, $\mathcal{N}$ is the largest saturated subsystem of $\mathcal{F}$ that contains each of the C_i for $1 \leqslant i \leqslant k$ as a normal subsystem.

5 | TAMENESS OF REALIZABLE FUSION SYSTEMS

We are now ready to show that realizable fusion systems are tame, assuming the classification of finite simple groups. This has already been shown in earlier papers for fusion systems of known simple groups (see Proposition 5.2). When $\mathcal{F}$ is the fusion system of an arbitrary finite p'-reduced $\mathcal{KC}$-group G, we will show that it is tame via a series of reductions based on an examination of the components of G.

We first restrict attention to tameness of fusion systems of finite simple groups. This was shown in most cases in earlier papers, and will be summarized below, but there were two cases whose proofs assumed earlier results that were in error.

Lemma 5.1. *Let (G, p) be one of the pairs $(\mathrm{He}, 3)$ or $(Co_1, 5)$, choose $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}^c_S(G)$. Then $\mathrm{Out}(\mathcal{L}) = 1$, and so $\mathcal{F}$ is tamely realized by G.*

Proof. Since p is odd, $\mathrm{Out}(\mathcal{F}) \cong \mathrm{Out}(\mathcal{L})$ in both cases. The simplest proof of this is given in [31, Theorem C] (and the sporadic groups are handled in Proposition 4.4 of that paper). A more general result is shown in [33, Theorem C] and [18, Theorem 1.1].

When $G = \mathrm{He}$ and $p = 3$, the argument in [36, p. 139] claimed (wrongly) that $\mathcal{F}$ is simple, but did not actually use this. Since S is extraspecial of order 27 and exponent 3 and $\mathrm{Out}_G(S) \cong D_8$, we have

$$D_8 \cong \mathrm{Out}_G(S) \leqslant \mathrm{Aut}(\mathcal{F})/\mathrm{Inn}(S) \leqslant N_{\mathrm{Out}(S)}(\mathrm{Out}_G(S)) \cong SD_{16}.$$

Elements in $N_{\mathrm{Aut}(S)}(\mathrm{Aut}_G(S)) \setminus \mathrm{Aut}_G(S)$ exchange subgroups of S of order 9 with nonisomorphic automizers, and hence do not normalize $\mathcal{F}$. So $\mathrm{Aut}(\mathcal{F}) = \mathrm{Aut}_G(S)$, and $\mathrm{Out}(\mathcal{F}) = 1$.

When $G = Co_1$ and $p = 5$, the proof that $\mathrm{Out}(\mathcal{F}) = 1$ in [36, p. 138] used the incorrect claim that $\mathcal{F}$ has a normal subsystem of index 2. So, we replace that argument with the following one. By [16, Theorem 5.1] and the correction in [43, p. 145], S contains a unique elementary abelian subgroup Q of order 5^3 and index 5, and $N_G(Q)/Q \cong \mathrm{Aut}_G(Q) \cong C_4 \times \Sigma_5$. Set $H = N_G(Q)$. By [36, Lemma 1.2(b)] and since $C_H(Q) = Q$, we have $|\mathrm{Out}(\mathcal{F})| \leqslant |\mathrm{Out}(H)|$. By [39, Lemma 1.2], there is an exact sequence

$$0 \longrightarrow H^1(H/Q; Q) \longrightarrow \mathrm{Out}(H) \longrightarrow N_{\mathrm{Out}(Q)}(\mathrm{Out}_H(Q))/\mathrm{Out}_H(Q),$$

and by [7, p. 110], there is a five-term exact sequence for the homology of H/Q as an extension of C_4 by Σ_5 that begins with

$$0 \longrightarrow H^1(\Sigma_5; H^0(C_4; Q)) \longrightarrow H^1(H/Q; Q) \longrightarrow H^0(\Sigma_5; H^1(C_4; Q)).$$

Since $H^0(C_4; Q) = H^1(C_4; Q) = 0$ (C_4 acts on $Q \cong C_5 \times C_5 \times C_5$ via multiplication by scalars), this proves that $H^1(H/Q; Q) = 0$. Also,

$$N_{\mathrm{Out}(Q)}(\mathrm{Out}_H(Q))/\mathrm{Out}_H(Q) \cong N_{GL_3(5)}(C_4 \times \Sigma_5)/(C_4 \times \Sigma_5)$$

is trivial since $GL_3(5) \cong C_4 \times PSL_3(5)$ and $GO_3(5) \cong \Sigma_5$ is a maximal subgroup of $PSL_3(5)$ (see, e.g., [22, Theorem 6.5.3]). So, $\mathrm{Out}(\mathcal{F}) = \mathrm{Out}(H) = 1$. □

We now summarize what we need to know here about tameness of fusion systems of finite simple groups.

Proposition 5.2. *Fix a known simple group G, choose $S \in \mathrm{Syl}_p(G)$, and assume that $S \ntrianglelefteq \mathcal{F}_S(G)$. Then, $\mathcal{F}_S(G)$ is tamely realized by some known simple group G^*.*

Proof. Set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}^c_S(G)$ for short. Note that G is nonabelian since $S \not\trianglelefteq \mathcal{F}_S(G)$.

Assume first that $G \cong A_n$ for some $n \geqslant 5$. By [2, Proposition 4.8], if $p = 2$ and $n \geqslant 8$ or if p is odd and $p^2 \leqslant n \equiv 0, 1 \pmod p$, then κ_G is an isomorphism. If p is odd and $p^2 < n \equiv k \pmod p$ where $2 \leqslant k \leqslant p-1$, then $\mathcal{F}$ is still tamely realized by A_n: $\mathrm{Out}(\mathcal{L}) = 1$ since $\mathcal{F}$ is isomorphic to the fusion system of Σ_n and also that of Σ_{n-k}. If $p = 2$ and $n = 6, 7$, then $\mathcal{F}$ is tamely realized by $A_6 \cong PSL_2(9)$ (and κ_{A_6} is an isomorphism). In all other cases, S is abelian and hence $S \trianglelefteq \mathcal{F}$.

If G is of Lie type in defining characteristic p, or if $p = 2$ and $G \cong {}^2F_4(2)'$, then by [12, Theorems A and D], κ_G is an isomorphism except when $p = 2$ and $G \cong SL_3(2)$. In this exceptional case, $\mathcal{F}$ is tamely realized by A_6 again.

If G is of Lie type in defining characteristic q_0 for some prime $q_0 \neq p$, then by [12, Theorem B], $\mathcal{F}$ is tamely realized by some other simple group G^* of Lie type. See also Tables 0.1– 0.3 in [12] for a list of which groups of Lie type do tamely realize their fusion system, and when they do not, which other groups they can be replaced by.

If G is a sporadic simple group (and $S \not\trianglelefteq \mathcal{F}$), then by [36, Theorem A] and Lemma 5.1, κ_G is an isomorphism except when (G, p) is one of the pairs $(M_{11}, 2)$ or $(\mathrm{He}, 3)$. If $(G, p) = (\mathrm{He}, 3)$, then by the same theorem, $|\mathrm{Out}(G)| = 2$ and $\mathrm{Out}(\mathcal{L}) = 1$, so $\mathcal{F}$ is still tamely realized by G. If $(G, p) = (M_{11}, 2)$, then $\mathcal{F}$ is the unique simple fusion system over SD_{16}, and is tamely realized by $G^* = PSU_3(5)$ (and κ_{G^*} is an isomorphism) by [2, Proposition 4.4]. □

The statements in the next proposition are very similar to results proven in [2], but except for part (a) are not stated there explicitly. Their proof consists mostly of repeating those arguments. Since many of the results referred to in [2, §2] require considering linking systems that are not centric, they depend in a crucial way on [2, Lemma 1.17], which states that $\mathrm{Out}(\mathcal{L}_0) \cong \mathrm{Out}(\mathcal{L})$ whenever $\mathcal{L}_0 \leqslant \mathcal{L}$ are linking systems associated to the same fusion system $\mathcal{F}$ and $\mathrm{Ob}(\mathcal{L}_0) \subseteq \mathrm{Ob}(\mathcal{L})$ are both $\mathrm{Aut}(\mathcal{F})$-invariant.

Proposition 5.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.*

(a) *If $\mathcal{F}/Z(\mathcal{F})$ is tamely realized by the finite p'-reduced $\mathscr{HC}$-group $\bar{G}$, then $\mathcal{F}$ is tamely realized by a finite p'-reduced $\mathscr{HC}$-group G such that $G/Z(G) \cong \bar{G}$.*

(b) *Assume $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ is a characteristic subsystem over $S_0 \trianglelefteq S$, with the property that $\mathcal{F}_0^{cr} \subseteq \mathcal{F}^c$. If $\mathcal{F}_0$ is tamely realized by a finite p'-reduced $\mathscr{HC}$-group G_0, then $\mathcal{F}$ is tamely realized by a finite p'-reduced $\mathscr{HC}$-group G such that $G_0 \trianglelefteq G$.*

(c) *If $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ is a characteristic subsystem of index prime to p, and $\mathcal{F}_0$ is tamely realized by a finite p'-reduced $\mathscr{HC}$-group G_0, then $\mathcal{F}$ is tamely realized by a finite p'-reduced $\mathscr{HC}$-group G such that $G_0 \trianglelefteq G$.*

(d) *If $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ is a characteristic subsystem of p-power index, $\mathcal{F}_0$ is tamely realized by a finite p'-reduced $\mathscr{HC}$-group G_0, and $Z(\mathcal{F}) = 1$, then $\mathcal{F}$ is tamely realized by a finite p'-reduced $\mathscr{HC}$-group G such that $G_0 \trianglelefteq G$.*

Proof.

(a) Assume that $\mathcal{F}/Z(\mathcal{F})$ is tamely realized by a finite p'-reduced $\mathscr{HC}$-group $\bar{G}$. Then $Z(\bar{G}) = Z(\mathcal{F}/Z(\mathcal{F}))$ by Proposition 4.11. So, by [2, Proposition 2.18], $\mathcal{F}$ is tamely realized by a finite p'-reduced group G such that $G/Z(G) \cong \bar{G}$. For each component C of G, the subgroup $CZ(G)/Z(G) \cong C/(Z(C) \cap Z(G))$ is a component of $\bar{G}$, and so, G is also a $\mathscr{HC}$-group.

(b) Assume $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ is characteristic over $S_0 \trianglelefteq S$ with $\mathcal{F}_0^{cr} \subseteq \mathcal{F}^c$. Let $\mathcal{H}_0$ be the set of all $P \in \mathcal{F}^c$ such that $P \leqslant S_0$, and let $\mathcal{H}$ be the set of all $P \leqslant S$ such that $P \cap S_0 \in \mathcal{H}_0$. For each $P \in \mathcal{H}$, $P \cap S_0 \in \mathcal{F}^c$ by assumption, and hence, $P \in \mathcal{F}^c$. Thus, $\mathcal{H} \subseteq \mathcal{F}^c$, and $\mathcal{F}_0^{cr} \subseteq \mathcal{H}_0$ since $\mathcal{F}_0^{cr} \subseteq \mathcal{F}^c$. So, by [2, Lemma 1.30] and the existence of a centric linking system associated to $\mathcal{F}$ (Theorem 1.9), there is a normal pair of linking systems $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ with object sets $\mathcal{H}_0$ and $\mathcal{H}$. Furthermore,

$$C_{\mathrm{Aut}_{\mathcal{L}}(S_0)}(\mathcal{L}_0) = \delta_{S_0}(C_S(\mathcal{L}_0)) \leqslant \delta_{S_0}(C_S(\mathcal{F}_0)) \leqslant \delta_{S_0}(C_S(S_0)) \leqslant \delta_{S_0}(S_0) \leqslant \mathrm{Aut}_{\mathcal{L}_0}(S_0) :$$

the equality and first inequality by Lemma 2.11(a), the third inequality since $S_0 \in \mathcal{F}_0^{cr} \subseteq \mathcal{F}^c$, and the other two by definition. So, $\mathcal{L}_0$ is centric in $\mathcal{L}$.

Assume that $\mathcal{F}_0$ is tamely realized by a finite p'-reduced $\mathscr{K}\mathscr{C}$-group G_0 with $S_0 \in \mathrm{Syl}_p(G_0)$. By Proposition 4.11, we have $Z(G_0) = Z(\mathcal{F}_0)$. Then, $\mathcal{L}_0 \cong \mathcal{L}_{S_0}^{\mathcal{H}_0}(G_0)$ (the full subcategory of $\mathcal{L}_{S_0}^c(G_0)$ with objects the set $\mathcal{H}_0$) by the uniqueness of linking systems. By definition, the sets of objects $\mathcal{H}_0$ and $\mathcal{H}$ are invariant under the actions of $\mathrm{Aut}(\mathcal{F}_0)$ and $\mathrm{Aut}(\mathcal{F})$, respectively. Also, $\mathcal{L}_0$ is $\mathrm{Aut}(\mathcal{L})$-invariant, since $\mathcal{F}_0$ is characteristic in $\mathcal{F}$, and $\mathcal{L}_0 = \pi^{-1}(\mathcal{F}_0)$ by [2, Lemma 1.30] (where $\pi : \mathcal{L} \longrightarrow \mathcal{F}$ is the structure functor for $\mathcal{L}$). All hypotheses in [2, Proposition 2.16] are thus satisfied, and so, $\mathcal{F}$ is tamely realized by some finite group G such that $G_0 \trianglelefteq G$ and $G/G_0 \cong \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0)$.

By the Frattini argument, $G = G_0 N_G(S_0)$, and hence,

$$N_G(S_0)/N_{G_0}(S_0) \cong G/G_0 \cong \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0).$$

Since $\mathrm{Aut}_{\mathcal{L}}(S_0) = N_G(S_0)/O^p(C_G(S_0))$ and similarly for $\mathcal{L}_0$, this proves that $O^p(C_G(S_0)) = O^p(C_{G_0}(S_0))$. Also,

$$C_{G_0}(S_0) = Z(S_0) \times O^p(C_{G_0}(S_0)) \qquad \text{and} \qquad C_G(S_0) = Z(S_0) \times O^p(C_G(S_0)) :$$

the last equality since $S_0 \in \mathcal{F}_0^{cr} \subseteq \mathcal{F}^c$. So $C_G(S_0) = C_{G_0}(S_0) \leqslant G_0$. In particular, since $[O_{p'}(G), G_0] \leqslant O_{p'}(G_0) = 1$, this implies that $O_{p'}(G) \leqslant C_G(S_0) \leqslant G_0$. So $O_{p'}(G) \leqslant O_{p'}(G_0) = 1$, and hence G is p'-reduced.

If C is a component of G, then since $G_0 \trianglelefteq G$, either C is a component of G_0 or $[C, G_0] = 1$ (see [3, 31.4] or [5, Lemma A.12]). Since $C_G(G_0) \leqslant C_G(S_0) \leqslant G_0$, and since all components of G contained in G_0 are subnormal in G_0 and hence in $\mathrm{Comp}(G_0)$, this shows that $\mathrm{Comp}(G) \subseteq \mathrm{Comp}(G_0)$, and hence, that G is also a $\mathscr{K}\mathscr{C}$-group.

(c) By [5, Lemma I.7.6(a)], we have $\mathcal{F}_0^c = \mathcal{F}^c$. (See also Definition 1.14(c).) Hence, (c) is a special case of (b).

(d) Assume that $\mathcal{F}_0$ has p-power index in $\mathcal{F}$ and $Z(\mathcal{F}) = 1$. By [9, Proposition 3.8(b)], a subgroup $P \leqslant S_0$ is $\mathcal{F}$-quasi-centric if and only if it is $\mathcal{F}_0$-quasi-centric. Let $\mathcal{H}$ be the set of all $P \leqslant S$ such that $P \cap S_0$ is $\mathcal{F}_0$-quasi-centric. Then $\mathcal{H} \subseteq \mathcal{F}^q$ since overgroups of quasi-centric subgroups are quasi-centric, and $\mathcal{H} \supseteq \mathcal{F}^{rc}$ by [2, Lemma 1.20(d)]. By Theorem 1.9, there is a unique linking system $\mathcal{L}$ associated to $\mathcal{F}$ with $\mathrm{Ob}(\mathcal{L}) = \mathcal{H}$, and by [9, Theorem 4.4], there is a unique linking system $\mathcal{L}_0 \leqslant \mathcal{L}$ associated to $\mathcal{F}_0$ with $\mathrm{Ob}(\mathcal{L}_0) = (\mathcal{F}_0)^q$. Then $\mathcal{L}_0 \trianglelefteq \mathcal{L}$: the condition on objects (Definition 1.12(a)) holds by construction, and the invariance condition (1.12(b)) holds by the uniqueness of $\mathcal{L}_0$.

By Lemma 2.11(c), there is an action of $\mathcal{L}/\mathcal{L}_0$ on $C_S(\mathcal{L}_0)$ such that $C_{C_S(\mathcal{F}_0)}(\mathcal{L}/\mathcal{L}_0) = Z(\mathcal{F})$, where $Z(\mathcal{F}) = 1$ by assumption. Also, $\mathcal{L}/\mathcal{L}_0 = \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0)$ is a p-group since $\mathcal{F}_0$ has p-power index in $\mathcal{F}$, so $C_S(\mathcal{L}_0) = 1$. Hence, $Z(\mathcal{F}_0) = 1$ and $C_{\mathrm{Aut}_{\mathcal{L}}(S_0)}(\mathcal{L}_0) = 1$ by Lemma 2.11(a), and, in particular, $\mathcal{L}_0$ is centric in $\mathcal{L}$.

If in addition, $\mathcal{F}_0$ is characteristic in $\mathcal{F}$, then each $\alpha \in \mathrm{Aut}(\mathcal{L})$ induces an element $\beta = \widetilde{\mu}_{\mathcal{L}}(\alpha) \in \mathrm{Aut}(\mathcal{F})$ (Definition 2.7), and $c_\beta(\mathcal{F}_0) = \mathcal{F}_0$ by assumption. Hence, $\alpha(\mathcal{L}_0) = \mathcal{L}_0$ by Proposition 2.6 and the uniqueness of the linking systems in [9, Theorem 4.4]. Also, by construction, $\mathrm{Ob}(\mathcal{L}_0)$ and $\mathrm{Ob}(\mathcal{L})$ are invariant under the actions of $\mathrm{Aut}(\mathcal{F}_0)$ and $\mathrm{Aut}(\mathcal{F})$.

Assume that $\mathcal{F}_0$ is tamely realized by a finite p'-reduced $\mathscr{K}\mathscr{C}$-group G_0. By Proposition 4.11, we have $Z(G_0) = Z(\mathcal{F}_0)$. By Theorem 1.9 (the uniqueness of centric linking systems), $\mathcal{L}_0 \cong \mathcal{L}^q_{S_0}(G_0)$. The hypotheses of [2, Proposition 2.16] thus hold, and so, $\mathcal{F}$ is tamely realized by some group G such that $G_0 \trianglelefteq G$ and $G/G_0 \cong \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0)$. In particular, G/G_0 is a p-group, so all components of G are in G_0, and $\mathrm{Comp}(G) = \mathrm{Comp}(G_0)$. Also, $O_{p'}(G) = O_{p'}(G_0) = 1$, and so, G is a p'-reduced $\mathscr{K}\mathscr{C}$-group. □

We are now ready to prove our main theorem. As explained in the introduction, Theorem 5.4, as well as Theorems 5.5 and 5.6, have been formulated so that their proofs are independent of the classification of finite simple groups.

Recall that by Lemma 4.2(b,c), the condition $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$ in the statement of Theorem 5.4 is satisfied whenever $\mathcal{E}$ is centric in $\mathcal{F}$, and these two conditions are, in fact, equivalent if $O_p(\mathcal{F}) = 1$.

Theorem 5.4. *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ be a normal pair of fusion systems over $T \trianglelefteq S$ such that $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$. Assume that $\mathcal{E}$ is realized by a finite p'-reduced group all of whose components are known quasi-simple groups. Then $\mathcal{F}$ is tamely realized by a finite p'-reduced group all of whose components are known quasi-simple groups.*

Proof. Let $\mathcal{S}$ be the set of all triples $(\mathcal{F}, \mathcal{E}, H)$ such that

- $\mathcal{E} \trianglelefteq \mathcal{F}$ are saturated fusion systems over finite p-groups $T \trianglelefteq S$ such that $\mathrm{Comp}(\mathcal{E}) = \mathrm{Comp}(\mathcal{F})$;
- H is a p'-reduced $\mathscr{K}\mathscr{C}$-group such that $T \in \mathrm{Syl}_p(H)$ and $\mathcal{E} = \mathcal{F}_T(H)$; and
- $\mathcal{F}$ is not tamely realized by any finite p'-reduced $\mathscr{K}\mathscr{C}$-group.

Assume that the theorem does not hold; that is, that $\mathcal{S} \neq \emptyset$. Let $(\mathcal{F}, \mathcal{E}, H) \in \mathcal{S}$ be such that $(|\mathrm{Mor}(\mathcal{E})|, |\mathrm{Mor}(\mathcal{F})|) \in \mathbb{N}^2$ is the smallest possible under the lexicographic ordering. In other words, there are no triples $(\mathcal{F}^*, \mathcal{E}^*, H^*)$ in $\mathcal{S}$ where $|\mathrm{Mor}(\mathcal{E}^*)| < |\mathrm{Mor}(\mathcal{E})|$; and among those where $|\mathrm{Mor}(\mathcal{E}^*)| = |\mathrm{Mor}(\mathcal{E})|$, there are none where $|\mathrm{Mor}(\mathcal{F}^*)| < |\mathrm{Mor}(\mathcal{F})|$.

We show in Step 1 that $O_p(\mathcal{F}) = 1$, and that H can be chosen to be a product of known finite simple groups. We then show in Step 2 that the components of $\mathcal{F}$ are all normal in $\mathcal{F}$, and reduce this to a contradiction in Step 3.

Step 1: Let $L_1, \dots, L_k$ be the components of H, and set $U_i = T \cap L_i \in \mathrm{Syl}_p(L_i)$. Set $\mathcal{D}_i = \mathcal{F}_{U_i}(L_i)$ and $C_i = O^{p'}(\mathcal{D}_i)$ for each i. Assume that the L_i are ordered so that for some m, C_i is quasi-simple if and only if $i \leqslant m$. We are thus in the situation of Proposition 4.9, with H, T, U_i, and $\mathcal{D}_i$ in the roles of G, S, T_i, and $\mathcal{E}_i$. So, $\mathrm{Comp}(\mathcal{F}) = \mathrm{Comp}(\mathcal{E}) = \{C_1, \dots, C_m\}$ by that proposition.

Set $U = U_1 \cdots U_m$ and $\mathcal{E}_0 = \mathcal{F}_U(L_1 \cdots L_m) \leqslant \mathcal{E}$. By Proposition 4.12(a), there is a unique minimal subsystem $\mathcal{C}^* \leqslant \mathcal{E}_0$ over U containing $O^{p'}(\mathcal{E}_0)$ that is realized by a p'-reduced $\mathscr{K}\mathscr{C}$-group, and C^* is realized by a central product H_0 of known finite p'-reduced quasi-simple groups. By

Proposition 4.12(b), $\mathcal{C}^*$ is characteristic in $\mathcal{E}$ and hence normal in $\mathcal{F}$. Hence, $(\mathcal{F}, \mathcal{C}^*, H_0) \in \mathcal{S}$, so $\mathcal{E} = \mathcal{C}^*$ by the minimality of $|\mathrm{Mor}(\mathcal{E})|$, and we can take $H = H_0$. In particular, $m = k$, and H is a central product of known finite p'-reduced quasi-simple groups.

Set $Q = O_p(\mathcal{F})$, and set $S_0 = N_S^{\mathrm{Inn}(Q)}(Q) = QC_S(Q)$ and $\mathcal{F}_0 = N_{\mathcal{F}}^{\mathrm{Inn}(Q)}(Q)$. Thus, $\mathcal{F}_0$ is a fusion subsystem over S_0. Since $Q^{\mathcal{F}} = \{Q\}$, the subgroup Q is fully $\mathrm{Inn}(Q)$-normalized in $\mathcal{F}$, and hence, $\mathcal{F}_0$ is saturated (see Definition I.5.1 and Theorem I.5.5 in [5]). Also, $\mathcal{F}_0$ is weakly normal in $\mathcal{F}$ by [2, Proposition 1.25(c)], and is normal since $C_S(S_0) \leqslant S_0$ (the extension condition holds). For each $\alpha \in \mathrm{Aut}(\mathcal{F})$, $\alpha(Q) = Q$, so $c_\alpha(\mathcal{F}_0) = \mathcal{F}_0$, and hence, $\mathcal{F}_0$ is characteristic in $\mathcal{F}$.

If $P \in \mathcal{F}_0^{cr}$, then $P \geqslant Q$ since $Q \trianglelefteq \mathcal{F}_0$ (see [5, Proposition I.4.5(a⇒b)]). So, for each $P^* \in P^{\mathcal{F}}$, $P^* \geqslant Q$, and $C_S(P^*) \leqslant C_S(Q) \leqslant S_0$. Thus, $C_S(P^*) = C_{S_0}(P^*) \leqslant P^*$ since $P^* \in \mathcal{F}_0^c$, so $P \in \mathcal{F}^c$. Thus, $\mathcal{F}_0^{cr} \subseteq \mathcal{F}^c$.

By Proposition 5.3(b) and since $\mathcal{F}$ is not tamely realized by any p'-reduced $\mathscr{K}\mathscr{C}$-group, the subsystem $\mathcal{F}_0$ is not tamely realized by any p'-reduced $\mathscr{K}\mathscr{C}$-group. Also, $\mathcal{E} \trianglelefteq N_{\mathcal{F}}^{\mathrm{Inn}(Q)}(Q) = \mathcal{F}_0$ by Lemma 4.14 (with $\mathcal{E}$ in the role of $\mathcal{C}^*$). Thus, $(\mathcal{F}_0, \mathcal{E}, H) \in \mathcal{S}$, and by the minimality assumption, we have $\mathcal{F} = \mathcal{F}_0$. So $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Inn}(Q)$.

Let $1 = Z_0(Q) \leqslant Z_1(Q) \leqslant Z_2(Q) \leqslant \cdots \leqslant Q$ be the upper central series of $Q = O_p(\mathcal{F})$. Thus, for each i, $Z_{i+1}(Q)/Z_i(Q) = Z(Q/Z_i(Q)) = Z(\mathcal{F}/Z_i(Q))$ since $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Inn}(Q)$. By Proposition 5.3(a), if $\mathcal{F}/Z_{i+1}(Q)$ is tamely realized by a finite p'-reduced $\mathscr{K}\mathscr{C}$-group G_{i+1}, then $\mathcal{F}/Z_i(Q)$ is tamely realized by a finite p'-reduced $\mathscr{K}\mathscr{C}$-group G_i. Since $\mathcal{F}$ is not tamely realized by any p'-reduced $\mathscr{K}\mathscr{C}$-group by assumption, we conclude that $\mathcal{F}/Q$ is not tamely realized by any p'-reduced $\mathscr{K}\mathscr{C}$-group either.

For each $P, R \leqslant C_S(Q)$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$, the morphism φ extends to some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ)$ since $Q \trianglelefteq \mathcal{F}$, and $\varphi|_Q = c_g|_Q$ for some $g \in Q$ since $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Inn}(Q)$. Hence, $c_g^{-1}\bar{\varphi}|_P = \varphi$ since $c_g|_{C_S(Q)} = \mathrm{Id}$. Thus, each morphism in $\mathcal{F}$ between subgroups of $C_S(Q)$ lies in $C_{\mathcal{F}}(Q)$, and so, $\mathcal{F}/Q \cong C_{\mathcal{F}}(Q)/Z(Q)$ by Lemma 1.22, applied with $C_{\mathcal{F}}(Q)$ in the role of $\mathcal{E}$.

Set $Z = Z(Q)$. We have now shown that $C_{\mathcal{F}}(Q)/Z$ is not realized by any p'-reduced $\mathscr{K}\mathscr{C}$-group. Also, $\mathcal{E} \trianglelefteq C_{\mathcal{F}}(Q)$ by Lemma 4.14 (applied with $\mathcal{E}$ in the role of $\mathcal{C}^*$), so $Z\mathcal{E} \trianglelefteq C_{\mathcal{F}}(Q)$ by Lemma 1.21, and $Z\mathcal{E}/Z \trianglelefteq C_{\mathcal{F}}(Q)/Z$ by Lemma 1.18. By Lemma 1.22, $Z\mathcal{E}/Z \cong \mathcal{E}/(Z \cap T)$, where $\mathcal{E}/(Z \cap T)$ is realized by $H/(Z \cap T)$ (see [15, Theorem 5.20]). So, $Z\mathcal{E}/Z$ is realized by a p'-reduced $\mathscr{K}\mathscr{C}$-group $H_0 \cong H/(Z \cap T)$. Also,

$$\mathrm{Comp}(Z\mathcal{E}/Z) = \{ZC_1/Z, \dots, ZC_k/Z\} = \mathrm{Comp}(C_{\mathcal{F}}(Q)/Z)$$

by Lemma 4.3. So $(C_{\mathcal{F}}(Q)/Z, Z\mathcal{E}/Z, H_0) \in \mathcal{S}$, and by the minimality assumption on $(\mathcal{F}, \mathcal{E}, H)$, we have $\mathcal{E} \cong Z\mathcal{E}/Z$ and $\mathcal{F} \cong C_{\mathcal{F}}(Q)/Z(Q)$, and thus, $O_p(\mathcal{F}) = Q = 1$.

To summarize, we have reduced to the case where $(\mathcal{F}, \mathcal{E}, H) \in \mathcal{S}$ satisfies:

$$\begin{aligned} &O_p(\mathcal{F}) = O_p(\mathcal{E}) = 1,\ \mathcal{E} = \mathcal{F}_T(H) \text{ where } H = L_1 \times \cdots \times L_k, \text{ and each } L_i \\ &\text{is a known finite simple group. Also, } \mathrm{Comp}(\mathcal{F}) = \mathrm{Comp}(\mathcal{E}) = \{C_1, \dots, C_k\} \\ &\text{where } T = U_1 \times \cdots \times U_k,\ U_i \in \mathrm{Syl}_p(L_i), \text{ and } C_i = O^{p'}(\mathcal{F}_{U_i}(L_i)). \end{aligned} \tag{5.1}$$

Step 2: Let $\mathcal{N} \trianglelefteq \mathcal{F}$ be the characteristic subsystem constructed in Proposition 4.15: a subsystem over $N = \bigcap_{i=1}^k N_S(U_i)$ normal in $\mathcal{F}$ and containing each component C_i as a normal subsystem. In particular, for each $\alpha \in \mathrm{Aut}_{\mathcal{N}}(T)$, $\alpha(U_i) = U_i$ for all $1 \leqslant i \leqslant k$.

For each $P \in \mathcal{N}^c$ and each $Q \in P^{\mathcal{F}}$, $Q \in \mathcal{N}^c$ since $\mathcal{N} \trianglelefteq \mathcal{F}$, and so $Q \geqslant Z(N)$. For each $1 \leqslant i \leqslant k$, we have $U_i \trianglelefteq N$ by definition of N, and hence $Q \cap U_i \neq 1$. Each $x \in C_S(Q)$ centralizes each

$Q \cap U_i$ and hence normalizes each subgroup U_i (recall that each element of S permutes the U_i). So, $C_S(Q) = C_N(Q) \leqslant Q$, and $P \in \mathcal{F}^c$. Thus, $\mathcal{N}^c \subseteq \mathcal{F}^c$. By Proposition 5.3(b) and since $\mathcal{F}$ is not tamely realized by a p'-reduced $\mathscr{K}\mathscr{C}$-group, $\mathcal{N}$ is not tamely realized by a p'-reduced $\mathscr{K}\mathscr{C}$-group either.

Now, $\mathcal{E}$ is weakly normal in $\mathcal{N}$ by [15, Proposition 8.17] and since $\mathcal{E} \trianglelefteq \mathcal{F}$ and $\mathcal{N} \leqslant \mathcal{F}$. By the definition of $\mathcal{N}$ in Proposition 4.15, if $T \leqslant P \leqslant N$, and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ is such that $\alpha|_T \in \mathrm{Aut}_{\mathcal{N}}(T)$, then $\alpha \in \mathrm{Aut}_{\mathcal{N}}(P)$. So, the extension condition for $\mathcal{E} \leqslant \mathcal{N}$ follows from that for $\mathcal{E} \trianglelefteq \mathcal{F}$, and hence, $\mathcal{E} \trianglelefteq \mathcal{N}$.

Thus, $(\mathcal{N}, \mathcal{E}, H) \in \mathcal{S}$, and $\mathcal{F} = \mathcal{N}$ by the minimality assumption. In other words, $(\mathcal{F}, \mathcal{E}, H) \in \mathcal{S}$ satisfies:

$$\text{(5.1) holds, and } C_i \trianglelefteq \mathcal{F} \text{ for each } 1 \leqslant i \leqslant k. \tag{5.2}$$

Step 3: Set $\mathcal{C} = E(\mathcal{F}) = C_1 \times \cdots \times C_k$. By Proposition 4.13 (applied with H and $\mathcal{E}$ in the roles of G and $\mathcal{F}$), there is a unique minimal normal fusion subsystem $\mathcal{C}^* = \mathcal{C}_1^* \times \cdots \times \mathcal{C}_k^* \trianglelefteq \mathcal{E}$ containing $\mathcal{C}$ that is realized by a product of known finite simple groups. Furthermore (by the same proposition), $\mathcal{C} = O^{p'}(\mathcal{C}^*)$, $\mathcal{C}^*$ is centric and characteristic in $\mathcal{E}$, and for each i,

- $C_i \leqslant \mathcal{C}_i^* \leqslant \mathcal{F}_{U_i}(L_i)$,
- $\mathcal{C}_i = O^{p'}(\mathcal{C}_i^*)$, and
- $\mathcal{C}_i^* = \mathcal{F}_{U_i}(H_i^*)$ where H_i^* is a known finite simple group.

Then, $\mathcal{C}^* \trianglelefteq \mathcal{F}_T(H) = \mathcal{E} \trianglelefteq \mathcal{F}$, and so, $\mathcal{C}^* \trianglelefteq \mathcal{F}$ by Lemma 2.4 and since $\mathcal{C}^*$ is characteristic in $\mathcal{E}$. Set $H^* = H_1^* \times \cdots \times H_k^*$, so that $\mathcal{C}^* = \mathcal{F}_T(H^*)$. Thus, $(\mathcal{F}, \mathcal{C}^*, H^*) \in \mathcal{S}$, and $\mathcal{E} = \mathcal{C}^*$ by the minimality assumption.

Let $\mathrm{Aut}^0(\mathcal{C}^*) \leqslant \mathrm{Aut}(\mathcal{C}^*)$ be the subgroup of all automorphisms that send each U_i to itself. Then $\mathrm{Aut}_{\mathcal{F}}(T) \leqslant \mathrm{Aut}^0(\mathcal{C}^*)$ by (5.2) and since $\mathcal{C}^* \trianglelefteq \mathcal{F}$. Each factor $\mathcal{C}_i^*$ is a full subcategory of $\mathcal{C}^*$ (contains all morphisms in $\mathcal{C}^*$ between subgroups of U_i), and hence each $\alpha \in \mathrm{Aut}^0(\mathcal{C}^*)$ sends each $\mathcal{C}_i^*$ to itself. So,

$$\mathrm{Aut}_{\mathcal{F}}(T)/\mathrm{Aut}_{\mathcal{C}^*}(T) \leqslant \mathrm{Aut}^0(\mathcal{C}^*)/\mathrm{Aut}_{\mathcal{C}^*}(T) \cong \prod_{i=1}^{k} \mathrm{Out}(\mathcal{C}_i^*).$$

By assumption, each $\mathcal{C}_i^*$ is realized by a known finite simple group, and hence is tamely realized by a known finite simple group by Proposition 5.2. Since $\mathrm{Out}(K)$ is solvable for each known finite simple group K (see [22, Theorem 7.1.1(a)]), the groups $\mathrm{Out}(\mathcal{C}_i^*)$ are also solvable. So, $\mathrm{Aut}_{\mathcal{F}}(T)/\mathrm{Aut}_{\mathcal{C}^*}(T)$ is solvable.

The hypotheses of [35, Theorem 5(b)] thus hold for the pair $\mathcal{C}^* \trianglelefteq \mathcal{F}$. By that theorem, there is a sequence $\mathcal{C}^* = \mathcal{F}_0 \trianglelefteq \mathcal{F}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{F}_m = \mathcal{F}$ of saturated fusion subsystems, for some $m \geqslant 0$, such that

(i) for each $0 \leqslant j < m$, $\mathcal{F}_j$ is normal of p-power index or index prime to p in $\mathcal{F}_{j+1}$ and $\mathcal{C}^* \trianglelefteq \mathcal{F}_j \trianglelefteq \mathcal{F}$; and

(ii) for each $1 \leqslant j \leqslant m$ and each $\alpha \in \mathrm{Aut}(\mathcal{F}_j)$ with $c_\alpha(\mathcal{C}^*) = \mathcal{C}^*$, we have $c_\alpha(\mathcal{F}_{j'}) = \mathcal{F}_{j'}$ for all $0 \leqslant j' < j$.

Recall that $\mathrm{Comp}(\mathcal{F}) = \{C_1, \dots, C_k\}$. For each $0 \leqslant j \leqslant m-1$, $\mathrm{Comp}(\mathcal{F}_j) \subseteq \mathrm{Comp}(\mathcal{F})$ since $\mathcal{F}_j \trianglelefteq \trianglelefteq \mathcal{F}$, and the opposite inclusion holds since $C_i \trianglelefteq \mathcal{C} \trianglelefteq \mathcal{C}^* \trianglelefteq \mathcal{F}_j$ for each i. Hence, $\mathcal{C}$ is characteristic in $\mathcal{F}_j$. For each $\alpha \in \mathrm{Aut}(\mathcal{F}_j)$, $\alpha|_T \in \mathrm{Aut}(\mathcal{C})$ since $\mathcal{C}$ is characteristic, and $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{C}^*)$

by Proposition 4.12(a). So, $c_\alpha(\mathcal{C}^*) = \mathcal{C}^*$, and hence $c_\alpha(\mathcal{F}_{j'}) = \mathcal{F}_{j'}$ for all $0 < j' < j$ by condition (ii) above.

In particular, this shows that $\mathcal{F}_j$ is characteristic in $\mathcal{F}_{j+1}$ for each j. Also, $Z(\mathcal{F}_j) \leqslant O_p(\mathcal{F}_j) = 1$ for each j by Lemma 2.3(b) and since $O_p(\mathcal{F}) = 1$ and $\mathcal{F}_j \trianglelefteq\trianglelefteq \mathcal{F}$. So, by Proposition 5.3(c,d), and since $\mathcal{F}_j$ has index prime to p or p-power index in $\mathcal{F}_{j+1}$ and $Z(\mathcal{F}_j) = 1$, if $\mathcal{F}_j$ is tamely realized by a finite p'-reduced $\mathcal{KC}$-group G_j, then $\mathcal{F}_{j+1}$ is tamely realized by a finite p'-reduced $\mathcal{KC}$-group $G_{j+1} \geqslant G_j$.

For each $1 \leqslant i \leqslant k$, $\mathcal{C}_i^*$ is tamely realized by some known finite simple group by Proposition 5.2, and so, $\mathcal{C}^* = \mathcal{F}_0$ is tamely realized by a product of known finite simple groups by Proposition 3.7(c). Hence, $\mathcal{F}_i$ is tamely realized by a p'-reduced $\mathcal{KC}$-group for each $1 \leqslant i \leqslant m$. This contradicts our assumption on $\mathcal{F} = \mathcal{F}_m$, and we conclude that $\mathcal{S} = \emptyset$. □

Note that Theorem A is just Theorem 5.4 without mentioning tameness.

We now list some special cases of Theorem 5.4.

Theorem 5.5. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. If all components of $\mathcal{F}$ are realized by known finite quasi-simple groups, then $\mathcal{F}$ is tamely realized by a finite p'-reduced group all of whose components are known quasi-simple groups.*

Proof. This is the special case of Theorem 5.4 where $\mathcal{E}$ is the generalized Fitting subsystem of $\mathcal{F}$. Note that $\mathcal{E}$ is realizable since it is the central product of its components (which are realizable by assumption) and a p-group. □

Our third theorem is the special case of Theorem 5.4 where $\mathcal{E} = \mathcal{F}$.

Theorem 5.6. *Let p be a prime, and let $\mathcal{F}$ be a fusion system over a finite p-group that is realized by a finite p'-reduced group all of whose components are known quasi-simple groups. Then $\mathcal{F}$ is tamely realized by a finite p'-reduced group all of whose components are known quasi-simple groups.*

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