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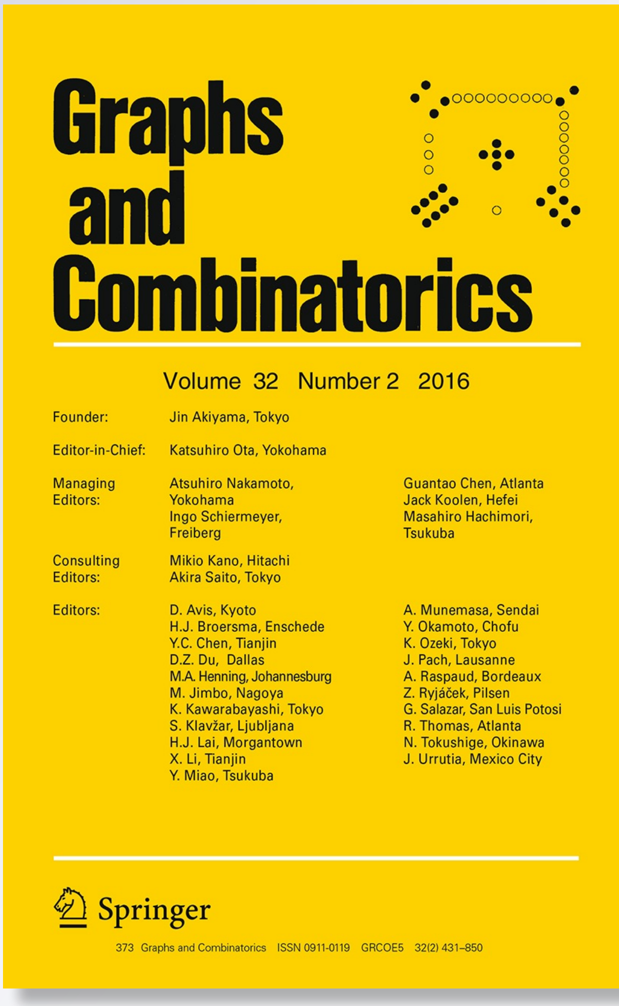

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ORIGINAL PAPER

Chromatic Polynomials of Simplicial Complexes

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Abstract In this note we consider s-chromatic polynomials for finite simplicial complexes. When $s = 1$, the 1-chromatic polynomial is just the usual graph chromatic polynomial of the 1-skeleton. In general, the s-chromatic polynomial depends on the s-skeleton and its value at r is the number of (r, s)-colorings of the simplicial complex.

Keywords Vertex coloring of simplicial complex · s-chromatic polynomial · s-chromatic lattice · s-Stirling number of second kind · Möbius function

Mathematics Subject Classification 05C15 · 05C31

1 Introduction

Let K be a finite simplicial complex with vertex set $V(K) \neq \emptyset$ and let $r \geq 1$ and $s \geq 1$ be two natural numbers. A map $\mathrm{col}\colon V(K) \to \{1, 2, \ldots, r\}$ is an (r, s)-*coloring* of K if there are no monochrome s-simplices in K. We write $\chi^s(K, r)$ for the number of (r, s)-colorings of K.

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Definition 1 The *s-chromatic polynomial* of K is the function $\chi^s(K, r)$ of r. The *s-chromatic number* of K, $\mathrm{chr}^s(K)$, is the minimal $r \geq 1$ with $\chi^s(K, r) > 0$.

This s-chromatic polynomial and the s-chromatic number of a simplicial complex are of course just special cases of the similar but much more general concepts for hypergraphs [9]. In the context of hypergraphs, our chromatic number is sometimes called the weak chromatic number. We restrict attention from hypergraphs to simplicial complexes and combinatorial manifolds in order to add a geometric flavor to a combinatorial problem and because of the connections to Davis–Januszkiewicz spaces [7], rational homotopy theory [6], and combinatorial topology [14]. By focusing on simplicial complex chromatic polynomials and weak colorings we detect phenomena that maybe would go unnoticed in the general context of hypergraph chromatic polynomials.

For instance, we note that the chromatic numbers of K form a descending sequence

$$\mathrm{chr}^1(K) \geq \mathrm{chr}^2(K) \geq \cdots \mathrm{chr}^d(K) \geq \mathrm{chr}^d(K) \geq \mathrm{chr}^{d+1}(K) = 1$$

terminating with $\mathrm{chr}^s(K) = 1$ when s is greater than the dimension $d = \dim K$. The chromatic numbers go down rather steeply because

$$\mathrm{chr}^{st}(K) \leq \left\lceil \frac{\mathrm{chr}^s(K)}{t} \right\rceil$$

as may be seen using rational homotopy [6, Theorem 2] or by noting that we get an $(\lceil r/t \rceil, st)$-coloring from any (r, s)-coloring by mixing batches of t colors from the original palette of r colors to obtain a palette of $\lceil r/t \rceil$ colors.

The theorem below shows that the the s-chromatic polynomial of K, $\chi^s(K, r)$, from Definition 1 is indeed a polynomial in r for fixed K and s. (The ith falling factorial in r is the polynomial $[r]_i = r(r-1)\dots(r-i+1)$.)

Theorem 1 *The s-chromatic polynomial of K is*

$$\chi^s(K, r) = \sum_{i=\mathrm{chr}^s(K)}^{|V(K)|} S(K, i, s)[r]_i$$

where $S(K, i, s)$ is the number of partitions of $V(K)$ into i blocks containing no s-simplex of K.

We introduce in Definition 6 the s-chromatic lattice $L^s(K)$ associated to K. Theorem 4 shows that the integer coefficients $S(K, i, s)$ are connected to the Möbius function of $L^s(K)$. There is more information and several examples illustrating features of the s-chromatic lattice in §2.3.

This is not the only way to determine the s-chromatic polynomial of a complex. Theorems 2 and 3 present two alternatives. Whereas Theorem 3 is a more or less obvious generalization of a known statement for 1-chromatic polynomials, the perspective of Theorem 2, expressing the s-chromatic polynomial as a sum of 1-chromatic polynomials, could be new.

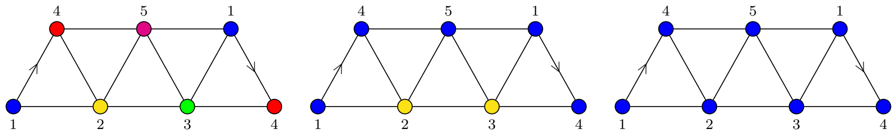


Fig. 1 A (5, 1)-, a (2, 2)-, and a (1, 3)-coloring of a 5-vertex triangulated Möbius band MB

Here is a simple example. Fig. 1 shows a triangulation MB of the Möbius band. To the left is a (5, 1)-, in the middle a (2, 2)-, and to the right a (1, 3)-coloring of MB. The three chromatic polynomials and chromatic numbers[1] of MB are

$$\chi^s(\mathrm{MB}, r) = \begin{cases} r^5 - 10r^4 + 35r^3 - 50r^2 + 24r & s = 1 \\ r^5 - 5r^3 + 5r^2 - r & s = 2 \\ r^5 & s \geq 3 \end{cases} \quad \mathrm{chr}^s(\mathrm{MB}) = \begin{cases} 5 & s = 1 \\ 2 & s = 2 \\ 1 & s \geq 3 \end{cases}$$

The 1-chromatic polynomial of MB is the falling factorial $[r]_5$ because the 1-skeleton is the complete graph on 5 vertices; see Examples 4 and 7 for details on the 2-chromatic polynomial. It is an interesting question what information is contained in the vector of chromatic polynomials of a simplicial complex. In Proposition 3 we observe that the chromatic polynomial vector determines the f-vector.

Although the higher s-chromatic polynomials for simplicial complexes are natural generalizations of 1-chromatic polynomials for graphs, there are in fact some significant structural differences between the cases $s = 1$ (graphs) and $s > 1$ (higher dimensional complexes). Just below Example 7 we list four general properties known to hold for all 1-chromatic polynomials and we observe that there is a single 2-chromatic polynomial (of Möbius' minimal triangulation of the torus) violating them all. For instance, the long conjectured and recently established fact that the absolute value of the coefficients of 1-chromatic polynomial are log-concave [12] does not hold for 2-chromatic polynomials.

On the other hand, we would like to draw attention to a property that just might hold for all s-chromatic polynomials for all $s \geq 1$. In all our examples (Examples 9–11, Example 12), the sequences $i \to S(K, i, s)$ are log-concave for fixed K and s. (Fig. 3 shows the graph of $i \to S(K, i, s)$ in a particular case where K is a 17-vertex triangulation of a 3-sphere.) We ask in Question 1 if this is a general phenomenon. Note that the answer to Question 1 is unknown even for graphs.

We end this note with a short discussion of chromatic uniqueness in §3, and in §4 we establish a kind of deletion-contraction relation for the coefficients $S(K, r, s)$.

1.1 Notation

We shall use the following notation throughout the paper:

- K A finite simplicial complex
- K^s The s-skeleton of K

[1] The computations behind the examples of this note were carried out in the computer algebra system Magma [3].

$F^s(K)$	The set of s-simplices K
$\lvert V \rvert$	The number of elements in the finite set V
$V(K)$	The vertex set $\bigcup K$ of K
$m(K)$	The number $\lvert V(K) \rvert$ of vertices in K
$D[V]$	The complete simplicial complex of *all* subsets of the finite set V
$[m]$	The finite set $\{1, \dots, m\}$ of cardinality m
$[r]_i$	The ith falling factorial polynomial $[r]_i = i!\binom{r}{i}$ in r
$P(a, b)$	The open interval (a, b) in the poset P

2 Three Ways to the s-Chromatic Polynomial of a Simplicial Complex

In this section we present three different approaches to the s-chromatic polynomial $\chi^s(K, r)$:

- Theorem 2 via 1-chromatic polynomials of graphs;
- Theorem 3 via the Möbius function for the s-chromatic lattice;
- Theorem 1 via the simplicial s-Stirling numbers of the second kind.

2.1 Block-Connected s-Independent Vertex Partitions

Let $s \geq 1$ be a natural number. We need the following definitions for our first result.

Definition 2 Let $B \subset V(K)$ be a set of vertices of K. Then

- B is *s-independent* if B contains no s-simplex of K;
- B is *connected* if $K \cap D[B]$ is a connected simplicial complex;
- the *connected components* of B are the maximal connected subsets of B.

Definition 3 Let P be a partition of $V(K)$.

- The *graph* $G_0(P)$ of P is the simple graph whose vertices are the blocks of P and with two blocks connected by and edge if their union is connected;
- The *block-connected refinement* P_0 of P is the refinement whose blocks are the connected components of the blocks of P;
- P is *block-connected* if the blocks of P are connected (ie if $P = P_0$);
- $\mathrm{BCP}^s(K)$ is the set of all block-connected s-independent partitions of $V(K)$.

Lemma 1 *Let P be a partition of $V(K)$. If two different blocks of the block-connected refinement P_0 are connected by an edge in the graph $G_0(P_0)$ of P then they lie in different blocks of P.*

Proof The connected components of the blocks of P are maximal with respect to connectedness. □

Recall that $\chi^1(G_0(P), r)$ is the 1-chromatic polynomial of the simple graph $G_0(P)$ of the partition P.

Theorem 2 *The s-chromatic polynomial for K is the sum*

$$\chi^s(K,r) = \sum_{P \in \mathrm{BCP}^s(K)} \chi^1(G_0(P), r)$$

of the 1*-chromatic polynomials and the s-chromatic number of K is the minimum*

$$\mathrm{chr}^s(K) = \min_{P \in \mathrm{BCP}^s(K)} \mathrm{chr}^1(G_0(P))$$

of the 1*-chromatic numbers for the graphs of all the block-connected s-independent partitions of $V(K)$.*

Proof Let col: $V(K) \to [r]$ be an (r,s)-coloring of K. The monochrome partition $P(\mathrm{col})$ of $V(K)$ is the s-independent partition whose blocks are the nonempty monochrome sets of vertices $\{\mathrm{col} = i\}$ for $i \in [r]$. The block-connected refinement $P(\mathrm{col})_0$ of the monochrome partition is a block-connected s-independent partition of K. The original coloring col of K is also a coloring of the graph $G_0(P(\mathrm{col})_0)$ of $P(\mathrm{col})_0$ for, by Lemma 1, distinct vertices of 1-simplices of this graph have distinct colors. We have shown that any (r,s)-coloring col of K induces an $(r,1)$-coloring col_0 of the graph $G_0(P(\mathrm{col})_0)$ of the block-connected refinement of the monochrome partition.

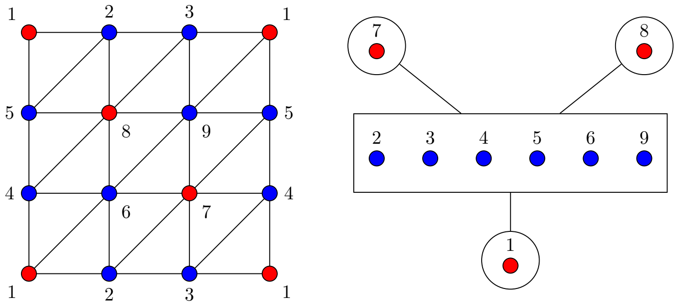
Let $P \in \mathrm{BCP}^s(K)$ be a block-connected s-independent partition of $V(K)$ and $\mathrm{col}_0\colon P \to \{1,\ldots,r\}$ an $(r,1)$-coloring of its graph $G_0(P)$. Then col_0 determines a map col: $V(K) \to [r]$ that is constant on the blocks of P. An s-simplex of K can not be monochrome under col as it intersects at least two different blocks of P connected by an edge of $G_0(P)$. Thus col is an (r,s)-coloring of K.

These two constructions are inverses of each other. □

Remark 1 (*The minimal block-connected s-independent partition*) Let $C_0 = \{\{v\} \mid v \in V(K)\}$ be the block-connected s-independent partition of $V(K)$ whose blocks are singletons. The graph $G_0(C_0) = K^1$ is the 1-skeleton of K. Thus the 1-chromatic polynomial of the 1-skeleton of K is always one of the polynomials in the sum of Theorem 2. If K is 1-dimensional, $\mathrm{BCP}^1(K)$ consists only of the partition C_0 and Theorem 2 simply says that the 1-chromatic polynomial of a simplicial complex is the 1-chromatic polynomial of its 1-skeleton.

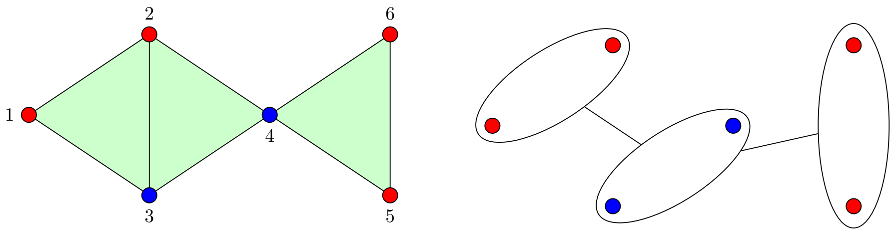
Example 1 (*The block-connected* 2*-independent partitions for* $D[3]$) The 2-simplex $D[3]$ has 4 block-connected 2-independent partitions C_0, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, and $\{\{3\},\{1,2\}\}$. The graph of C_0 is the complete graph K_3, the 1-skeleton of $D[3]$. The graphs of the other three partitions are all the complete graph K_2. Thus the 2-chromatic polynomial of $D[3]$ is $\chi^2(D[3],r) = \chi^1(K_3,r) + 3\chi^1(K_2,r) = [r]_3 + 3[r]_2 = [r]_2(r+1) = r^3 - r$ and the 2-chromatic number is $\mathrm{chr}^2(D[3]) = 2$.

Example 2 (*A* $(2,2)$*-coloring and the graph of the block-connected refinement of its monochrome partition*) The picture below illustrates a $(2,2)$-coloring of a 9-vertex triangulation of the torus



and its corresponding graph. There are 6937 block-connected partitions of the vertex set, and 3 of them has the graph shown above. The 2-chromatic polynomial is $21[r]_2 + 742[r]_3 + 3747[r]_4 + 4908[r]_5 + 2295[r]_6 + 444[r]_7 + 36[r]_8 + [r]_9 = [r]_2(r^7 + r^6 - 17r^5 + 10r^4 + 82r^3 - 116r^2 - 23r + 67)$ and the 2-chromatic number is 2.

Example 3 (*The* $(r, 2)$*-colorings of a simplicial complex* K) Let K be the pure 2-dimensional complex with facets $F^2(K) = \{\{1, 2, 3\}, \{2, 3, 4\}, \{4, 5, 6\}\}$.

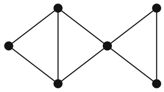
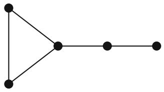
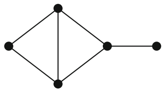
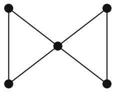
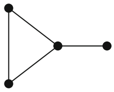


The picture shows a $(2, 2)$-coloring of K and the corresponding $(2, 1)$-coloring of the associated graph, $G_0(P_0)$, the block connected refinement of the monochrome partition $P = \{\{1, 2, 5, 6\}, \{3, 4\}\}$. Table 1 shows the graphs $G_0(P)$ for all block connected partitions $P \in \mathrm{BCP}^2(K)$. For each graph, the table records its 1-chromatic polynomial and its 1-chromatic number. The 2-chromatic polynomial of K is $\chi^2(K, 2) = 15[r]_2 + 73[r]_3 + 62[r]_4 + 15[r]_5 + [r]_6 = [r]_2(r-1)(r+1)(r^2 + r - 1)$ and the 2-chromatic number is $\mathrm{chr}^2(K) = 2$.

Example 4 (*The* $(r, 2)$*-colorings of the* Möbius *band*) The set $\mathrm{BCP}^2(\mathrm{MB})$ of block-connected 2-independent partitions of the triangulated Möbius band MB (Fig. 1) has 36 elements. There are 5, 5, 15, 10, 1 partitions in $\mathrm{BCP}^2(\mathrm{MB})$ realizing the partitions [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1] of the integer $|V(\mathrm{MB})| = 5$. All associated graphs are complete graphs. This yields the 2-chromatic polynomial $\chi^2(\mathrm{MB}, r) = 5[r]_2 + 20[r]_3 + 10[r]_4 + [r]_5 = [r]_2(r^3 + r^2 - 4r + 1) = r^5 - 5r^3 + 5r^2 - r$ and the 2-chromatic number is $\mathrm{chr}^2(\mathrm{MB}) = 2$.

Remark 2 (*The* $\mathcal{S}$*-chromatic polynomial of* K) Let $\mathcal{S}$ be a set of connected subcomplexes of K. A set $B \subset V(K)$ of vertices is $\mathcal{S}$-independent if B is not a superset of any member of $\mathcal{S}$. Let $\mathrm{BCP}^{\mathcal{S}}(K)$ be the set of $\mathcal{S}$-independent partitions of $V(K)$. An

Table 1 The graphs for the block-connected partitions in $\mathrm{BCP}^2(K)$

# in $\mathrm{BCP}^2(K)$	$G_0(P)$	$\chi^1(G_0(P), r)$	$\mathrm{chr}^1(G_0(P))$
1		$r(r-1)^2(r-2)^3$	3
1		$r(r-1)^3(r-2)$	3
3		$r(r-1)^2(r-2)^2$	3
4		$r(r-1)^2(r-2)^2$	3
16		$r(r-1)^2(r-2)$	3
3		$r(r-1)^3$	2
12		$r(r-1)^2$	2

$(r, \mathcal{S})$-coloring is a map $V(K) \to \{1, \ldots, r\}$ such that $|\mathrm{col}(S)| > 1$ for all $S \in \mathcal{S}$. The number of $(r, \mathcal{S})$-colorings of K is

$$\chi^{\mathcal{S}}(K, r) = \sum_{P \in \mathrm{BCP}^{\mathcal{S}}(K)} \chi^1(G_0(P), r)$$

as one sees by an obvious generalization of Theorem 2. An (r, s)-coloring of K is an $(r, \mathcal{S})$-coloring of K where $\mathcal{S} = F^s(K)$ is the set of s-simplices.

2.2 The s-Chromatic Linear Program

In the paper [15, §10] Read explains how to construct a linear program with minimal value equal to the s-chromatic number $\mathrm{chr}^s(K)$ of K.

Definition 4 $M^s(K)$ is the set of all maximal s-independent subsets of $V(K)$.

Let A be the $(m(K) \times |M^s(K)|)$-matrix

$$A(v, M) = \begin{cases} 1 & v \in M \\ 0 & v \notin M \end{cases}$$

recording which vertices $v \in V(K)$ belong to which maximal s-independent sets $M \in M^s(K)$. Now the s-chromatic number

$$\operatorname{chr}^s(K) = \min \left\{ \sum_{M \in M^s(K)} x(M) \mid x\colon M^s(K) \to \{0,1\}, \forall v \in V(K)\colon \right. \\ \left. \sum_{M \in M^s(K)} A(v, M)x(M) \geq 1 \right\}$$

is the minimal value of the objective function $\sum_{M \in M^s(K)} x(M)$ in $|M^s(K)|$ variables $x\colon M^s(K) \to \{0,1\}$, taking values 0 or 1, and $m(K)$ constraints $\sum_{M \in M^s(K)} A(v, M)x(M) \geq 1$, $v \in V(K)$.

2.3 The s-Chromatic Lattice

Our approach here simply follows Rota's classical method for computing chromatic polynomials from Möbius functions of lattices [16, §9]. We need some terminology in order to characterize the monochrome loci for colorings of K. Recall that $F^s(K)$ is the set of s-simplices of K.

Definition 5 Let $S \subset F^s(K)$ be a set of s-simplices of K.

- The equivalence relation $\sim$ is the smallest equivalence relation in S such that $s_1 \cap s_2 \neq \emptyset \implies s_1 \sim s_2$ for all $s_1, s_2 \in S$;
- The connected components of S are the equivalence classes under $\sim$;
- $\pi_0(S)$ is the set of connected components of S;
- S is *connected* if it has at most one component;
- $V(S) = \bigcup S$ is the *vertex set* of S;
- $\pi(S)$ is the partition of $V(K)$ whose blocks are the vertex sets of the connected components of S together with the singleton blocks $\{v\}$, $v \in V(K) - V(S)$, of vertices not in any simplex in S;
- S is *closed* if S contains any s-simplex in K contained in the vertex set of S, ie if

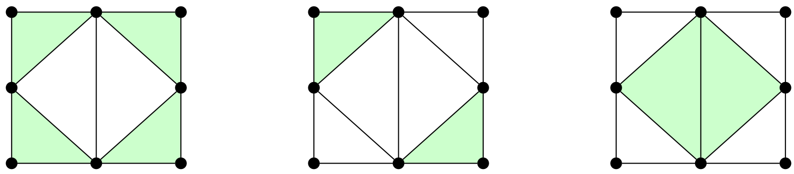
$$\left\{\sigma \in F^s(K) \mid \sigma \subset V(S)\right\} = S$$

- The *closure* of S is the smallest closed set of s-simplices containing S.

For instance, the empty set $S = \emptyset$ of 0 s-simplices is connected with 0 connected components. If $K = D[4]$, the set $\{\{1,2\}, \{2,4\}\}$ of 1-simplices is connected while $\{\{1,2\}, \{3,4\}\}$ has the two components $\{\{1,2\}\}$ and $\{\{3,4\}\}$.

A set of s-simplices is closed if and only if it equals its closure. For instance in $F^2(D[5])$, the set $\{\{1,2,3\}, \{3,4,5\}\}$ is not closed because its closure is the set of all 2-simplices in $D[5]$. The empty set of s-simplices, any set of just one s-simplex, and any set of disjoint s-simplices are closed.

In this picture the green set of 2-simplices is



connected and not closed, closed and not connected, closed and connected, respectively.

The partition $\pi(S)$ has $|\pi(S)| = |\pi_0(S)| + m(K) - |V(S)|$ blocks.

Lemma 2 *Let S be a set of s-simplices in K and S_0 a connected component of S. Then S_0 is closed if and only if*

$$\left\{\sigma \in F^s(K) \mid \sigma \subset V(S_0)\right\} \subset S$$

Proof Since the condition is certainly necessary we only need to see that it is sufficient. Let σ be an s-simplex in K with all its vertices in $V(S_0)$. Then σ lies in S by assumption. But σ is equivalent to all elements of the equivalence class S_0. Thus $\sigma \in S_0$. □

Lemma 3 *Let S and T be sets of s-simplices in K.*

1. *If S and T are closed, so is $S \cap T$.*
2. *If S and T have closed connected components, so does $S \cap T$.*

Proof (1) Let σ be an s-simplex of K and suppose that $\sigma \subset V(S \cap T)$. Then $\sigma \subset V(S)$ an $\sigma \subset V(T)$ so that $\sigma \in S$ and $\sigma \in T$ as S and T are closed.
(2) Let R be a connected component of $S \cap T$. Let S_0 be the connected component of S containing R and T_0 be the connected component of T containing R. Then $R \subset S_0 \cap T_0$. Suppose that $\sigma \in F^s(K)$ is an s-simplex with $\sigma \subset V(R)$. Then $\sigma \subset V(S_0 \cap T_0)$ so $\sigma \in S_0 \cap T_0$ by (1) as the connected components S_0 and T_0 are assumed to be closed. In particular, $\sigma \in S \cap T$. According to Lemma 2, the connected component R is closed. □

Definition 6 The s-chromatic lattice of K is the set $L^s(K)$ of all subsets of $F^s(K)$ with closed connected components. $L^s(K)$ is a partially ordered by set inclusion.

The set $L^s(K)$ contains the empty set $\emptyset$ of s-simplices and the set $F^s(K)$ of all s-simplices. These two elements of $L^s(K)$ are distinct when K has dimension at least s and then $S = F^s(K)$ is the only subset of $F^s(K)$ whose associated partition $\pi(S)$ of $V(K)$ has just one block.

Corollary 1 *$L^s(K)$ is a finite lattice with $\widehat{0} = \emptyset$, $\widehat{1} = F^s(K)$, and meet $S \wedge T = S \cap T$.*

Proof If $S, T \in L^s(K)$ then $S \cap T$ is also in $L^s(K)$ by Lemma 3 and this is clearly the greatest lower bound of S and T. It is now a standard result that $L^s(K)$ is a finite lattice [19, Proposition 3.3.1]. The join $S \vee T$ of $S, T \in L^s(K)$ is the intersection of all supersets $U \in L^s(K)$ of $S \cup T$. □

Example 5 (*The s-chromatic lattice* $L^s(D[m])$) The closed and connected elements of the s-chromatic lattice $L^s(D[m])$ of the complete simplex $D[m]$ on $m > s$ vertices are $\emptyset$ and the sets $F^s(D[k])$ of all s-simplices in the $\binom{m}{k}$ subcomplexes isomorphic to $D[k]$ for $s < k \leq m$. The map $S \to \pi(S)$ is an isomorphism between the lattice $L^s(D[m])$ and the lattice, ordered by refinement, of all partitions of the set $[m]$ into blocks of size $> s$ or 1. The least element, $\widehat{0} = (1)\dots(m)$, is the partition with m blocks and the greatest element, $\widehat{1} = (1\dots m)$, the partition with 1 block. $L^s(D[m])$ is not a graded lattice [19, p 99] in general when $s \geq 2$. To see this, observe that the 2-chromatic lattices $L^2(D[3])$, $L^2(D[4])$, and $L^2(D[5])$ are graded but the lattice $L^2(D[6])$ is not graded as it contains two maximal chains

$$\widehat{0} = (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (1234)(5)(6) < (12345)(6) < (123456) = \widehat{1}$$
$$\widehat{0} = (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (123)(456) < (123456) = \widehat{1}$$

of unequal length. In contrast, the 1-chromatic lattice of any finite simplicial complex is always graded and even geometric [16, §9, Lemma 1].

Remark 3 (*The* Möbius *function for the s-chromatic lattices* $L^s(D[m])$) Our discussion of the Möbius function for the lattice $L^s(D[m])$ echoes the exposition of the Möbius function for the geometric lattice $L^1(D[m])$ of *all* partitions from [19, Example 3.10.4].

Let $w\colon [m] \to \mathbf{N}$ be a function that to every element of $[m]$ associates a natural number, thought of as a weight function. We write $w = 1^{i_1}2^{i_2}\dots r^{i_r}$, or something similar, for the weight function w defined on the set $[m]$ of cardinality $m = \sum_j i_j$ and mapping i_j elements to j for $1 \leq j \leq r$. The map w extends to a map, also called w, defined on the set of all nonempty subsets X of $[m]$ given by $w(X) = \sum_{x \in X} w(x)$. Let $L^s_m(w)$ be the lattice of all partitions of the set $[m]$ into blocks X that are singletons or have weight $w(X) > s$. The non-singleton blocks of the meet $\sigma \wedge \tau$ of two partitions $\sigma, \tau \in L^s_m(w)$ are the subsets of weight $> s$ of the form $S \cap T$ where S is a block in σ and T a block in τ. Write $\mu^s_m(w)$ for the Möbius function of $L^s_m(w)$.

In particular, $L^s_m(1^m)$ is a synonym for $L^s(D[m])$ and we are primarily interested in the Möbius function $\mu^s_m(1^m)$ of the uniform weight $w = 1^m$. However, the computation of this Möbius function will involve the Möbius functions of other weights as well. We shall therefore discuss the Möbius functions $\mu^s_m(w)$ for general weight functions w.

Suppose that $\sigma \in L^s_m(w)$, $\sigma < \widehat{1}$, is a partition of $[m]$ into singleton blocks or blocks of weight $> s$. Let $w(\sigma)$ be the restriction of w to the set of blocks of σ. Thus $w(\sigma)(X) = \sum_{x \in X} w(x)$ for any block X of σ. Then the interval $[\sigma, \widehat{1}]$ in $L^s_m(w)$ has the form

$$[\sigma, \widehat{1}] = L^s_{|\sigma|}(w(\sigma))$$

and hence $\mu^s_m(w)(\sigma, \widehat{1}) = \mu^s_{|\sigma|}(w(\sigma))(\widehat{0}, \widehat{1})$. More generally, suppose that $\sigma < \tau$ for some $\tau \in L^s_m(w)$. Assume that the partition τ has blocks τ_j. Let σ_j be the set of those blocks of σ that intersect the block τ_j of τ. Let $w(\sigma_j)$ be the restriction of $w(\sigma)$ to σ_j. The interval $[\sigma, \tau]$ in $L^s_m(w)$ is isomorphic to

$$[\sigma, \tau] \cong \prod_j L^s_{|\sigma_j|}(w(\sigma_j))$$

and therefore the value of the Möbius function on the pair (σ, τ)

$$\mu^s_m(w)(\sigma, \tau) = \prod_j \mu^s_{|\sigma_j|}(w(\sigma_j))\left(\widehat{0}, \widehat{1}\right)$$

by the product theorem for Möbius functions [19, Proposition 3.8.2]. We conclude that the complete Möbius functions on all the lattices $L^s_m(w)$, are actually determined by the values $\mu^s_m(w)(\widehat{0}, \widehat{1})$ of these Möbius functions on just $(\widehat{0}, \widehat{1})$. See Equation (2.3) for more information about these Euler characteristics.

For the following it is convenient to name the elements of the domain $[m]$ of w in such a way that the element m carries minimal weight. Assume that $a_m = (1 \dots m-1)(m)$ is an element of $L^s_m(w)$, ie that $w(1) + \cdots + w(m-1) > s$. We shall determine the set of lattice elements x with $x \wedge a_m = \widehat{0}$. There is only one solution to this equation with $x \leq a_m$ and that is $x = \widehat{0}$. As the other solutions satisfy $x \not\leq a_m$, they must have a block that contains m and at least one other element. It follows that the solutions $x \neq \widehat{0}$ are all elements of the form

$$x = (x_1 \dots x_t m)(\cdot) \cdots (\cdot) \text{ with } \begin{cases} w(x_1) > s - w(m) & t = 1 \\ s \geq w(x_1) + \cdots + w(x_t) > s - w(m) & t > 1 \end{cases}$$

where all blocks but the unique block containing m are singletons. There are $t+1$ elements in the block containing m where t is some number in the range $1 \leq t \leq s$. (All the solutions $x \neq \widehat{0}$ are atoms in the lattice $L^s_m(w)$.) Since we are in a lattice, the Möbius function satisfies the equation [19, Corollary 3.9.3]

$$\mu^s_m(w)\left(\widehat{0}, \widehat{1}\right) = -\sum_{\substack{x \wedge a_m = \widehat{0} \\ x \neq \widehat{0}}} \mu^s_m(w)\left(x, \widehat{1}\right)$$

which translates to

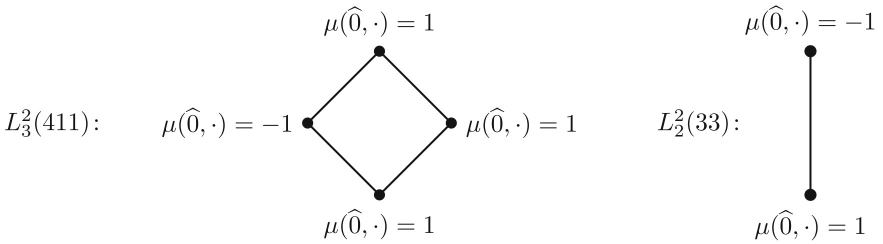
$$\begin{aligned} \mu^s_m(w)\left(\widehat{0}, \widehat{1}\right) &= -\sum_{\substack{x \wedge a_m = \widehat{0} \\ x \neq \widehat{0}}} \mu^s_{|x|}(w(x))\left(\widehat{0}, \widehat{1}\right) \\ &= -\sum_{\substack{1 \leq x_1 \leq m-1 \\ w(x_1) > s - w(m)}} \mu^s_{m-1}(w(x_1 m)w(\cdot)\cdots w(\cdot))\left(\widehat{0}, \widehat{1}\right) \\ &\quad -\sum_{1 < t \leq s} \sum_{\substack{1 \leq x_1, \dots, x_t \leq m-1 \\ s \geq w(x_1) + \cdots + w(x_t) > s - w(m)}} \mu^s_{m-t}(w(x_1 \dots x_t m))w(\cdot)\cdots w(\cdot))\left(\widehat{0}, \widehat{1}\right) \end{aligned} \tag{2.1}$$

This describes a recursive procedure for computing all values of the Möbius function on the weight lattices $L^s_m(w)$.

As an illustration we compute $\mu_6^2(1^6)(\widehat{0},\widehat{1})$. Using (2.1) twice gives

$$\mu_6^2(1^6)\left(\widehat{0},\widehat{1}\right) = -10\mu_4^2(3111)\left(\widehat{0},\widehat{1}\right) = 10\left(\mu_3^2(411)\left(\widehat{0},\widehat{1}\right) + \mu_2^2(33)\left(\widehat{0},\widehat{1}\right)\right)$$

The lattices $L_4^2(411)$ and $L_2^2(33)$ have 4 and 2 elements, respectively, and they look like



so that $\mu_3^2(411)(\widehat{0},\widehat{1}) = 1$ and $\mu_2^2(33)(\widehat{0},\widehat{1}) = -1$. Therefore $\mu_6^2(1^6)(\widehat{0},\widehat{1}) = 0$.

We remind the reader of the well-known fact that $\mu_m^s(w)(\widehat{0},\widehat{1})$ is the reduced Euler characteristic of the open interval $L_m^s(w)(\widehat{0},\widehat{1})$ between $\widehat{0}$ and $\widehat{1}$ in the lattice $L_m^s(w)$.

Proposition 1 [16, §6] [19, Proposition 3.8.5] *Let $x < y$ be two elements in a finite poset. The value of the Möbius function on the pair (x, y) is the reduced Euler characteristic of the open interval (x, y).*

Proof Write μ be the Möbius function of P and E for Euler characteristic. The closed interval from x to y has Euler characteristic 1 since it has a smallest element. Thus

$$\begin{aligned} 1 = \mathrm{E}([x,y]) = \sum_{a,b\in[x,y]} \mu(a,b) &= \sum_{a,b\in(x,y)} \mu(a,b) \\ &+ \sum_{a\in[x,y]} \mu(a,y) + \sum_{b\in[x,y]} \mu(x,b) - \mu(x,y) \\ &= \mathrm{E}((x,y)) + 0 + 0 - \mu(x,y) = \mathrm{E}((x,y)) - \mu(x,y) \end{aligned}$$

which means that $\mu(x, y) = \widetilde{\mathrm{E}}((x, y))$. □

Define the *s-monochrome set* of a map col: $V(K) \to [r] = \{1, \ldots, r\}$ to be the set

$$M^s(\mathrm{col}) = \{\sigma \in F^s(K) \mid |\mathrm{col}(\sigma)| = 1\}$$

of all monochrome s-simplices in K. Note that the map col is not necessarily an (r, s)-coloring; indeed, col is an (r, s)-coloring of K if and only if $M^s(\mathrm{col}) = \emptyset$.

Lemma 4 *The s-monochrome set $M^s(\mathrm{col})$ of any map $\mathrm{col}\colon V(K) \to [r]$ is an element of the s-chromatic lattice $L^s(K)$.*

Proof Let S be a connected component of $M^s(\mathrm{col})$. Since S is connected, all vertices in S have the *same* color. Let $\sigma \in F^s(K)$ be an s-simplex of K such that $\sigma \subset V(S)$. The σ is monochrome: $\sigma \in M^s(\mathrm{col})$. By Lemma 2, S is closed. □

Theorem 3 *The number of* (r, s)*-colorings of* K *is*

$$\chi^s(K, r) = \sum_{T \in L^s(K)} \mu\left(\widehat{0}, T\right) r^{|\pi(T)|}$$

where μ *the Möbius function for the* s*-chromatic lattice* $L^s(K)$*.*

Proof For any $B \in L^s(K)$, let $\chi(K, B, r, s)$ be the number of maps $\mathrm{col}\colon V(K) \to [r]$ with $M^s(\mathrm{col}) = B$. We want to determine $\chi(K, \emptyset, r, s) = \chi^s(K, r)$. For any $A \in L^s(K)$,

$$r^{|\pi(A)|} = \sum_{A \leq B} \chi(K, B, r, s)$$

because there are $r^{|\pi_0(A)|} r^{m(K) - |V(A)|} = r^{|\pi(A)|}$ maps $\mathrm{col}\colon V(K) \to [r]$ with $A \leq M^s(\mathrm{col})$. Equivalently,

$$\sum_{B \geq A} \mu(A, B) r^{|\pi(B)|} = \chi(K, A, r, s)$$

by Möbius inversion [19, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A = \widehat{0}$. □

The defining rules for the Möbius function of the poset $L^s(K)$ [19, 3.7]

- $\mu(S, S) = 1$ for all $S \in L^s(K)$
- $\sum_{R \leq S \leq T} \mu(R, S) = 0$ when $R \lneqq T$
- $\mu(R, S) = 0$ when $R \not\leq S$

imply that $\mu(\widehat{0}, \widehat{0}) = 1$ and $\mu(\widehat{0}, \{\sigma\}) = -1$ for every s-simplex $\sigma \in F^s(K)$.

Corollary 2 *The highest degree terms of the* s*-chromatic polynomial are*

$$\chi^s(K, r) = r^{m(K)} - f_s(K) r^{m(K)-s} + \cdots$$

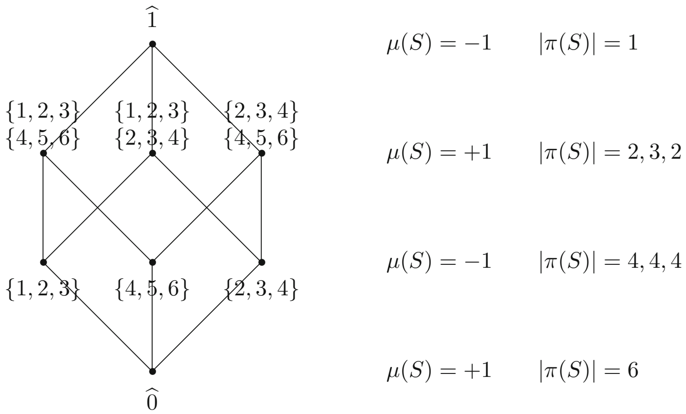
Thus the s*-chromatic polynomial determines* $f_0(K)$ *and* $f_s(K)$*.*

Proof The s-chromatic polynomial is

$$\chi^s(K, r) = \mu\left(\widehat{0}, \widehat{0}\right) r^{f_0(K)} + \sum_{\sigma \in F^s(K)} \mu\left(\widehat{0}, \{\sigma\}\right) r^{f_0(K)-s} + \cdots$$

where $\mu(\widehat{0}, \widehat{0}) = 1$ and $\mu(\widehat{0}, \{\sigma\}) = -1$ for all s-simplices σ of K. □

Example 6 Consider the 2-dimensional complex K from Example 3. The 2-chromatic lattice $L^2(K)$ of K

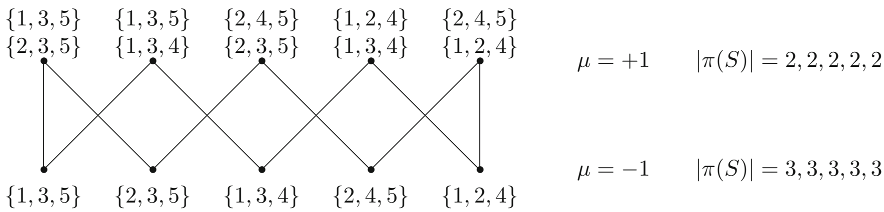


consists of *all* subsets of $F^2(K)$. The 2-chromatic polynomial is

$$\chi^2(K, r) = r^6 - r^4 - r^4 - r^4 + r^2 + r^3 + r^2 - r = r^6 - 3r^4 + r^3 + 2r^2 - r$$

K has $\chi^2(K, 2) = 30$ (2, 2)-colorings and $\chi^2(K, 3) = 528$ (3, 2)-colorings.

Example 7 The triangulation MB of the Möbius band with f-vector $f(\text{MB}) = (5, 10, 5)$ shown in Fig. 1 has the following (reduced) 2-chromatic lattice $L^2(\text{MB}) - \{\widehat{0}, \widehat{1}\}$



and 2-chromatic polynomial

$$\chi^2(\text{MB}, r) = r^5 - 5r^3 + 5r^2 - r$$

The lattice $L^2(\text{MB})$ is graded but it is still not semi-modular [19, Proposition 3.3.2]: The meet and join of $a = \{\{2, 3, 5\}\}$ and $b = \{\{1, 3, 4\}\}$ are $a \wedge b = \widehat{0}$ and $a \vee b = \widehat{1}$. Thus a and b cover $a \wedge b$ but $a \vee b$ covers neither a nor b.

Example 8 Let MT be Möbius's minimal triangulation of the torus with f-vector $f(\text{MT}) = (7, 21, 14)$ and P2 the triangulation of the projective plane with f-vector

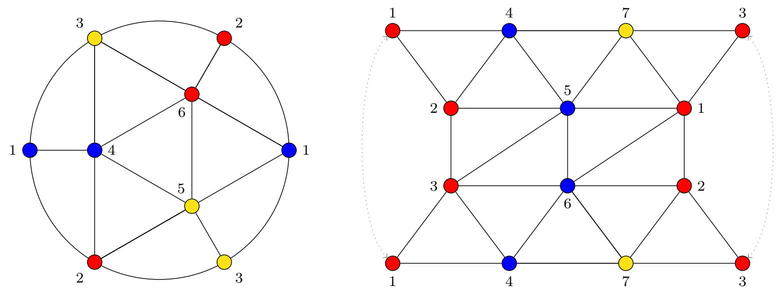


Fig. 2 (3, 2)-colorings of P2 and MT

f(P2) = (1, 6, 15, 10) shown in Fig. 2 (decorated with (3, 2)-colorings). The chromatic polynomials of these two simplicial complexes are

$$\chi^1(\mathrm{MT}, r) = [r]_7, \qquad \chi^2(\mathrm{MT}, r) = r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r$$
$$\chi^1(\mathrm{P2}, r) = [r]_6, \qquad \chi^2(\mathrm{P2}, r) = r^6 - 10r^4 + 15r^3 - 6r^2$$

In both cases, the 1-skeleton is the complete graph on the vertex set. The chromatic numbers are $\mathrm{chr}^1(\mathrm{MT}) = 7$, $\mathrm{chr}^1(\mathrm{P2}) = 6$, and $\mathrm{chr}^2(\mathrm{MT}) = 3 = \mathrm{chr}^2(\mathrm{P2})$.

The chromatic polynomials of simple graphs (the 1-chromatic polynomials of simplicial complexes) are known to have these properties:

– The coefficients are sign-alternating [16, §7, Corollary]
– The coefficients are log-concave (Definition 9) in absolute value [12]
– There are no negative roots and no roots between 0 and 1 [20]
– $\chi^1(K, m(K)) > e\chi^1(K, m(K) - 1)$ [8]

In contrast, the 2-chromatic polynomial

$$\chi^2(\mathrm{MT}, r) = r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r = [r]_3(r+1)\left(r^3 + 2r^2 - 9r + 3\right)$$

has none of these properties.

2.4 The s-Chromatic Polynomial in Falling Factorial Form

Theorem 1 provides an interpretation of the coefficients of the falling factorial $[r]_i$ in the s-chromatic polynomial of the simplicial complex K.

Definition 7 $S(K, r, s)$ is the number of partitions of $V(K)$ into r s-independent blocks.

We think of $S(K, r, s)$ as an s-Stirling number of the second kind for the simplicial complex K. If $s > \dim(K)$, then there are no s-simplices in K and all partitions of $V(K)$ are s-independent, so that $S(K, r, s)$ is the Stirling number of the second kind

$S(m(K), r)$ [19, p 33]. We now explain the general relation between the simplicial Stirling numbers and the usual Stirling numbers of the second kind.

Define the *s-monochrome set* of a partition P of $V(K)$ to be the set

$$M^s(P) = \{\sigma \in F^s(K) \mid \sigma \text{ is contained in a block of } P\}$$

of all s-simplices entirely contained in one of the blocks of P. The set $M^s(P)$ is an element of the s-chromatic lattice $L^s(K)$ by Lemma 4.

Theorem 4 *The number of partitions of $V(K)$ into r s-independent blocks is*

$$S(K, r, s) = \sum_{T \in L^s(K)} \mu\left(\widehat{0}, T\right) S(|\pi(T)|, r)$$

where μ is the Möbius function for the s-chromatic lattice $L^s(K)$.

Proof For any $B \in L^s(K)$, let $S(K, B, r, s)$ be the number of partitions P of $V(K)$ into r blocks with monochrome set $M^s(P) = B$. We want to determine $S(K, \emptyset, r, s) = S(K, r, s)$. For any $A \in L^s(K)$,

$$S(|\pi(A)|, r) = \sum_{A \leq B} S(K, B, r, s)$$

because there are $S(|\pi(A)|, r)$ partitions P of $V(K)$ into r blocks with $A \leq M^s(P)$. Equivalently,

$$\sum_{A \leq B} \mu(A, B) S(|\pi(B)|, r) = S(K, A, r, s)$$

by Möbius inversion [19, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A = \widehat{0}$. □

Proof (Proof of Theorem 1) We simply follow the proof of the similar statement for chromatic polynomials for graphs [15, Theorem 15]. When $r \geq i$ we can get an (r, s)-coloring out of one of the $S(K, i, s)$ partitions of $V(K)$ into i s-independent blocks by choosing i out of the r colors and assigning them to the i blocks. There are $\binom{r}{i}$ ways of choosing the i out of r colors and $i!$ ways of coloring i blocks in i colors. The number of (r, s)-colorings of K in exactly i colors is thus

$$S(K, i, s)\binom{r}{i} i! = S(K, i, s)[r]_i$$

so that

$$\chi^s(K, r) = \sum_{i=1}^{m(K)} S(K, i, s)[r]_i$$

is the total number of (r, s)-colorings of K. □

Corollary 3 *The reduced Euler characteristic of the open interval* $(\widehat{0}, \widehat{1})$ *in the s-chromatic lattice* $L^s(K)$ *is*

$$\mu(L^s(K))\left(\widehat{0}, \widehat{1}\right) = \sum_{i=\mathrm{chr}^s(K)}^{m(K)} (-1)^{i-1}(i-1)!S(K,i,s)$$

when $\dim K \geq s$.

Proof Compare the terms of degree 1 of the two expressions

$$\sum_{T\in L^s(K)} \mu(L^s(K))\left(\widehat{0}, T\right) r^{|\pi(T)|} = \sum_{i=\mathrm{chr}^s(K)}^{m(K)} S(K,i,s)[r]_i \tag{2.2}$$

from Theorems 3 and 1 for the s-chromatic polynomial of K. Since $\dim K \geq s \geq 1$ the s-chromatic lattice $L^s(K)$ has a unique element T with $|\pi(T))| = 1$, namely the set $T = F^s(K)$ of all s-simplices of K. □

We observe that

$$\begin{aligned}\sum_i S(K,i,s)[r]_i &= \sum_i \sum_T \mu\left(\widehat{0}, T\right) S(|\pi(T)|, i)[r]_i \\ &= \sum_T \mu\left(\widehat{0}, T\right) \sum_i S(|\pi(T)|, i)[r]_i = \sum_T \mu\left(\widehat{0}, T\right) r^{|\pi(T)|}\end{aligned}$$

so that Theorem 4 implies Theorem 1.

The s-chromatic number of K is immediately visible with the s-chromatic polynomial in factorial form because

$$\mathrm{chr}^s(K) = \min\{i \mid S(K,i,s) \neq 0\}$$

is the lowest degree of the nonzero terms. The positive integer sequence

$$\chi^s(K, \mathrm{chr}^s(K)), \ldots, \chi^s(K, m(K)) = 1$$

has no internal zeros. (If there is a partition of $V(K)$ into r blocks not containing any s-simplex of K and $r < m(K)$, then split one of the blocks with more than one vertex into two sub-blocks to get a partition of $V(K)$ into $r+1$ blocks containing no s-simplices of K.)

Proposition 2 *Let K be a subcomplex of L and assume that $V(K) = V(L)$.*

1. $S(K,r,s) \geq S(L,r,s)$ *for all* r.
2. *If* $S(K,r,s) = S(L,r,s)$ *for some* r *with* $\frac{1}{s}(|V|-1) \leq r \leq |V|-s$, *then* $K^s = L^s$.

Proof (1) Let V be the vertex set of K and L. Write $\mathcal{S}(K, r, s)$ and $\mathcal{S}(L, r, s)$ for the set of partitions of V into r blocks containing no s-simplex of K or L, respectively. Then $\mathcal{S}(L, r, s) \subseteq \mathcal{S}(K, r, s)$ for all r and s. Thus $S(L, r, s) \leq S(K, r, s)$.
(2) Suppose that $\sigma \in F^s(L) - F^s(K)$ is an s-simplex of L that is not an s-simplex of K. Any partition of the form

$$\{\sigma\} \cup \tau, \qquad \tau \in \mathcal{S}(D[V - \sigma], r - 1, s),$$

is in $\mathcal{S}(K, r, s) - \mathcal{S}(L, r, s)$. The set $\mathcal{S}(D[V - \sigma], r - 1, s)$ is nonempty when

$$\mathrm{chr}^s(D[V - \sigma]) = \left\lceil \frac{|V| - s - 1}{s} \right\rceil \leq r - 1 \leq |V| - s - 1$$

and thus $S(K, r, s)$ is strictly greater than $S(L, r, s)$ when $\frac{|V|-1}{s} \leq r \leq |V| - s$. □

Remark 4 ($S(K, r, s)$ *for the complete simplex* $K = D[m]$) For any finite set M, let $S(M, r, s)$ stand for $S(D[M], r, s)$ (Definition 7), the number of partitions of the set M into r blocks containing at most s elements. Let us even write $S(m, r, s)$ in case $M = [m]$, $m \geq 1$, $r, s \geq 0$.

Clearly, $S(m, r, s)$ is nonzero only when $m/s \leq r \leq m$. Also, $S(m, r, s) = S(m, r)$ when r is among the s numbers $m - s + 1, \ldots, m$.

The recurrence relation

$$S(m, r, s) = \sum_{j=m-s}^{m-1} \binom{m-1}{j} S(j, r - 1, s)$$

can be used to compute these numbers. Alternatively, one may use

$$S(m + 1, r + 1, s) = S(m, r, s) + (r + 1)S(m, r + 1, s) + \binom{m}{s} S(m - s, r, s)$$

from [5]. Table 2 shows $S(m, r, s)$ for small m; the number $S(m, r, s)$ is in row s and column r in the chromatic table (Definition 8) for $D[m]$. All the red numbers are usual Stirling numbers of the second kind.

According to Theorem 1, the numbers $S(m, r, s)$ determine the s-chromatic polynomial in falling factorial form of the complete simplex on m vertices

$$\chi^s(D[m], r) = \sum_{i=\lceil m/s \rceil}^{m} S(m, i, s)[r]_i$$

and, according to Corollary 3, they also determine the reduced Euler characteristic

$$\widetilde{E}(L^s(D[m])) = \mu_m^s(1^m)\left(\widehat{0}, \widehat{1}\right) = \sum_{i=\lceil m/s \rceil}^{m} (-1)^{i-1}(i - 1)!S(m, i, s)$$

Table 2 Chromatic tables for complete simplices $D[m]$ for $m = 2, \dots, 7$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 6 & 1 \\ 0 & 7 & 6 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 15 & 10 & 1 \\ 0 & 10 & 25 & 10 & 1 \\ 0 & 15 & 25 & 10 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 15 & 45 & 15 & 1 \\ 0 & 10 & 75 & 65 & 15 & 1 \\ 0 & 25 & 90 & 65 & 15 & 1 \\ 0 & 31 & 90 & 65 & 15 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 105 & 105 & 21 & 1 \\ 0 & 0 & 175 & 315 & 140 & 21 & 1 \\ 0 & 35 & 280 & 350 & 140 & 21 & 1 \\ 0 & 56 & 301 & 350 & 140 & 21 & 1 \\ 0 & 63 & 301 & 350 & 140 & 21 & 1 \end{pmatrix}$$

of the s-chromatic lattice $L^s(D[m])$ on the full simplex on m vertices.

More generally, if $w : M \to \mathbf{N}$ is a function on M with natural numbers as values, let $S(M, w, r, s)$ be the number of partitions of M into admissible blocks, where we declare a block admissible if it is a singleton or it has weight at most s. (Then $S(m, r, s) = S([m], 1^m, r, s)$ occur when $M = [m]$ and $w = 1^m$ places weight 1 on all elements.) Any such partition is a partition of M into blocks of weight at most s, and therefore $S(M, w, r, s) \leq S(|M|, r, s)$. In particular, $S(M, w, r, s)$ is nonzero only when $|M|/s \leq r \leq \|M|$. The recurrence relation

$$S(M, w, r, s) = \sum_{\substack{\emptyset \neq J \subset M - \{\max(M)\} \\ M - J \text{ admissible}}} S(J, w|J, r-1, s)$$

provides a means to compute these numbers.

The weighted version of Eq. (2.2) for $K = D[m]$,

$$\sum_{\sigma \in L^s_m(w)} \mu^s_m(w) \left(\widehat{0}, \sigma\right) r^{|\sigma|} = \sum_{i=\lceil m/s \rceil}^{m} S\left([m], w, i, s\right) [r]_i$$

implies, by comparing coefficients of first degree terms, the expression

$$\mu^s_m(w) \left(\widehat{0}, \widehat{1}\right) = \sum_{i=\lceil m/s \rceil}^{m} (-1)^{i-1}(i-1)! S([m], w, i, s) \tag{2.3}$$

for the Euler characteristic of the weighted lattice $L^s_m(w)$ from Remark 3.

Remark 5 (*Euler characteristics of s-chromatic lattices for full simplices*) Fix $s \geq 1$. The reduced Euler characteristics $\widetilde{E}(L^s(D[m+1]))$, $m \geq s$, of the s-chromatic lattices of the full simplices satisfy the relation

$$\left(\sum_{k=0}^{s} \frac{x^k}{k!}\right)\left(1 + \sum_{m=s}^{\infty} \widetilde{E}(L^s(D[m+1]))\frac{x^m}{m!}\right) = \sum_{k=0}^{s-1} \frac{x^k}{k!}$$

due to Martin Wedel Jacobsen. For $s = 1, 2, 3, 4, 5$ and $m \geq s$ the reduced Euler characteristics $\widetilde{E}(L^s(D[m+1]))$ are

- $1, 2, -6, 24, -120, 720, -5040, 40320, -362880, 3628800, -39916800, 479001600, -6227020800, 87178291200, \ldots$
- $1, 3, -6, 0, 90, -630, 2520, 0, -113400, 1247400, -7484400, 0, 681080400, -10216206000, 81729648000, \ldots$
- $1, 4, -10, 20, -70, 560, -4200, 25200, -138600, 924000, -8408400, 84084000, -798798000, 7399392000, \ldots$
- $1, 5, -15, 35, -70, 0, 2100, -23100, 173250, -1051050, 5255250, -15765750, -105105000, 2858856000, \ldots$

For $s = 1$ we obtain the sequence $(-1)^m m!$, $m \geq 1$, of reduced Euler characteristics of the lattice of partitions of $[m+1]$ [19, Example 3.10.4]. For $s = 2$, we recognize the sequence A009014 from the On-Line Encyclopedia of Integer Sequences (OES). The remaining three sequences apparently do not match any sequences of the OES.

Because any simplicial complex K is a subcomplex of the complete simplex $D[m(K)]$ on its vertex set, we have

$$S(m(K), r) \geq S(K, r, s) \geq S(m(K), r, s), \qquad 1 \leq r \leq m(K) \tag{2.4}$$

Moreover, these inequalities are equalities for the s highest values $m(K) - s + 1, \ldots, m(K)$ of r. Thus the s terms of highest falling factorial degree in the s-chromatic polynomial of K

$$\chi^s(K, r) = \sum_{i=0}^{m(K)-s} S(K, i, s)[r]_i + \sum_{i=m(K)-s+1}^{m(K)} S(m(K), i)[r]_i$$

are given by the s Stirling numbers $S(m(K), m(K) - s + 1), \ldots, S(m(K), m(K))$ of the second kind. These coefficients depend only on the size of the vertex set of K. We shall next show that the coefficient number $s + 1$ counted from above, $S(K, m(K) - s, s)$, informs about the number $f_s(K)$ of s-simplices in K.

Proposition 3 $S(K, m(K) - s, s) = S(m(K), m(K) - s) - f_s(K)$. *If* $S(K, m(K) - s, s) = S(m(K), m(K) - s, s)$ *then* $K^s = D[m(K)]^s$.

Proof The only partitions of the $S(m, m - s)$ partitions of $V(K)$ into $m - s$ blocks that are not s-independent are those consisting of one s-simplex of K together with singleton blocks. If $S(K, m(K) - s, s) = S(D[m(K)], m(K) - s, s)$ then $f_s(K) = f_s(D[m(K)])$ so $K^s = D[m(K)]^s$. [This is a special case of Proposition 2.(2)]. $\square$

Definition 8 The *chromatic table*, $\chi(K)$, of K is the $(\dim(K) \times m(K))$-table with $S(K, r, s)$ in row s and column r.

This means that row s in the chromatic table lists the coefficients of the s-chromatic polynomial. The chromatic table of a 3-dimensional simplicial complex K, for instance, looks like this

	$r=1$	$r=2$	$\dots$	$r=m-3$	$r=m-2$	$r=m-1$	$r=m$
$S(K,\cdot,1)$	$S(K,1,1)$	$S(K,2,1)$	$\dots$	$S(K,m-3,1)$	$S(K,m-2,1)$	$S(m,m-1)-f_1$	$S(m,m)=1$
$S(K,\cdot,2)$	$S(K,1,2)$	$S(K,2,2)$	$\dots$	$S(K,m-3,2)$	$S(m,m-2)-f_2$	$S(m,m-1)$	$S(m,m)=1$
$S(K,\cdot,3)$	$S(K,1,3)$	$S(K,2,3)$	$\dots$	$S(m,m-3)-f_3$	$S(m,m-2)$	$S(m,m-1)$	$S(m,m)=1$

where the red entries in row s are Stirling numbers of the second kind $S(m,r)$ for $m-s+1 \le r \le m$, and the blue entry in row s is $S(m(K),m(K)-s)-f_s(K)$.

Example 9 The chromatic tables of the 2-dimensional simplicial complexes from Examples 3, 7, and 8 are

$$\chi(K)=\begin{pmatrix}0&0&2&10&7&1\\0&15&73&62&15&1\end{pmatrix}\qquad \chi(\mathrm{MB})=\begin{pmatrix}0&0&0&0&1\\0&5&20&10&1\end{pmatrix}$$
$$\chi(\mathrm{MT})=\begin{pmatrix}0&0&0&0&0&0&1\\0&0&84&231&126&21&1\end{pmatrix}\quad \chi(\mathrm{P2})=\begin{pmatrix}0&0&0&0&0&1\\0&0&45&55&15&1\end{pmatrix}$$

The red entries in column r are Stirling numbers $S(m,r)$ and they are independent of the row index. The blue entry in row s and column $m-s$, which equals $S(m-s,s)-f_s(K)$, detects if K has maximal s-skeleton by Proposition 3.

Example 10 Let $K=\mathrm{AS3}$ be Altshuler's peculiar triangulation of the 3-sphere with f-vector $f=(10,45,70,35)$ [1]. The 1-chromatic polynomial is $\chi^1(\mathrm{AS3},r)=[r]_{10}$ as K^1 is the complete graph on 10 vertices. The chromatic table is

$$\chi(\mathrm{AS3})=\begin{pmatrix}0&0&0&0&0&0&0&0&0&1\\0&0&0&1360&8475&10355&4200&680&45&1\\0&26&4320&25915&38550&22152&5845&750&45&1\end{pmatrix}$$

The blue numbers determine the f-vector

$$f(\mathrm{AS3})=(10,S(10,9)-\chi(\mathrm{AS3})_{19},S(10,8)-\chi(\mathrm{AS3})_{28},S(10,7)-\chi(\mathrm{AS3})_{37})$$

The column number of the first nonzero term in each row tells us that $\mathrm{chr}^1(\mathrm{AS3})=10$, $\mathrm{chr}^2(\mathrm{AS3})=4$, and $\mathrm{chr}^3(\mathrm{AS3})=2$.

Example 11 The nonconstructible, nonshellable 3-sphere $S^3_{17,74}$, $f=(17,91,148,74)$, found by Lutz [13], has

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$
$s=1$	0	0	0	0	0	0	0	88	3089
$s=2$	0	0	36	702475	82949364	1075420155	3827766587	5493687086	3876597169
$s=3$	0	422	4319865	338438489	3903094622	14292381565	22946854806	19158310796	9202775199

	$r=10$	$r=11$	$r=12$	$r=13$	$r=14$	$r=15$	$r=16$	$r=17$
$s=1$	23017	55285	54973	25941	6210	762	45	1
$s=2$	1507939074	346346664	48855523	4302470	235026	7672	136	1
$s=3$	2708454744	507528561	61784524	4903589	249826	7820	136	1

as its chromatic table. Figure 3 shows a semi-logarithmic plot of the simplicial Stirling numbers $S(S^3_{17,74}, r, s)$.

The triangulation Σ^3_{16}, $f = (16, 106, 180, 90)$, of the Poincaré homology 3-sphere constructed by Björner and Lutz [2, Theorem 5] has

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
$s=1$	0	0	0	0	0	0	0	0
$s=2$	0	0	0	4589	2974411	69671411	300475213	442354547
$s=3$	0	3	845561	70005500	701299653	2158716508	2888730959	2000811501

	$r=9$	$r=10$	$r=11$	$r=12$	$r=13$	$r=14$	$r=15$	$r=16$
$s=1$	0	0	0	0	28	44	14	1
$s=2$	292864435	100793551	19546606	2225261	150095	5840	120	1
$s=3$	792553648	190527025	28730056	2750278	165530	6020	120	1

as its chromatic table.

Observe that all the above chromatic tables have strictly log-concave rows.

Definition 9 [17] A finite sequence $a_1, a_2, \ldots, a_N$ of $N \geq 3$ nonnegative integers is strictly log-concave if $a_{i-1}a_{i+1} < a_i^2$ for $1 < i < N$ (and log-concave if $a_{i-1}a_{i+1} \leq a_i^2$).

It has been conjectured that the sequence of coefficients of the 1-chromatic polynomial of a simple graph in falling factorial form, $r \to S(K, r, 1), \mathrm{chr}^1(K) \leq r \leq m(K)$, is log-concave [4, Conjecture 3.11]. More generally, one may ask

Question 1 Is the finite sequence of simplicial Stirling numbers

$$r \to S(K, r, s), \qquad \mathrm{chr}^s(K) \leq r \leq m(K),$$

log-concave for fixed K and s?

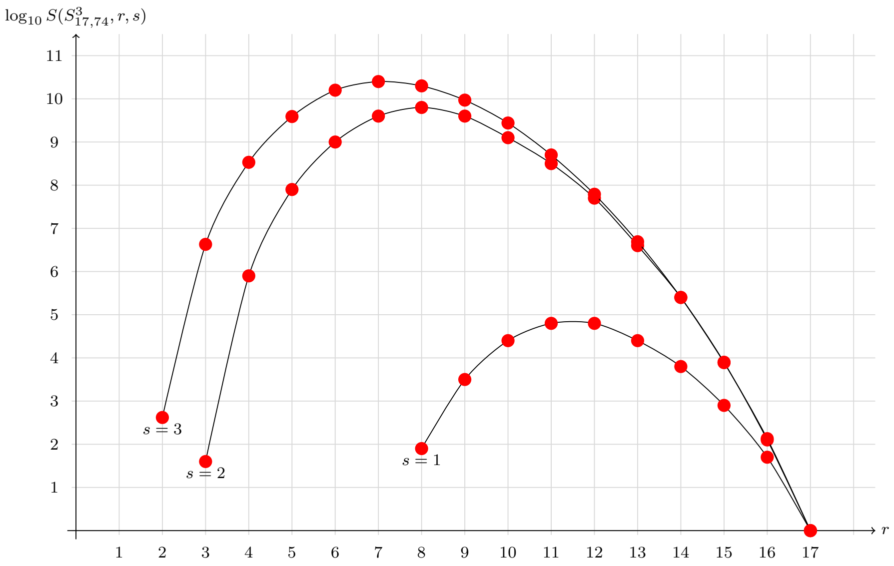


Fig. 3 The simplicial Stirling numbers for $S^3_{17,74}$

We already noted that *none* of the general properties of chromatic polynomials of simple graphs listed just below Example 7 holds for chromatic polynomials of higher dimensional simplicial complexes. Thus we have the bizarre situation that the only property that might hold for chromatic polynomials of any dimension, log-concavity of the simplicial Stirling numbers, is not even known to hold for graphs!

Note that the Stirling numbers of the second kind, which are upper bounds for the simplicial Stirling numbers $S(K, r, s)$ by the inequalities (2.4), are log-concave in r [17, Corollary 2].

We already examined Question 1 in Examples 9–11 and in Fig. 3. We shall now examine Question 1 on spherical boundary complexes of cyclic n-polytopes.

Definition 10 The boundary of the m-vertex cyclic n-polytope, $\partial\mathrm{CP}(m, n)$, $m > n$, is the $(n-1)$-dimensional simplicial complex on the ordered set $[m]$ with the following facets: An n-subset σ of $[m]$ is a facet if and only if between any two elements of $[m] - \sigma$ there is an even number of vertices in σ.

By Gale's Evenness Theorem [11], the simplicial complex $\partial\mathrm{CP}(m, n)$ triangulates the boundary of the cyclic n-polytope on m vertices. Thus $\partial\mathrm{CP}(m, n)$ is a simplicial $(n-1)$-sphere on m vertices and it is $\lfloor n/2\rfloor$-neighborly in the sense that $\partial\mathrm{CP}(m, n)$ has the same s-skeleton as the full simplex on its vertex set when $s < \lfloor n/2\rfloor$.

Example 12 (*Cyclic polytopes with* log-*concave simplicial Stirling numbers of the second kind*) The chromatic tables of the simplicial 3-spheres $\partial\mathrm{CP}(m, 4)$ on $m = 6, 7, 8, 9, 10$ vertices are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 21 & 47 & 15 & 1 \\ 0 & 16 & 81 & 65 & 15 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 28 & 147 & 112 & 21 & 1 \\ 0 & 21 & 238 & 336 & 140 & 21 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 50 & 393 & 582 & 226 & 28 & 1 \\ 0 & 29 & 654 & 1533 & 1030 & 266 & 28 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 94 & 1062 & 2523 & 1719 & 408 & 36 & 1 \\ 0 & 36 & 1729 & 6471 & 6591 & 2619 & 462 & 36 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 180 & 2980 & 10200 & 10777 & 4225 & 680 & 45 & 1 \\ 0 & 46 & 4445 & 25960 & 38550 & 22152 & 5845 & 750 & 45 & 1 \end{pmatrix}$$

All rows are strictly log-concave. As $\partial\mathrm{CP}(m,4)^1 = D[m]^1$, the 1-chromatic number $\mathrm{chr}^1(\partial\mathrm{CP}(m,4)) = m$, and it is not difficult to see that the 2-chromatic number $\mathrm{chr}^2(\partial\mathrm{CP}(m,4))$ is 2 if m is even and 3 if m is odd [7].

Right multiplication with the upper triangular matrix $([j]_i)_{1\leq i,j\leq m(K)}$ with $[j]_i = \binom{j}{i}i! = \frac{j!}{(i-j)!}$ in row i and column j transforms, by the formula $\chi^s(K,j) = \sum_{\mathrm{chr}^s(K)\leq i\leq j} S(K,i,s)[j]_i$ of Theorem 1, the chromatic table into the $(\dim(K)\times m(K))$-matrix

$$\chi(K)([j]_i)_{1\leq i,j\leq m(K)} = \left(\chi^s(K,j)\right)_{\substack{1\leq s\leq \dim(K)\\ 1\leq j\leq m(K)}}$$

with the $m(K)$ values $\chi^s(K,j)$, $1\leq j\leq m(K)$, of the s-chromatic polynomial in row s. This matrix of chromatic polynomial values appears also to have log-concave rows. Note, however, that the chromatic polynomial of a graph is not log-concave for all integer values of r even though it is log-concave above a certain threshold value depending on the maximal degree of the graph [10,18].

3 Chromatic Uniqueness

In this section we briefly discuss to what extent simplicial complexes are determined by their chromatic polynomials. Proposition 3 shows that the chromatic table of a simplicial complex determines its f-vector.

Definition 11 K is *chromatically unique* if it is determined up to isomorphism by its chromatic table.

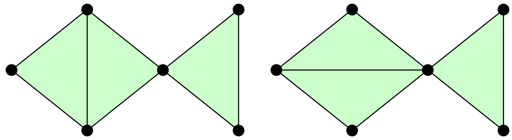
In Lemma 5 below, $K \amalg L$ is the disjoint union and $K\vee L$ the one-point union of K and L. The proof is identical to the one for the similar statements about chromatic polynomials for simple graphs [15, Theorems 2, 3].

Lemma 5 *If K and L are finite simplicial complexes then*

$$\chi^s(K\amalg L,r) = \chi^s(K,r)\chi^s(L,r), \qquad \chi^s(K\vee L,r) = \frac{\chi^s(K,r)\chi^s(L,r)}{r}$$

for all r and all $s\geq 0$.

The two nonisomorphic simplicial complexes



are not chromatically unique as they have identical chromatic tables

$$\begin{pmatrix} 0 & 0 & 2 & 10 & 7 & 1 \\ 0 & 15 & 73 & 62 & 15 & 1 \end{pmatrix}$$

by Lemma 5. (These two complexes are, however, PL-isomorphic.)

On the other hand, Proposition 2.(2) immediately implies that the s-skeleton of a full simplex is chromatically unique (in a very strong sense).

Proposition 4 *If K has the same s-chromatic polynomial as a full simplex $D[N]$, then K and $D[N]$ have isomorphic s-skeleta.*

Proof If K and $D[N]$ have the same s-chromatic polynomial for some $s \geq 1$, then K has N vertices (Corollary 2), and, since $\chi^s(K, N-s) = \chi^s(D[N], N-s)$, the s-skeleton of K is isomorphic to the s-skeleton of the full simplex on N vertices [Proposition 2.(2)]. □

4 Recurrence Relations for Simplicial Stirling Numbers

This section contains two recurrence relations for simplicial Stirling numbers. We begin with the most simple version going by 0-simplices. It is clear that

$$S(K,1,s) = \begin{cases} 1 & s > \dim(K) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

when $r = 1$. When $r > 1$, fix a vertex v_0 of K. Then

$$\begin{aligned} S(K,r,s) &= \sum_{\substack{U_0 \subsetneq V(K),\ v_0 \in U_0 \\ U_0 \text{ is } s\text{-independent}}} S\left(K \cap D[V(K) - U_0], r-1, s\right) \\ &= \sum_{\substack{\emptyset \neq U \subsetneq V(K) - \{v_0\} \\ V(K) - U \text{ is } s\text{-independent}}} S\left(K \cap D[U], r-1, s\right) \end{aligned}$$

To see the, let P be partition of $V(K)$ into r s-independent subsets. Let U_0 be the block containing v_0. The other blocks in P form a partition P_0 of $K \cap D[V(K) - U_0]$ into $r-1$ s-independent subsets. The map $P \leftrightarrow (P_0, U_0)$ is a bijection. This explains the first equality. The second equality is simply obtained by writing U for $V(K) - U_0$.

We state next a slightly different recurrence relation going by s-simplies rather than vertices.

Let $\sigma \in F^s(D[V(K)]) - F^s(K)$ be an $(s+1)$-subset of $V(K)$ that itself is not in K but all its proper subsets are in K. Define $K + \sigma$ to be the s-dimensional simplicial complex whose set of s-simplices is $F^s(K+\sigma) = F^s(K) \cup \{\sigma\}$ and with the same $(s-1)$-skeleton as K.

Proposition 5 (Deletion–contraction relation) *The simplicial Stirling numbers of the second kind for $K + \sigma$ are*

$$S(K+\sigma, r, s) = S(K, r, s) - \sum_{\substack{U \subseteq V(K) - \sigma \\ \forall \tau \in F^s(K):\, U \cap \tau \neq \emptyset}} S(K \cap D[U], r-1, s)$$

when $r > 1$.

Proof Using the notation of Theorem 4 we write $S(K, B, r, s)$ for the number of partitions P of $V(K)$ into r blocks with monochrome set $M^s(P) = B \in L^s(K)$. The equation

$$S(K+\sigma, \emptyset, r, s) = S(K, \emptyset, r, s) - S(K+\sigma, \sigma, r, s)$$

expresses the fact that an s-independent partition of K is also an s-independent partition of $K + \sigma$ unless its monochrome set is $\{\sigma\}$. We now focus on the second term on the right hand side.

For any finite simplicial complex L and any $\sigma \in F^s(L)$ we have

$$\begin{aligned} S(L, \sigma, r, s) &= \sum_{\substack{\sigma \subseteq U_0 \subseteq V(L) \\ \{\tau \in F^s(L) \mid \tau \subseteq U_0\} = \{\sigma\}}} S(L \cap D[V(L) - U_0], \emptyset, r-1, s) \\ &= \sum_{\substack{U \subseteq V - \sigma \\ \{\tau \in F^s(L) \mid U \subseteq V - \tau\} = \{\sigma\}}} S(L \cap D[U], \emptyset, r-1, s) \\ &= \sum_{\substack{U \subseteq V(L) \\ \{\tau \in F^s(L) \mid U \cap \tau = \emptyset\} = \{\sigma\}}} S(L \cap D[U], r-1, s) \\ &= \sum_{\substack{U \subseteq V(L) - \sigma \\ \forall \tau \in F^s(L) - \{\sigma\}:\, U \cap \tau \neq \emptyset}} S(L \cap D[U], r-1, s) \end{aligned}$$

To see this, let P be a partition of $V(L)$ into r blocks with $M^s(P) = \{\sigma\}$. Let U_0 be the block of P containing σ and let P_0 be the partition of $V(L) - U_0$ into $r-1$ blocks containing no s-simplex of L. (The partition P_0 is not empty as $r > 1$.) The first equality comes from the bijection $P \leftrightarrow (U_0, P_0)$. The second equality is obtained by writing U for $V(L) - U_0$. The third and fourth equality are simple rewritings.

Applying the above equality to $K + \sigma$ gives

$$S(K+\sigma, \sigma, r, s) = \sum_{\substack{U \subseteq V(K) - \sigma \\ \forall \tau \in F^s(K):\, U \cap \tau \neq \emptyset}} S(K \cap D[U], \emptyset, r-1, s)$$

because $(K+\sigma)\cap D[U] = K\cap D[U]$ when $U \subseteq V(K)-\sigma$. This finishes the proof of the proposition. □

Proposition 5 can be used inductively to compute s-chromatic polynomials.

Example 13 We consider the same 2-dimensional simplicial complexes as in Example 9. Proposition 5 leads to the following integral relations

$$\begin{aligned}
S(K,r,2) &= S(6,r) - 6S(1,r-1) - 8S(2,r-1) - 2S(3,r-1)\\
&\quad - S(3,r-1,2),\ \ r \geq \mathrm{chr}^2(K) = 2\\
S(\mathrm{MB},r,2) &= S(5,r) - 5S(1,r-1) - 5S(2,r-1), r \geq \mathrm{chr}^2(\mathrm{MB}) = 2\\
S(\mathrm{MT},r,2) &= S(7,r) - 7S(1,r-1) - 21S(2,r-1) - 35S(3,r-1)\\
&\quad - 14S(4,r-1) + 7S(2,r-2),\ \ r \geq \mathrm{chr}^2(\mathrm{MT}) = 3\\
S(\mathrm{P2},r,2) &= S(6,r) - 6S(1,r-1) - 15S(2,r-1) - 10S(3,r-1),\\
&\quad r \geq \mathrm{chr}^2(\mathrm{P2}) = 3
\end{aligned}$$

between Stirling numbers of the second kind.

The familiar recurrence relation $S(m,r) = S(m-1,r-1) + rS(m-1,r)$ for the usual Stirling numbers of the second kind does not readily apply to simplicial Stirling numbers. The closest analogue may be

$$\begin{aligned}
S(K,r,s) &= S(K\cap D[V(K)-\{v_0\}], r-1, s)\\
&\quad + \sum_{P\in\mathcal{S}(K\cap D[V(K)-\{v_0\}],r,s)} |\{B\in P \mid B\cup\{v_0\} \text{ is } s\text{-independent in } K\}|
\end{aligned}$$

where v_0 is a vertex of K and $\mathcal{S}(K\cap D[V(K)-\{v_0\}], r, s)$ is the *set* of partitions P of the vertex set of the simplicial complex $K\cap D[V(K)-\{v_0\}]$ into r s-independent subsets.

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