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Equivariant Euler characteristics of partition posets

Jesper M. Møller

Institut for Matematiske Fag, Københavns Universitet, Universitetsparken 5, DK-2100 København, Denmark

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ABSTRACT

The first part of this paper deals with the combinatorics of equivariant partitions of finite sets with group actions. In the second part, we compute all equivariant Euler characteristics of the Σ_n-poset of non-extreme partitions of an n-set.

1. Introduction

Let G be a finite group, Π a finite G-poset, and $r \geq 1$ a natural number. For any homomorphism $X : \mathbf{Z}^r \to G$, write Π^X for the sub-poset consisting of all elements of Π fixed by all elements in the image of X. The *rth equivariant reduced Euler characteristic* of the G-poset Π was defined by Atiyah and Segal [2] as the normalized sum

$$\widetilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(\Pi^X) \tag{1.1}$$

of the reduced Euler characteristics $\widetilde{\chi}(\Pi^X)$ (Definition 3.1(1)) of the X-fixed sub-poset Π^X as X runs through the set of all homomorphisms of $\mathbf{Z}^r$ to G, or, equivalently, the set of r-tuples of commuting elements of G. Two extreme examples are the following. When the poset $\Pi = \emptyset$ is empty, $\widetilde{\chi}_r(\emptyset, G) = -|\mathrm{Hom}(\mathbf{Z}^r, G)|/|G|$, and when Π has a least or greatest element, $\widetilde{\chi}_r(\Pi, G) = 0$ for all $r \geq 1$.

We are here particularly interested in G-posets of partitions of G-sets. For a finite G-set S, let $\Pi(S)$ denote the G-lattice of partitions of S and $\Pi^*(S) = \Pi(S) - \{\widehat{0}, \widehat{1}\}$ its proper part, the sub-G-poset of

E-mail address: moller@math.ku.dk.
URL: http://www.math.ku.dk/~moller.

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non-extreme partitions obtained by removing the discrete partition $\widehat{0}$ and the indiscrete partition $\widehat{1}$. The rth equivariant reduced Euler characteristic of the partition G-poset $\Pi^*(S)$ is the normalized sum

$$\widetilde{\chi}_r(\Pi^*(S), G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(\Pi^*(S)^X) \tag{1.2}$$

of the reduced Euler characteristics of the sub-posets $\Pi^*(S)^X$ of non-extreme X-partitions of S as X ranges over the set of commuting r-tuples of elements of G.

Eq. (1.2) above highlights the relevance of Euler characteristics of G-partitions of G-sets. The first part of this paper, dealing with the combinatorics of posets of G-partitions of G-sets, addresses this issue. The main result here, Theorem 3.9, identifies the reduced Euler characteristic $\widetilde{\chi}(\Pi^*(S)^G)$ as a G-Stirling number of the first kind.

In the second part we compute the equivariant reduced Euler characteristics $\widetilde{\chi}_r(\Pi^*(S), G)$ in the archetypical case where $G = \Sigma_n$ is the symmetric group of degree n and $S = \Sigma_{n-1}\backslash\Sigma_n$ the standard n-element right Σ_n-set. The Σ_n-poset $\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)$ consists of all non-extreme partitions of the n-set. The main result, Theorem 1.3 below, describes the equivariant reduced Euler characteristics $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ for all $r \geq 1$ and all $n \geq 1$. (It is convenient to declare $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ to mean 1 for all $r \geq 1$ when $n = 1$ even though the equivariant reduced Euler characteristics actually equal -1 in these cases.)

Let π_k, $k \geq 0$, be the multiplicative functions given by $\pi_k(n) = n^k$ for all $n \geq 1$ and ι_2 the multiplicative function given by $\iota_2(n) = n$ if $n = 1, 2, 4, \dots$ is a power of 2 and $\iota_2(n) = 0$ otherwise.

Theorem 1.3. *The rth reduced equivariant Euler characteristic of the Σ_n-poset $\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)$ is*

$$\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = c_r(n)/n, \quad n \geq 1, r \geq 1$$

where the multiplicative function c_r is the Dirichlet inverse

$$c_r = (\iota_2 * \pi_1 * \cdots * \pi_{r-1})^{-1}$$

of the iterated Dirichlet convolution of the function ι_2 and the $r-1$ functions π_k for $0 < k < r$.

The corollary below presents two alternative and more explicit views on the rth equivariant reduced Euler characteristics of Theorem 1.3. Let $b_0 = \varepsilon$ be the multiplicative Dirichlet unit function $\varepsilon = 1, 0, 0, \dots$ and for $r \geq 1$, let $b_r(n)$ and $\lambda_r(n)$ be the multiplicative functions whose values on prime powers $n = p^e$ are

$$b_r(p^e) = (-1)^e p^{\binom{e}{2}} \binom{r}{e}_p, \qquad \lambda_r(p^e) = \binom{e+r-1}{e}_p$$

where $\binom{n}{k}_p$ refers to a p-binomial coefficient (Eq. (6.2)). We note in Corollary 6.8 that $\lambda_r(n)$ is the number of subgroups of $\mathbf{Z}^r$ of index n and that b_r and that λ_r are reciprocal under Dirichlet convolution, $b_r * \lambda_r = \varepsilon$.

Corollary 1.4. *Fix $r \geq 1$.*

(1) $(c_r * \lambda_r)(n) = (-1)^{n+1}$ *for all $n \geq 1$.*
(2) $\sum_{d|n} d\widetilde{\chi}_r(\Pi^*(\Sigma_{d-1}\backslash\Sigma_d), \Sigma_d)\lambda_r(n/d) = (-1)^{n+1}$ *for all $n \geq 1$.*
(3) *The multiplicative function $c_r(n)/n = \widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ takes value $b_{r-1}(2^e) - b_{r-1}(2^{e-1})$ on an even prime power $n = 2^e$, $e > 0$, and value $b_{r-1}(p^e)$ on an odd prime power $n = p^e$, $e \geq 0$.*

The equation of Corollary 1.4(2) provides a recurrence relation for the rth equivariant reduced Euler characteristic function $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ when regarding the rth subgroup enumeration function $\lambda_r(n)$ known. Eq. (7.5) makes explicit the fact that the values given in Corollary 1.4(3) completely determine the equivariant reduced Euler characteristics $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ for all r and n. See Fig. 2 for concrete numerical values of $c_r(n)/n = \widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ for small r and n. The proofs of Theorem 1.3 and Corollary 1.4 are in Sections 5 and 7.

$$\begin{tikzcd}
S_G(iS,(j-1)T) \arrow[dr, "\cdot \mathcal{S}_G(S,T)"'] & S_G(iS,T) \arrow[d, "\cdot j|\mathcal{O}_G(S,T)|"] \\
& S_G((i+1)S,T)
\end{tikzcd}$$

Fig. 1. Recurrence for G-Stirling numbers of the second kind.

$c_r(n)/n$	$n=1$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$r=1$	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	−2	−1	1	−1	2	−1	0	0	2	−1	−1	−1	2	1
3	1	−4	−4	5	−6	16	−8	−2	3	24	−12	−20	−14	32	24
4	1	−8	−13	21	−31	104	−57	−22	39	248	−133	−273	−183	456	403
5	1	−16	−40	85	−156	640	−400	−190	390	2496	−1464	−3400	−2380	6400	6240

Fig. 2. The equivariant reduced Euler characteristics $c_r(n)/n = \widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ for $1 \leq r \leq 5$ and $1 \leq n \leq 15$.

This paper contains six more sections. Section 2 introduces basic concepts as listed in Definition 2.1. We focus on the poset $\Pi(S)^G$ of G-partitions of a G-set S as this a premier ingredient in the definition of equivariant Euler characteristics. We see in Corollary 2.9 that $\Pi(1\backslash G)^G$, the poset of G-partitions of the right G-set G, is the poset $\mathcal{S}_G$ of subgroups of G. Thus subgroup posets are special cases of G-partition posets. More importantly, Lemma 2.5, a result from Arone [1], states that $\Pi^*(S)^G$ is contractible unless S is an isotypical G-set, a G-set in which all G-orbits are isomorphic. Corollary 3.7 emphasizes the consequence that the sub-poset $\Pi^{*+\mathrm{iso}}(S)^G$ of non-extreme isotypical G-partitions (Definition 2.1(9)) is homotopy equivalent to the full poset $\Pi^*(S)^G$ of non-extreme G-partitions. In the following we shall therefore concentrate on posets of isotypical G-partitions of isotypical G-sets. If S is now a G-orbit and i a natural number, we write iS for the isotypical G-set with iG-orbits all isomorphic to S.

Section 3 discusses the reduced Euler characteristic $\widetilde{\chi}(\Pi^*(iS)^G) = \widetilde{\chi}(\Pi^{*+\mathrm{iso}}(iS)^G)$ of the (isotypical) G-partition poset of an isotypical G-set iS. Definition 3.8 introduces G-Stirling numbers of the second and first kind. These numbers are equivariant versions of the classical Stirling numbers. Just as the classical Stirling number of the second kind, $S(n,k)$, counts the number of partitions of an n-set with k blocks, the G-Stirling number of the second kind, $S_G(iS, jT)$, counts the number of isotypical G-partitions of the isotypical G-set iS with isotypical block G-set isomorphic to jT. The G-Stirling numbers of the second kind are determined by the table of marks for G and the classical Stirling numbers (Proposition 3.13). The G-Stirling table of the first kind is the inverse of the G-Stirling table of the second kind. Example 3.18 contains the concrete numerical values for the Σ_3-Stirling numbers. The main result of Section 3 is Theorem 3.9 asserting that the Euler characteristics $\widetilde{\chi}(\Pi^*(iS)^G)$ are (special) G-Stirling numbers of the first kind. Corollary 3.17 specializes to the abelian case as this suffices when computing equivariant Euler characteristics by means of Eq. (1.2).

Section 4, even though not needed for the proof of Theorem 1.3, contains additional information about G-Stirling numbers of the first kind. Eq. (4.5) shows that G-Stirling numbers of the first kind are related to the Möbius function on the poset of G-partitions just as the classical Stirling numbers of the first kind are related to the Möbius function on the poset of classical partitions [19, Example 3.10.4].

In Section 5 we use the material of the previous sections to prove Theorem 1.3. In the following Section 6 we identify the multiplicative sequences $n \to \widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ of rth equivariant Euler characteristics as Dirichlet convolutions of simpler multiplicative arithmetic sequences.

For a prime q, the rth *q-primary equivariant Euler characteristic* of a G-poset Π, while defined in purely combinatorial terms (Eq. (7.1)), turns out to be the Euler characteristic of the homotopy orbit space $|\Pi|_{hG}$ seen through the eyes of the rth Morava K-theory $K(r)$ at q (Remark 7.2). The q-primary equivariant reduced Euler characteristics of the Σ_n-poset $\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)$ are determined in the main result, Theorem 7.4, of the final Section 7. See Figs. 3–4 for concrete numerical values.

We end the introduction with a brief comment on the boolean case to put the results of this paper into context. (See [5,21] for much more general results.) Let $B(\Sigma_{n-1}\backslash\Sigma_n)$ be the Σ_n-lattice of subsets of the n-set. The reduced Euler characteristic of the sub-Σ_n-poset of non-empty and proper subsets, $B^*(\Sigma_{n-1}\backslash\Sigma_n)$, is $\widetilde{\chi}(B^*(\Sigma_{n-1}\backslash\Sigma_n)) = (-1)^n$ [19, Example 3.8.3] and the rth equivariant reduced Euler

$c^2_r(2^d)/2^d$	$d=0$	1	2	3	4	5
$r=1$	1	−1	0	0	0	0
2	1	−2	1	0	0	0
3	1	−4	5	−2	0	0
4	1	−8	21	−22	8	0
5	1	−16	85	−190	184	−64

Fig. 3. The rth 2-primary equivariant reduced Euler characteristics $c^2_r(n)/n = \widetilde{\chi}^2_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$, $1 \leq r \leq 5$, for $n = 2^d$, $0 \leq d \leq 5$, a power of 2.

$c^3_r(3^d)/3^d$	$d=0$	1	2	3	4
$r=1$	1	0	0	0	0
2	1	−1	0	0	0
3	1	−4	3	0	0
4	1	−13	39	−27	0
5	1	−40	390	−1080	729

$c^5_r(5^d)/5^d$	$d=0$	1	2	3	4
$r=1$	1	0	0	0	0
2	1	−1	0	0	0
3	1	−6	5	0	0
4	1	−31	155	−125	0
5	1	−156	4030	−19500	−15625

Fig. 4. The rth q-primary equivariant reduced Euler characteristics $c^q_r(n)/n = \widetilde{\chi}^q_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$, $1 \leq r \leq 5$, for $n = q^d$, $0 \leq d \leq 4$, a power of $q = 3, 5$.

characteristic is

$$\widetilde{\chi}_r(B^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = \frac{1}{n!}\sum_{X\in\mathrm{Hom}(\mathbf{Z}^r,\Sigma_n)}(-1)^{|\Sigma_{n-1}\backslash\Sigma_n/X|}$$

where $\Sigma_{n-1}\backslash\Sigma_n/X$ is the orbit set for the action of $X(\mathbf{Z}^r)$ on $\Sigma_{n-1}\backslash\Sigma_n$. (When $n = 0$ we interpret these equivariant reduced Euler characteristics as 1 for all $r \geq 1$.) The result [5, Theorem 2.1] [20, 5.13.(d) p 113] or the orbit counting formula of [23, Theorem 1] show that the generating function for fixed $r \geq 1$ is

$$\sum_{n\geq 0}\widetilde{\chi}_r(B^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)u^n = \prod_{d\geq 1}(1-u^d)^{\lambda_{r-1}(d)}.$$

This identity paired with [20, Exercise 5.13, pp 76,111–113] reveal that the sequences $(\widetilde{\chi}_r(B^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n))_{n\geq 0}$ of rth equivariant reduced Euler characteristics and $(|\mathrm{Hom}(\mathbf{Z}^r, \Sigma_n)|/|\Sigma_n|)_{n\geq 0} = (|\mathrm{Hom}(\mathbf{Z}^{r-1}, \Sigma_n)/\Sigma_n|)_{n\geq 0}$ of conjugacy classes of $(r-1)$-tuples of commuting elements in Σ_n are reciprocal under convolution. We may view this observation and its reformulation

$$\sum_{i=0}^{n}\binom{n}{i}|\mathrm{Hom}(\mathbf{Z}^r, \Sigma_{n-i})|\sum_{X\in\mathrm{Hom}(\mathbf{Z}^r,\Sigma_i)}(-1)^{|\Sigma_{n-1}\backslash\Sigma_n/X|} = 0, \quad r > 0,\ n > 0$$

as the boolean analogs to Corollary 1.4(1)–(2).

2. Partitions of G-sets

We start by listing for easy reference a collection of basic definitions some of which will be detailed below.

Definition 2.1. Let G be a finite group and S a finite right G-set.

(1) A *partition* π of S is an equivalence relation on S. The *blocks* of π are the equivalence classes of π and $\pi\backslash S$ is the set of blocks. For any $x \in S$, $[x]_\pi = \{y \in S \mid x\pi y\}$, or simply $[x]$, is the π-block of x.
(2) $\Pi(S)$ is the G-lattice of all partitions of S and $\Pi^*(S) = \Pi(S) - \{\widehat{0}, \widehat{1}\}$ the G-poset of all partitions of S but the discrete and the indiscrete partitions, $\widehat{0} = \{\{x\} \mid x \in S\}$ and $\widehat{1} = \{S\}$.
(3) A partition π of S is a *G-partition* if $x\pi y \iff (xg)\pi(yg)$ holds for all $x, y \in S$ and $g \in G$.
(4) The *block set* $\pi\backslash S$ of a G-partition π of S is a G-set and $\pi\backslash S/G$ is the set of G-orbits of π-blocks.
(5) $\Pi(S)^G$ is the lattice of all G-partitions of S and $\Pi^*(S)^G = \Pi(S)^G - \{\widehat{0}, \widehat{1}\}$ the poset of all G-partitions of S but the discrete and indiscrete partitions (which are G-partitions).

(6) The *isotropy subgroup* at $x \in S$ is the subgroup ${}_xG = \{g \in G \mid xg = x\}$ of G.
(7) If π is a G-partition and $B \in \pi\backslash S$ a block of π, the *block isotropy subgroup* at B is the isotropy subgroup ${}_BG$ at B for the G-action on the set $\pi\backslash S$ of π-blocks.
(8) The G-set S is *isotypical* if all isotropy subgroups are conjugate.
(9) The G-partition $\pi \in \Pi(S)^G$ is *isotypical* if the G-set $\pi\backslash S$ of π-blocks is isotypical. $\Pi^{\mathrm{iso}}(S)^G$ is the poset of all isotypical G-partitions and $\Pi^{*+\mathrm{iso}}(S)^G = \Pi^{\mathrm{iso}}(S)^G - \{\widehat{0}, \widehat{1}\}$ the poset of all isotypical G-partitions of S but the discrete partition (which is isotypical precisely when S is isotypical) and the indiscrete partition (which is isotypical).
(10) $\mathcal{S}_G$ is the poset of subgroups and $[\mathcal{S}_G]$ the set of conjugacy classes of subgroups of G. Also, ζ_G is the poset incidence function ($\zeta_G(H, K) = 1$ if $H \le K$ and $\zeta_G(H, K) = 0$ otherwise), and $\mu_G = \zeta_G^{-1}$ the Möbius function of $\mathcal{S}_G$ [19, Section 3.7].
(11) $\overline{\mathcal{O}}_G$ is the (Burnside) category of finite right G-sets with surjective G-maps as morphisms. $\overline{\mathcal{O}}_G^{\mathrm{iso}}$ is the full subcategory of $\overline{\mathcal{O}}_G$ generated by all isotypical finite right G-sets. The orbit category $\mathcal{O}_G$ is the full subcategory of $\overline{\mathcal{O}}_G$ generated by all G-orbits (transitive right G-sets).

When H and K are subgroups of G and $N_G(H, K) = \{g \in G \mid H^g \le K\}$ denotes the *transporter set*, the bijection

$$N_G(H, K)/K \xrightarrow{\cong} \mathcal{O}_G(H\backslash G, K\backslash G)$$

takes the left coset $gK \in N_G(H, K)/K$ to the right G-map $H\backslash G \to K\backslash G$ between G-orbits given by $Hx \to Kg^{-1}x$. (If $g_1 \in N_G(H, K)$ and $g_2 \in N_G(K, L)$ for some subgroup $L \le G$, then the composition $H \to Kg_1^{-1} \to Lg_2^{-1}g_1^{-1} = L(g_1g_2)^{-1}$ in $\mathcal{O}_G$ is the morphism defined by $g_1g_2 \in N_G(H, L)$.) The bijection

$$N_G(H, K)/K \xrightarrow{\cong} (K\backslash G)^H$$

is induced by the map that takes any $g \in G$ with $H^g \le K$ to the coset Kg^{-1}. The *mark* of H on K,

$$\mathrm{TOM}_G(H, K) = |N_G(H, K)/K| = |\mathcal{O}_G(H\backslash G, K\backslash G)| = |(K\backslash G)^H| \tag{2.2}$$

is the number of morphisms in the category $\mathcal{O}_G$ with domain $H\backslash G$ and codomain $K\backslash G$ or, equivalently, the number of elements of the G-orbit $K\backslash G$ fixed by H. These numbers depend only on the conjugacy classes of the subgroups H and K. The *table of marks* for G is the matrix $\mathrm{TOM}_G = (\mathrm{TOM}_G(H, K))_{H,K\in[\mathcal{S}_G]}$ of marks. The set $[\mathcal{O}_G]$ of isomorphism classes of right G-orbits corresponds bijectively to the set $[\mathcal{S}_G]$ of conjugacy classes of subgroups of G [6, Theorem 1.3.(b)].

The set $\Pi(S)$ of partitions of S is partially ordered by refinement [19, Example 3.3.1]:

$$\pi_1 \le \pi_2 \iff \forall x \in S\colon [x]_{\pi_1} \subseteq [x]_{\pi_2} \iff \overline{\mathcal{O}}_G(\pi_1\backslash S, \pi_2\backslash S) \ne \emptyset.$$

The *meet* of π_1 and π_2 is the partition $\pi_1 \wedge \pi_2$ with blocks $[x]_{\pi_1\wedge\pi_2} = [x]_{\pi_1} \cap [x]_{\pi_2}$, $x \in S$. The discrete partition is $\widehat{0}$ with blocks $[x]_{\widehat{0}} = \{x\}$, $x \in S$, block set $\widehat{0}\backslash S = S$, and the indiscrete partition is $\widehat{1}$ with block $[x]_{\widehat{1}} = S$, $x \in S$, and block set $\widehat{1}\backslash S = \{*\}$ of cardinality 1.

The set $\Pi(S)$ of partitions of S is a right G-lattice: For any partition π of S and any $g \in G$, πg is the partition given by $x(\pi g)y \iff (xg)\pi(yg)$. Then $[x]_{\pi g}g = \{yg \mid x(\pi g)y\} = \{yg \mid (yg)\pi(xg)\} = \{y \mid y\pi(xg)\} = [xg]_\pi$. Obviously,

$$\begin{aligned} &\pi \text{ is a } G\text{-partition} \iff \forall g \in G\colon \pi g = \pi \iff \forall g \in G \forall x \in X\colon [x]_\pi g = [xg]_\pi \\ &\quad \iff \forall g \in G \forall b \in \pi : bg \in \pi. \end{aligned}$$

Thus the fixed lattice, $\Pi(S)^G$, for this G-action on $\Pi(S)$ is the lattice of all G-partitions.

Proposition 2.3. *Let π be a G-partition of S.*

(1) *There is a right G-action on the set $\pi\backslash S$ of π-blocks such that $S \to \pi\backslash S$ is a G-map.*
(2) ${}_xG \le {}_{[x]}G$ *for any $x \in S$.*
(3) ${}_{xg}G = ({}_xG)^g$ *and* ${}_{[xg]}G = ({}_{[x]}G)^g$
(4) ${}_{xg}G \le ({}_{[x]}G)^g$ *for any $x \in S$ and any $g \in G$.*

Proof. The G-action on $\pi\backslash S$ is given by $[x]g = [xg]$ for all $x \in S$ and $g \in G$. $\square$

Definition 2.4. Let P be a sub-poset of a lattice. An element c of P is a contractor for P if $x \vee c \in P$ or $x \wedge c \in P$ for all $x \in P$.

If c is a contractor for P then $x \leq x \vee c \geq c$ or $x \geq x \wedge c \leq c$ are homotopies between the identity map of P and the constant map c.

Lemma 2.5 *([1, Lemma 7.1]). $\Pi^*(S)^G$ is contractible unless S is an isotypical G-set.*

Proof. Let ω_G be the G-partition represented by the G-map $S \to S/G$ to the G-set of G-orbits and θ_G the G-partition represented by the G-map $S \to S/G \to \cong\backslash S/G$ to the set of isomorphism classes of G-orbits. Equivalently, $x\omega_G y$ if and only if x and y are in the same G-orbit, and $x\theta_G y$ if and only if x and y have conjugate isotropy subgroups. We shall prove that θ_G is a contractor (Definition 2.4) for $\Pi^*(S)^G$ when S is not isotypical.

We first make some small observations. Obviously, $\omega_G \leq \theta_G$. The G-action is trivial if and only if $\omega_G = \widehat{0}$. The G-action is isotypical if and only if $\theta_G = \widehat{1}$. If the G-action is trivial, all isotropy subgroups are equal to G, and therefore $\theta_G = \widehat{1}$. We may summarize these observations in a string of implications

$$\theta_G = \widehat{0} \Longrightarrow \omega_G = \widehat{0} \iff \forall x \in S\colon {}_xG = G \Longrightarrow \theta_G = \widehat{1} \iff S \text{ is isotypical.}$$

Let π be any G-partition of S. We claim that

$$\pi \wedge \theta_G = \widehat{0} \Longrightarrow \pi = \widehat{0}. \tag{2.6}$$

To see this first note that

$$\forall x, y \in S\colon x\pi y \Longrightarrow y \cdot {}_xG \subseteq [y]_{\pi\wedge\theta_G}.$$

Indeed, let $x\pi y$ and $g \in {}_xG$. Then $y\pi(yg)$ for $y\pi x$, $x = xg$, and $(xg)\pi(yg)$. Thus y and yg are both in $[y]_\pi$ and in $[y]_{\theta_G}$. Now assume that $\pi \wedge \theta_G = \widehat{0}$. Then

$$\forall x, y \in S\colon x\pi y \Longrightarrow {}_xG \leq {}_yG$$

for the block $[y]_{\pi\wedge\theta_G} = [y]_{\widehat{0}} = \{y\}$ consists of y alone which forces $yg = y$ for all $g \in {}_xG$. This can be sharpened to

$$\forall x, y \in S\colon x\pi y \Longrightarrow {}_xG = {}_yG$$

as the equivalence relation π is symmetric, of course. Now, when x and y have the same isotropy subgroups, x and y belong to the same block under θ_G. Thus we have shown $\pi \leq \theta_G$. Then $\pi = \pi \wedge \theta_G = \widehat{0}$. This proves claim (2.6).

Suppose that S is not isotypical. Then $\theta_G \neq \widehat{0}, \widehat{1}$ and θ_G belongs to the poset $\Pi^*(S)^G$. From claim (2.6) we know that $\pi \wedge \theta_G \neq \widehat{0}$ for all $\pi \in \Pi^*(S)^G$. Thus θ_G is a contractor for $\Pi^*(S)^G$. $\square$

See Example 2.10 for an isotypical G-set S for which $\Pi^*(S)^G$ is contractible.

In the following, when π is an element of the poset Π we write π/Π for the sub-poset of all $\lambda \in \Pi$ with $\pi \leq \lambda$. (See Definition 3.1(2) for a more detailed presentation of this notation.)

The *block functor*

$$\Pi(S)^G \to \overline{\mathcal{O}}_G \tag{2.7}$$

takes a G-partition $\pi \in \Pi(S)^G$ to its block G-set $\pi\backslash S$. If $\pi_1 \leq \pi_2$, there is an induced surjection $\pi_1\backslash S \to \pi_2\backslash S$ of block G-sets as any block of π_1 lies in a block of π_2. Since $\widehat{0} \leq \pi \leq \widehat{1}$ there are G-maps $S = \widehat{0}\backslash\pi \to \pi\backslash S \to \widehat{1}\backslash S = G\backslash G$. Observe that if $\pi_1 \leq \pi_2$ and the block sets $\pi_1\backslash S$ and $\pi_2\backslash S$ are isomorphic then $\pi_1 = \pi_2$ by the pigeon-hole principle. Thus G-partitions with isomorphic block G-sets are incomparable.

The block functor is the left adjoint functor in the adjoint equivalence [15, Definitions 2.1.1, 2.2.5, 1.3.5]

$$\Pi(S)^G \underset{R}{\overset{L}{\rightleftarrows}} S/\overline{\mathcal{O}}_G \qquad L(\pi) = (S \to \pi\backslash S) \qquad \{\varphi^{-1}(t) \mid t \in T\} = R(S \xrightarrow{\varphi} T)$$

between the poset of G-partitions of S and the coslice under S of $\overline{\mathcal{O}}_G$ [15, Example 2.3.3]. From this perspective, G-partitions of S are surjective G-maps with domain S. When S is an isotypical G-set we get an induced adjoint equivalence

$$\Pi^{\mathrm{iso}}(S)^G \underset{R}{\overset{L}{\rightleftarrows}} S/\overline{\mathcal{O}}_G^{\mathrm{iso}}$$

for the isotypical case.

Proposition 2.8. *For any G-partition $\pi \in \Pi(S)^G$ there are isomorphisms of posets*

$$\pi/\Pi(S)^G \xrightarrow{\cong} \Pi(\pi\backslash S)^G, \qquad \pi/\Pi^{\mathrm{iso}}(S)^G \xrightarrow{\cong} \Pi^{\mathrm{iso}}(\pi\backslash S)^G$$

where we in the second case assume that π is isotypical.

Proof. This isomorphism takes a G-partition $\lambda \in \pi/\Pi(S)^G$ to the G-partition $\pi\backslash\lambda = R(\pi\backslash S \to \lambda\backslash S)$ whose blocks are the fibers of the G-map $\pi\backslash S \to \lambda\backslash S$. The converse takes a G-partition λ of $\pi\backslash S$ to $R(S \to \pi S \to \lambda\backslash(\pi\backslash S))$. The block set of $\pi\backslash\lambda$ is the block set of λ, $(\pi\backslash\lambda)\backslash(\pi\backslash S) = \lambda\backslash S$. In particular, λ is an isotypical G-partition of S if and only if $\pi\backslash\lambda$ is an isotypical G-partition of $\pi\backslash S$. $\square$

Corollary 2.9 (*[24, Lemma 3]*)**.** *Let H be a subgroup of G. There is an isomorphism of posets*

$$H/\mathcal{S}_G \xrightarrow{\cong} \Pi(H\backslash G)^G.$$

The blocks of the G-partition corresponding to the supergroup K of H are the fibers of the G-map $H\backslash G \to K\backslash G$ taking coset H to coset K.

The G-partition of the G-orbit $H\backslash G$ corresponding to the supergroup K of H has $|G : K|$ blocks $[Hg] = \{Hgx \mid x \in K^g\}$, $g \in G$, of size $|K : H|$. The special case where $H = 1$ is the trivial subgroup shows that the subgroup poset (Definition 2.1(10)) $\mathcal{S}_G \cong \Pi(1\backslash G)^G = \Pi^{\mathrm{iso}}(1\backslash G)^G$ is a special case of a partition poset.

Example 2.10 (*An Isotypical G-set S Such That $\Pi^*(S)^G$ is Contractible*)**.** Suppose that G has a nontrivial Frattini subgroup $\Phi(G)$ [7, Chp 5, Section 1]. The right G-set $1\backslash G$ is transitive and hence isotypical. But still the poset $\Pi^*(1\backslash G)^G = 1//\mathcal{S}_G//G$ of non-identity proper subgroups of G (Corollary 2.9) is contractible as $\Phi(G)$ is a contractor [7, Chp 5, Theorem 1.1]. (This example was pointed out to me by Matthew Gelvin.)

3. Euler characteristics of posets of G-partitions of isotypical G-sets

We first fix some general notation.

Definition 3.1. Let Π be a finite poset and let a and b be elements of Π.

(1) The Euler characteristic and the reduced Euler characteristic of Π are

$$\chi(\Pi) = \sum_{i=0}^{\infty} (-1)^i f_i(\Pi), \qquad \widetilde{\chi}(\Pi) = \chi(\Pi) - 1$$

where $f_i(\Pi)$ is the number of length i chains in Π [19, Equation (19) p 120].

(2) We write

$$a/\Pi = \{p \in \Pi \mid a \le p\} \qquad a//\Pi = \{p \in \Pi \mid a < p\} \qquad k^a(\Pi) = -\widetilde{\chi}(a//\Pi)$$
$$\Pi/b = \{p \in \Pi \mid p \le b\} \qquad \Pi//b = \{p \in \Pi \mid p < b\} \qquad k_b(\Pi) = -\widetilde{\chi}(\Pi//b)$$

for the coslice of Π under a (the left ideal generated by a), the proper coslice of Π under a, and the weighting at a, and, dually, the slice of Π over b (the right ideal generated by b), the proper slice of Π over b, and the coweighting at b [14, Definition 1.10]. Also,

$$a/\Pi/b = \{p \in \Pi \mid a \le p \le b\} = [a, b] \qquad a//\Pi//b = \{p \in \Pi \mid a < p < b\} = (a, b)$$

are the closed or open intervals from a to b.

The Euler characteristic of Π is the sum

$$\sum_{a\in\Pi} k^a(\Pi) = \chi(\Pi) = \sum_{b\in\Pi} k_b(\Pi)$$

of the values of the weighting or coweighting [14, Definition 2.2].

The zeta function $\zeta_\Pi : \Pi \times \Pi \to \mathbf{Z}$ of Π is defined by $\zeta_\Pi(a, b) = 1$ if $a \le b$ and $\zeta_\Pi(a, b) = 0$ otherwise [19, p 114]. The Möbius function $\mu_\Pi : \Pi \times \Pi \to \mathbf{Z}$ of Π [19, Section 3.7] is determined by the relations

$$\forall a, b \in \Pi : \sum_{\pi\in\Pi} \mu_\Pi(a, \pi)\zeta_\Pi(\pi, b) = \delta_{a,b}$$

where, as usual, $\delta_{a,a} = 1$ and $\delta_{a,b} = 0$ if $a \ne b$. The Möbius function satisfies $\mu_\Pi(a, a) = 1$, $\mu_\Pi(a, b) = \widetilde{\chi}(a//\Pi//b)$ for $a < b$, and $\mu_\Pi(a, b) = 0$ otherwise. In the case where $a < b$, this is implied by Philip Hall's theorem on chains [19, Proposition 3.8.5], for instance as expressed by the identities

$$\begin{aligned} 1 = \chi(a/\Pi//b) &= \sum_{a\le x,y<b} \mu_\Pi(x, y) = \sum_{a\le y<b} \mu_\Pi(a, y) + \sum_{a<x,y<b} \mu_\Pi(x, y) \\ &= -\mu_\Pi(a, b) + \sum_{a\le y\le b} \mu_\Pi(a, y) + \chi(a//\Pi//b) = -\mu_\Pi(a, b) + \chi(a//\Pi//b). \end{aligned}$$

In case Π has a least element $\widehat{0}$ and a greatest element $\widehat{1} \ne \widehat{0}$,

$$1 = \chi(\Pi//\widehat{1}) = \sum_{a\in\Pi//\widehat{1}} k^a(\Pi//\widehat{1}) = \sum_{\widehat{0}\le a<\widehat{1}} -\widetilde{\chi}(a//\Pi//\widehat{1}) = \sum_{\widehat{0}\le a<\widehat{1}} -\mu_\Pi(a, \widehat{1}) \tag{3.2}$$
$$1 = \chi(\widehat{0}//\Pi) = \sum_{b\in\widehat{0}//\Pi} k_b(\widehat{0}//\Pi) = \sum_{\widehat{0}<b\le\widehat{1}} -\widetilde{\chi}(\widehat{0}//\Pi//b) = \sum_{\widehat{0}<b\le\widehat{1}} -\mu_\Pi(\widehat{0}, b) \tag{3.3}$$

as $\Pi//\widehat{1}$ has a $\widehat{0}$ and $\widehat{0}//\Pi$ a $\widehat{1}$. The weighting for $\Pi//\widehat{1}$ (coweighting for $\widehat{0}//\Pi$) restricts to a weighting (coweighting) for $\widehat{0}//\Pi//\widehat{1}$.

We now specialize to partition posets $\Pi(S)$ of right G-sets S.

Proposition 3.4 (*Coslices in $\Pi(S)^G$ and weightings in $\Pi(S)^G//\widehat{1}$*). *For any G-partition π of the right G-set S*

$$\pi/\Pi(S)^G = \Pi(\pi\backslash S)^G, \qquad \pi//\Pi(S)^G//\widehat{1} = \Pi^*(\pi\backslash S)^G \quad (\text{when } \pi < \widehat{1}).$$

The weighting for $\Pi(S)^G//\widehat{1}$ at $\pi < \widehat{1}$ is

$$k^\pi(\Pi(S)^G//\widehat{1}) = -\widetilde{\chi}(\Pi^*(\pi\backslash S)^G).$$

The open interval $\pi//\Pi(S)^G//\widehat{1}$ is contractible and the weighting $k^\pi(\Pi(S)^G//\widehat{1})$ is zero unless π is an isotypical G-partition (Definition 2.1 (9)).

Proof. The first identity comes from Proposition 2.8. If π is a non-isotypical G-partition then $\Pi^*(\pi\backslash S)$ is contractible by Lemma 2.5 and its Euler characteristic $k^\pi(\Pi(S)^G//\widehat{1})$ is zero. □

Recall from Definition 2.1(4) that if π is a G-partition of the right G-set S, then $\pi\backslash S$ is the G-set of π-blocks and $\pi\backslash S/G$ the set of G-orbits of π-blocks. For any π-block $B \in \pi\backslash S$, ${}_BG$ is the G-isotropy subgroup at B (Definition 2.1(6)) and $BG \in \pi\backslash S/G$ the G-orbit through B. Thus ${}_BG\backslash G$ and BG are isomorphic right G-sets. $\Pi(B)^{{}_BG}$ is the poset of ${}_BG$-partitions of the right ${}_BG$-set B. The symbol

$$\prod_{BG\in\pi\backslash S/G} \Pi(B)^{{}_BG}$$

denotes the product of all the posets $\Pi(B)^{{}_BG}$ as the π-blocks B in $\pi\backslash S$ range over a complete set of representatives for the set of G-orbits BG in $\pi\backslash S/G$. See the proof of Lemma 2.5 for the definition of the G-partition θ_G.

Proposition 3.5 (*Slices in $\Pi(S)^G$ and coweightings in $\widehat{0}//\Pi(S)^G$*). *For any G-partition π of the right G-set S*

$$\Pi(S)^G/\pi = \prod_{BG\in\pi\backslash S/G} \Pi(B)^{{}_BG}, \qquad \widehat{0}//\Pi(S)^G//\pi = \Big(\prod_{BG\in\pi\backslash S/G} \Pi(B)^{{}_BG}\Big)^* \quad (\text{when } \widehat{0} < \pi).$$

The coweighting for $\widehat{0}//\Pi(S)^G$ at $\widehat{0} < \pi$ is

$$k_\pi(\widehat{0}//\Pi(S)^G) = -\prod_{\substack{BG\in\pi\backslash S/G\\ |B|>1}} \widetilde{\chi}(\Pi^*(B)^{{}_BG}).$$

The open interval $\widehat{0}//\Pi(S)^G//\pi$ is contractible and the coweighting $k_\pi(\widehat{0}//\Pi(S)^G)$ is zero unless $\pi \le \theta_G$.

Proof. Let π be a G-partition and B one its blocks. Observe first that the blocks contained in B of a G-partition $\lambda \le \pi$ determine all blocks of λ contained in any of the blocks of the orbit BG through B for the G-action on $\pi\backslash S$.

Let B be a block, with isotropy subgroup ${}_BG$, of the G-partition π. Let λ be a ${}_BG$-partition of B. Extend λ to a G-partition of the orbit BG of B in π by $[xg]_\lambda = [x]_\lambda g$. We must argue that this extension is well-defined. Suppose that $x_1g_1 = x_2g_2$ for some $x_1, x_2 \in B$ and $g_1, g_2 \in G$. We must show that $[x_1]_\lambda g_1 = [x_2]_\lambda g_2$. We have $x_2 = x_2g_2g_2^{-1} = x_1g_1g_2^{-1}$. From $B = [x_2]_\pi = [x_1g_1g_2^{-1}]_\pi = [x_1]_\pi g_1g_2^{-1} = Bg_1g_2^{-1}$ we get that $g_1g_2^{-1}$ stabilizes the block B. As λ is a ${}_BG$-partition, $[x_1]_\lambda g_1 = [x_1]_\lambda g_1g_2^{-1}g_2 = [x_1g_1g_2^{-1}]_\lambda g_2 = [x_2]_\lambda g_2$ as we wanted.

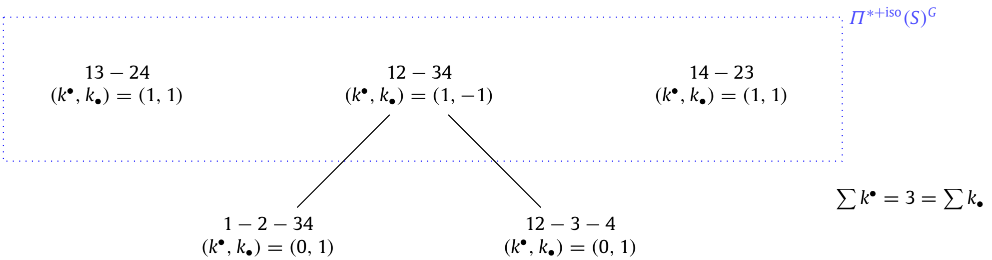
Conversely, if λ is a G-partition and $\lambda \le \pi$ then the blocks of λ inside a fixed block B of π form a ${}_BG$-partition of B, of course.

According to Quillen the reduced Euler characteristic is multiplicative: $\widetilde{\chi}((\prod L_i)^*) = \prod \widetilde{\chi}(L_i^*)$ for lattices L_i of more than one element [1, Proposition 2.8].

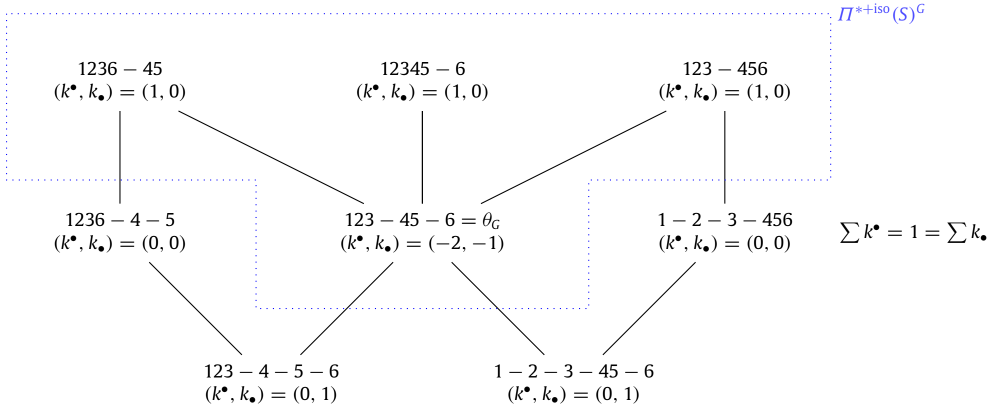
If the block B of partition π consists of a single element of S, then also the partition poset $\Pi(B)$ consists of a single element so it can be omitted from the poset product $\prod_{BG\in\pi\backslash S/G} \Pi(B)^{{}_BG}$.

Note that $\pi \le \theta_G$ means that all isotropy subgroups within all blocks of π are conjugate. If $\pi \not\le \theta_G$, there is a block B of π that is a non-isotypical ${}_BG$-set. Then the product of the contractors for the blocks of π is a contractor for $(\prod \Pi(B)^{{}_BG})^*$. $\square$

Example 3.6 (*Two Examples of G-partition Posets with weightings and coweightings*). The poset $\Pi^*(S)^G$ of non-extreme G-partitions for $S = \{1, 2, 3, 4\}$ and $G = \langle(1, 2)(3, 4)\rangle \le \Sigma_4$ (isotypical):



The poset $\Pi^*(S)^G$ of non-extreme G-partitions for $S = \{1, \dots, 6\}$ and $G = \langle (1, 2, 3), (4, 5) \rangle \leq \Sigma_6$ (non-isotypical):



The sub-posets $\Pi^{*+\mathrm{iso}}(S)^G$ of isotypical G-partitions are indicated with dotted lines. The weighting for $\Pi^*(S)^G$ restricts to a weighting for $\Pi^{*+\mathrm{iso}}(S)^G$.

Corollary 3.7. *The inclusion $\Pi^{*+\mathrm{iso}}(S)^G \hookrightarrow \Pi^*(S)^G$ is a homotopy equivalence.*

Proof. Note that if $\pi \in \Pi^*(S)^G$ is not isotypical then the proper coslice $\pi//\Pi^*(S)^G = \pi//\Pi(S)^G//\widehat{1}$ is contractible by Proposition 3.4. The corollary now follows immediately from Bouc's theorem [4, Proposition 4]. $\square$

Because of Lemma 2.5 and Corollary 3.7 we now restrict attention to isotypical G-partitions of isotypical G-sets.

Definition 3.8. Let S and T be G-orbits.

(1) For any natural number $i \geq 1$, $iS = \coprod_i S$ is the isotypical G-set with i G-orbits isomorphic to S.
(2) The type of an isotypical G-partition $\pi \in \Pi^{\mathrm{iso}}(iS)^G$ is the isomorphism class in $\overline{\mathcal{O}}_G^{\mathrm{iso}}$ of its block G-set $\pi\backslash(iS)$.
(3) $\Sigma_G(iS, jT) = \{\pi \in \Pi^{\mathrm{iso}}(iS)^G \mid \pi\backslash iS \cong jT\}$ is the antichain in $\Pi^{\mathrm{iso}}(iS)^G$ of isotypical G-partitions of iS of type jT.
(4) The G-Stirling number of the second kind at (iS, jT) is the number $|\Sigma_G(iS, jT)|$ of isotypical G-partitions of iS of type jT.
(5) The G-Stirling table of the second kind in degree n is the square $(n|[\mathcal{O}_G]| \times n|[\mathcal{O}_G]|)$-matrix

$$S_G = \big((|\Sigma_G(iT_1, jT_2)|)_{1\leq i,j\leq n}\big)_{T_1,T_2\in[\mathcal{O}_G]}$$

of G-Stirling numbers of the second kind. The G-Stirling table of the second kind in degree n of the G-orbit S is the submatrix

$$S_G(S) = \big((|\Sigma_G(iT_1, jT_2)|)_{1\leq i,j\leq n}\big)_{T_1,T_2\in[S/\mathcal{O}_G]}$$

of the G-Stirling table S_G.
(6) The G-Stirling table of the first kind in degree n is the inverse $s_G = S_G^{-1}$ of the G-Stirling table of the second kind in degree n. The G-Stirling number of the first kind at (iS, jT) is the (iS, jT)-entry, $s_G(iS, jT)$, of s_G. The G-Stirling table of the first kind in degree n of the G-orbit S is the inverse $s_G(S) = S_G(S)^{-1}$ of the G-Stirling table $S_G(S)$ of the second kind in degree n of the G-orbit S.
(7) The isotypical G-Bell number of the isotypical G-set iS

$$B_G(iS) = |\Pi^{\mathrm{iso}}(iS)^G| = \sum_{j,T} |\Sigma_G(iS, jT)|$$

is the total number of isotypical G-partitions of iS.

The G-Stirling number of the second kind, $S_G(iS, jT)$, is the number of G-surjections of iS onto jS up to G-automorphisms of jS. The G-Stirling tables are lower triangular with the convention that the G-orbits are listed with increasing size. We regard the Stirling tables, S_G and s_G, in degree n as two-variable functions: $S_G(iS, jT) = |\Sigma_G(iS, jT)|$ and $s_G(iS, jT)$ are the (iS, jT)-entries in the respective tables S_G and s_G when $1 \le i, j \le n$ and S, T are (conjugacy classes of) G-orbits. See Example 3.12 for 1-Stirling tables and Example 3.18 for Σ_3-Stirling tables. The G-Stirling tables are the G-Stirling tables of the right G-set $1\backslash G$: $S_G = S_G(1\backslash G)$ and $s_G = s_G(1\backslash G)$.

We now observe that the first column in the G-Stirling table of the first kind computes Euler characteristics of G-partition posets of isotypical right G-sets.

Theorem 3.9. *Let S be a G-orbit and n a natural number such that $|nS| > 1$. The reduced Euler characteristic of the poset of non-extreme G-partitions of the isotypical G-set nS*

$$\widetilde{\chi}(\Pi^*(nS)^G) = s_G(nS,\, 1G\backslash G)$$

is the G-Stirling number of the first kind at $(nS,\, 1G\backslash G)$.

Proof. The weighting for $\Pi^*(nS)^G$ restricts to a weighting for $\Pi^{*+\mathrm{iso}}(nS)^G$. If we in Eq. (3.2) insert the values of the weighting from Proposition 3.4 we get

$$1 = \sum_{|kT|>1} -\widetilde{\chi}(\Pi^*(kT)^G) S_G(nS,\, kT) \tag{3.10}$$

with T ranging over the set of isomorphism classes of G-orbits and $k \ge 1$ over natural numbers with $|kT| > 1$. This may be restated in matrix notation as

$$S_G \begin{bmatrix} 0 \\ -\widetilde{\chi}(\Pi^*(2G\backslash G)^G) \\ \vdots \\ -\widetilde{\chi}(\Pi^*(sH\backslash G)^G) \\ \vdots \\ -\widetilde{\chi}(\Pi^*(n1\backslash G)^G) \end{bmatrix}_{\substack{H\in[\mathcal{S}_G] \\ 1\le s\le n}} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad S_G = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ \vdots & & & \\ 1 & & & \end{bmatrix} \tag{3.11}$$

where S_G is the G-Stirling table of the second kind in degree n. The reason for the 0 at the top of the left column vector is that the trivial orbit $1G\backslash G$ does not figure in Eq. (3.10) but it is recorded as the first row and column of the Stirling table S_G. All entries, $S_G(sH\backslash G,\, 1G\backslash G)$, of the first column of S_G on the left side equal 1. Eq. (3.11) gives

$$S_G \begin{bmatrix} 1 \\ \widetilde{\chi}(\Pi^*(2G\backslash G)^G) \\ \vdots \\ \widetilde{\chi}(\Pi^*(sH\backslash G)^G) \\ \vdots \\ \widetilde{\chi}(\Pi^*(n1\backslash G)^G) \end{bmatrix} = S_G \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + S_G \begin{bmatrix} 0 \\ \widetilde{\chi}(\Pi^*(2G\backslash G)^G) \\ \vdots \\ \widetilde{\chi}(\Pi^*(sH\backslash G)^G) \\ \vdots \\ \widetilde{\chi}(\Pi^*(n1\backslash G)^G) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and we just apply the inverse of S_G to this equation to finish the proof. $\square$

Eq. (4.5) will later reveal that Theorem 3.9 is but a special case of a more general connection between G-Stirling numbers of the first kind and values of Möbius functions for posets of isotypical G-partitions of isotypical G-sets.

Example 3.12 (*Stirling Tables of the Trivial Group 1*)**.** The 1-Stirling tables of the second and first kind in degree 4 are the matrices

$$S_1 = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 3 & 1 & \\ 1 & 7 & 6 & 1 \end{bmatrix}, \qquad s_1 = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 2 & -3 & 1 & \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

of classical Stirling numbers $S_1(n,k) = S(n,k) = |\{\pi \in \Pi(n1) \mid |\pi| = k\}|$ and $s_1(n,k) = s(n,k)$ of the second and first kind [19, pp 33–36]. We recover, as a special instance of Theorem 3.9, the result of [19, Example 3.10.4] that the reduced Euler characteristic $\widetilde{\chi}(\Pi^*(n1))$ of the non-extreme partitions of an n-element set is the Stirling number of the first kind $s(n,1) = (-1)^{n-1}(n-1)!$ when $n \geq 2$.

Proposition 3.13. *Let S and T be G-orbits and i, j natural numbers. The G-Stirling number of the second kind*

$$S_G(iS, jT) = \frac{|\mathcal{O}_G(S,T)|^i}{|\mathcal{O}_G(T,T)|^j} S(i,j)$$

is determined by TOM_G *(Eq. (2.2)) and the classical Stirling numbers $S(i,j)$ of the second kind (Example 3.12).*

Proof. There is a bijection between $\mathcal{O}_G(S,T)^i \times \{1,\dots,j\}^i$ and the set of G-maps $iS \to jT$. This bijection takes (φ, τ) to the G-map $(s,k) \to (\varphi_k(s), \tau_k)$, $s \in S$, $1 \leq i \leq k$. The surjective G-maps correspond to pairs (φ,τ) where τ is surjective. Thus $|\overline{\mathcal{O}}_G(iS, jT)| = |\mathcal{O}_G(S,T)|^i S(i,j)j!$. The G-Stirling number of the second kind $S_G(iS, jT)$ is the number $|\overline{\mathcal{O}}_G(iS,jT)|/|\overline{\mathcal{O}}_G(jT,jT)|$ of G-surjections of iS onto jT counted up to G-automorphisms of jT. □

Remark 3.14 (*Consequences of* Proposition 3.13)**.** Suppose that $i = 1 = j$ and that $S = H\backslash G$, $T = K\backslash G$ for subgroups H and K of G. Then $\Pi^{\mathrm{iso}}(H\backslash G)^G = \Pi(H\backslash G)^G = H/\mathcal{S}_G = [H,G]$ is the poset of supergroups of H in G by Corollary 2.9. The antichain $\Sigma_G(H\backslash G, K\backslash G) = H/[K]$ is the set of supergroups of H conjugate to K and the G-Stirling number of the second kind $S_G(H\backslash G, K\backslash G) = |N_G(H,K)/N_G(K)| = |H/[K]|$ is the number of supergroups of H conjugate to K. For this reason, the G-Stirling table of the second kind in degree 1 is also called the table of conjugate supergroups. (We shall discuss the G-Stirling numbers of the first kind $s_G(H\backslash G, K\backslash G)$ in Remark 4.6.)

Let $\Delta^n(S,T) = \mathrm{diag}(|\mathcal{O}_G(S,T)|, |\mathcal{O}_G(S,T)|^2, \dots, |\mathcal{O}_G(S,T)|^n)$ be the diagonal $(n\times n)$-matrix given by the first n powers of the mark $|\mathcal{O}_G(S,T)|$ (Eq. (2.2)), $S^n = (S(i,j))_{1\leq i,j\leq n}$ the $(n\times n)$-matrix of the first Stirling numbers of the second kind, and $s^n = (s(i,j))_{1\leq i,j\leq n}$ its inverse given by the first Stirling numbers of the first kind (Example 3.12). Proposition 3.13 shows that the G-Stirling tables in degree n are the block matrices

$$S_G = \begin{pmatrix} \Delta^n(S_1,S_1) & 0 & 0 \\ \Delta^n(S_2,S_1) & \Delta^n(S_2,S_2) & 0 \\ \Delta^n(S_3,S_1) & \Delta^n(S_3,S_2) & \Delta^n(S_3,S_3) \end{pmatrix} \begin{pmatrix} S^n & 0 & 0 \\ 0 & S^n & 0 \\ 0 & 0 & S^n \end{pmatrix}$$
$$\times \begin{pmatrix} \Delta^n(S_1,S_1) & 0 & 0 \\ 0 & \Delta^n(S_2,S_2) & 0 \\ 0 & 0 & \Delta^n(S_3,S_3) \end{pmatrix}^{-1}$$

$$s_G = \begin{pmatrix} \Delta^n(S_1,S_1) & 0 & 0 \\ 0 & \Delta^n(S_2,S_2) & 0 \\ 0 & 0 & \Delta^n(S_3,S_3) \end{pmatrix} \begin{pmatrix} s^n & 0 & 0 \\ 0 & s^n & 0 \\ 0 & 0 & s^n \end{pmatrix}$$
$$\times \begin{pmatrix} \Delta^n(S_1,S_1) & 0 & 0 \\ \Delta^n(S_2,S_1) & \Delta^n(S_2,S_2) & 0 \\ \Delta^n(S_3,S_1) & \Delta^n(S_3,S_2) & \Delta^n(S_3,S_3) \end{pmatrix}^{-1}$$

where we for simplicity assume that $[\mathcal{O}_G] = \{S_1, S_2, S_3\}$ contains only three isomorphism classes of G-orbits. The identity for s_G translates into the recurrence

$$(s_G(iS, jS))_{1\le i,j\le n} = \Delta^n(S,S)s^n\Delta^n(S,S)^{-1}$$
$$\sum_{T\in[\mathcal{O}_G]} (s_G(iS, jT))_{1\le i,j\le n}\Delta^n(T,U) = 0, \quad S \neq U \in [\mathcal{O}_G]$$

for the G-Stirling table of the first kind. The first of these identities states that $s_G(iS, jS) = |\mathcal{O}_G(S,S)|^{i-j}s(i,j)$.

When G is abelian

$$S_G(nS, kT) = \begin{cases} |T|^{n-k}S(n,k) & \mathcal{O}_G(S,T) \neq \emptyset \\ 0 & \mathcal{O}_G(S,T) = \emptyset \end{cases} \tag{3.15}$$

because $|\mathcal{O}_G(S,T)|$ equals $|T|$ when nonzero. In general, combining the familiar recurrence relation $S(n+1,k) = kS(n,k) + S(n,k-1)$ for the classical Stirling numbers of the second kind [19, Equation (23)] with Proposition 3.13, establishes the recurrence relation

$$S_G((n+1)S, kT) = k|\mathcal{O}_G(S,T)|S_G(nS, kT) + S_G(S,T)S_G(nS,(k-1)T)$$

for the G-Stirling numbers of the second kind for any finite group G (Fig. 1).

Lemma 3.16. *If $H \trianglelefteq G$ is normal in G, then $\widetilde{\chi}(\Pi^*(nH\backslash G)^G) = \widetilde{\chi}(\Pi^*(nH\backslash G)^{H\backslash G})$ for all $n \ge 1$.*

Proof. H acts trivially on $H\backslash G$ as $Hgh = Hghg^{-1}g = Hg$ for all $h \in H, g \in G$. Thus a partition of $nH\backslash G$ is a G-partition if and only if it is a $H\backslash G$-partition and $\widetilde{\chi}(\Pi^*(nH\backslash G)^G) = \widetilde{\chi}(\Pi^*(nH\backslash G)^{H\backslash G})$. □

Recall from Definition 2.1(10) that μ_G denotes the Möbius function of the subgroup poset $\mathcal{S}_G$.

Corollary 3.17. *If G is abelian, $\widetilde{\chi}(\Pi^*(nH\backslash G)^G) = \mu_G(H,G)|H\backslash G|^{n-1}s(n,1)$ for all $n \ge 1$ and all subgroups $H \le G$.*

Proof. Since G is abelian, $S_G(iH\backslash G, jK\backslash G) = |G:K|^{i-j}S(i,j)$ by Eq. (3.15), and the G-Stirling table of the second kind in degree n is the block matrix

$$S_G = \left((\zeta_G(H,K)|G:K|^{i-j}S(i,j))_{1\le i,j\le n}\right)_{H,K\in[\mathcal{S}_G]}.$$

The vector $\left((\widetilde{\chi}(iH\backslash G)^G)_{1\le i\le n}\right)_{H\in[\mathcal{S}_G]}$ is by Theorem 3.9 the first column ($K = G$) in the G-Stirling table

$$s_G = \left((\mu_G(H,K)|G:H|^{i-j}s(i,j))_{1\le i,j\le n}\right)_{H,K\in[\mathcal{S}_G]}$$

of the first kind. □

Example 3.18. The symmetric group Σ_3 has $|[\mathcal{O}_{\Sigma_3}]| = 4$ orbits, $S_1 = \Sigma_3\backslash\Sigma_3, S_2 = A_3\backslash\Sigma_3$, $S_3 = C_2\backslash\Sigma_3$, and $S_6 = 1\backslash\Sigma_3$, of sizes 1, 2, 3, 6. The table of marks (Eq. (2.2)) $\mathrm{TOM}_{\Sigma_3} = (|\mathcal{O}_{\Sigma_3}(S,T)|)$, the Stirling table S_{Σ_3} of the second kind in degree 1 (the table of conjugate supergroups), and its inverse, the Stirling table s_{Σ_3} of the first kind in degree 1, are

$$(|\mathcal{O}_{\Sigma_3}(S,T)|)_{S,T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 6 \end{pmatrix}, \qquad S_{\Sigma_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix},$$
$$s_{\Sigma_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 3 & -1 & -3 & 1 \end{pmatrix}.$$

The Σ_3-Stirling table S_{Σ_3} of the second kind in degree 3 is

$S_{\Sigma_3}(S,T)$	$T = 1S_1$	$2S_1$	$3S_1$	$1S_2$	$2S_2$	$3S_2$	$1S_3$	$2S_3$	$3S_3$	$1S_6$	$2S_6$	$3S_6$
$S = 1S_1$	1	0	0									
$2S_1$	1	1	0									
$3S_1$	1	3	1									
$1S_2$	1	0	0	1	0	0						
$2S_2$	1	1	0	2	1	0						
$3S_2$	1	3	1	4	6	1						
$1S_3$	1	0	0				1	0	0			
$2S_3$	1	1	0				1	1	0			
$3S_3$	1	3	1				1	3	1			
$1S_6$	1	0	0	1	0	0	3	0	0	1	0	0
$2S_6$	1	1	0	2	1	0	9	9	0	6	1	0
$3S_6$	1	3	1	4	6	1	27	81	27	36	18	1

and the Σ_3-Stirling table s_{Σ_3} of the first kind in degree 3 is

$s_{\Sigma_3}(S,T)$	$T = 1S_1$	$2S_1$	$3S_1$	$1S_2$	$2S_2$	$3S_2$	$1S_3$	$2S_3$	$3S_3$	$1S_6$	$2S_6$	$3S_6$
$S = 1S_1$	1	0	0									
$2S_1$	−1	1	0									
$3S_1$	2	−3	1									
$1S_2$	−1	0	0	1	0	0						
$2S_2$	2	−1	0	−2	1	0						
$3S_2$	−8	6	−1	8	−6	1						
$1S_3$	−1	0	0				1	0	0			
$2S_3$	1	−1	0				−1	1	0			
$3S_3$	−2	3	−1				2	−3	1			
$1S_6$	3	0	0	−1	0	0	−3	0	0	1	0	0
$2S_6$	−18	9	0	6	−1	0	18	−9	0	−6	1	0
$3S_6$	216	−162	27	−72	18	−1	−216	162	−27	72	−18	1

As the row sums of S_{Σ_3} are isotypical Σ_3-Bell numbers (Definition 3.8(7)), we see from the three last rows that the total number of isotypical Σ_3-partitions of the free Σ_3-sets are $B_{\Sigma_3}(nS_6) = 6, 30, 206$ for $n = 1, 2, 3$. The row sums for s_{Σ_3} are zero, except for the first row, because the first column of S_{Σ_3} contains only 1s. By Theorem 3.9, the first column of s_{Σ_3} contains the reduced Euler characteristics $\widetilde{\chi}(\Pi^*(iS)^{\Sigma_3})$ for Σ_3-partition posets of the isotypical Σ_3-sets $S = S_1, S_2, S_3, S_6$; for instance, $\widetilde{\chi}(\Pi^*(2S_3)^{\Sigma_3}) = 1$ (and as this reduced Euler characteristic is not divisible by $|S_3|s(2,1) = -3$, Corollary 3.17 cannot be extended to general non-abelian groups). By Remark 4.6, the Σ_3-Stirling numbers $s_{\Sigma_3}(S_6, S_i)$ of the first kind inform about Möbius numbers of subgroups of Σ_3.

The Stirling tables in degree 3 of the Σ_3-orbit S_2 are the submatrices, $S_{\Sigma_3}(S_2)$ and $s_{\Sigma_3}(S_2)$, of S_{Σ_3} or s_{Σ_3} with indices iT for $T \in [S_2/\mathcal{O}_{\Sigma_3}] = \{S_1, S_2\}$ and $i = 1, 2, 3$.

Remark 3.19. A finite poset is *graded* if all maximal chains have the same length [19, Section 3.1]. Example 3.6 shows that $\Pi(S)^G$ is not graded in general. Neither the sub-poset $\Pi^{\mathrm{iso}}(S)^G$ of isotypical G-partitions is graded in general as subgroup posets $\Pi^{\mathrm{iso}}(1\backslash G)^G = \mathcal{S}_G$ most often are not graded.

4. More about G-Stirling numbers of the first kind

This section contains additional information, not needed for the proof of Theorem 1.3, about G-Stirling numbers of the first kind.

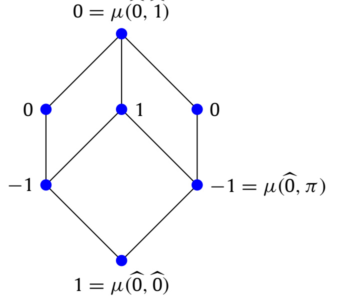
Let $\varPi$ be a finite poset with $\widehat{0}$ and $\widehat{1}$. Suppose that there is a set $\{A_i\}$ of antichains A_i in $\varPi$ and integers $M_{ij} \geq 0$ such that

$$\varPi = \coprod A_i, \quad \forall \pi \in A_i \colon |\pi / A_j| = M_{ij}. \tag{4.1}$$

This means that $\varPi$ decomposes as a disjoint union of antichains A_i such that the number of elements $\lambda \in A_j$ with $\pi \leq \lambda$ is independent of the choice of $\pi \in A_i$. If $\widehat{0} \in A_i$ then $A_i = \{\widehat{0}\}$, and if $\widehat{1} \in A_j$ then $A_j = \{\widehat{1}\}$, as A_i and A_j are antichains. By extending the partial order on the set of antichains [19, Section 3.1] to a linear order we can assume that the matrix (M_{ij}) is upper triangular (lower triangular with the opposite order) with 1s in the diagonal.

Example 4.2. Below is the Hasse diagram of a poset $\varPi = \coprod_{1 \leq i \leq 4} A_i$ divided into 4 disjoint antichains, indicated by height above $\widehat{0}$, satisfying (4.1). We let μ be the Möbius function for $\varPi$ and at each element π of $\varPi$ the value of $\mu(\widehat{0}, \pi)$ is given.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 1 \end{pmatrix}$$



$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

$$\Big(\sum_{\pi \in A_i} \mu(\widehat{0}, \pi)\Big)_{1 \leq i \leq 4} = (0, 1, -2, 1)$$

Proposition 4.3 explains the identity $(0, 1, -2, 1)M = (0, 0, 0, 1)$.

Proposition 4.3. *Assume (4.1) and let μ be the Möbius function for $\varPi$ [19, Section 3.7]. Then*

$$\sum_i \sum_{\pi \in A_i} M_{ij} \mu(\widehat{0}, \pi) = \begin{cases} 1 & \widehat{0} \in A_j \\ 0 & \widehat{0} \notin A_j. \end{cases}$$

Proof. The sum equals $\sum_{\pi \in A_j} \sum_{\widehat{0} \leq \lambda \leq \pi} \mu(\widehat{0}, \lambda)$ which evaluates to 1 if $A_j = \{\widehat{0}\}$ and to $\sum_{\pi \in A_j} 0 = 0$ otherwise. □

Let now S be a G-orbit and i a natural number. The lattice, $\varPi^{\mathrm{iso}}(iS)^G$, $i \geq 1$, of isotypical G-partitions of the isotypical G-set iS is generally not graded by Remark 3.19 and there is no characteristic polynomial as there is for the classical poset of partitions of the n-set [19, Example 3.10.4]. Instead, the set-up of Eq. (4.1) applies to $\varPi^{\mathrm{iso}}(iS)^G$ since

$$\varPi^{\mathrm{iso}}(iS)^G = \coprod_{\substack{1 \leq j \leq i \\ T \in [\mathcal{O}_G]}} \varSigma_G(iS, jT), \quad \forall \pi \in \varSigma_G(iS, jT) \colon |\pi / \varSigma_G(iS, kU)| \overset{\text{Proposition 2.8}}{=} S_G(jT, kU)$$

and the matrix $M = S_G(S)$ is the G-Stirling table of the second kind in degree i of the G-orbit S (Definition 3.8(5)).

Corollary 4.4. *Let $\mu_\varPi$ be the Möbius function of $\varPi^{\mathrm{iso}}(iS)^G$. Fix a G-orbit U and a natural number $k \geq 1$. The sum*

$$\sum_{jT} S_G(jT, kU) \sum_{\pi \in \varSigma_G(iS, jT)} \mu_\varPi(\widehat{0}, \pi)$$

equals 1 if iS and kU are isomorphic and 0 otherwise.

Corollary 4.4 states that the G-Stirling numbers of the first kind

$$s_G(iS, jT) = \sum_{\pi \in \Sigma_G(iS,jT)} \mu_\Pi(\widehat{0}, \pi) \tag{4.5}$$

are given by the values of the Möbius function for $\Pi^{\mathrm{iso}}(iS)^G$ on the antichain $\Sigma_G(iS, jT)$; see [19, Example 3.10.4] for the case of classical partitions.

Consider the special case of Eq. (4.5) where $jT = 1G\backslash G$ is the trivial G-orbit. Since the set $\Sigma_G(iS, 1G\backslash G)$ of isotypical G-partitions of type $1G\backslash G$ contains only the indiscrete partition $\widehat{1}$ of iS, the equation states that $s_G(iS, 1G\backslash G) = \mu_\Pi(\widehat{0}, \widehat{1})$, where, as noted just below Definition 3.1, $\mu_\Pi(\widehat{0}, \widehat{1}) = \widetilde{\chi}(\Pi^{*+\mathrm{iso}}(iS)^G)$, is the reduced Euler characteristic of $\widehat{0}//\Pi^{\mathrm{iso}}(iS)^G//\widehat{1} = \Pi^{*+\mathrm{iso}}(iS)^G$. This special case of Eq. (4.5) thus provides an alternative proof of Theorem 3.9.

Remark 4.6 (*G-Stirling Tables in Degree 1*). Suppose that $i = 1$ and that $S = H\backslash G$ for some subgroup H of G. We noted in Remark 3.14 that $\Pi^{\mathrm{iso}}(H\backslash G)^G = [H, G] = H/\mathcal{S}_G$ is the poset of supergroups of H in G. Let K be such a supergroup of H. Eq. (4.5) states in this case that the G-Stirling number of the first kind in degree 1

$$s_G(H\backslash G, K\backslash G) = \sum_{L \in H/[K]} \mu_G(H, L)$$

is a sum of values for the Möbius function μ_G of $\mathcal{S}_G$. The sum ranges over the set of supergroups of H conjugate to K. The fact that $S_G s_G$ and $s_G S_G$ equal the identity matrix implies the general rule that

$$\sum_{K,V \colon H \le K \le V \in [U]} \mu_G(K, V) = 0 = \sum_{K,V \colon H \le K \le V \in [U]} \mu_G(H, K)$$

whenever $1 \le H \lneqq U \le G$. When H is a normal subgroup of G we learn from

$$s_G(H\backslash G, K\backslash G) = \sum_{L \in [K]} \mu_G(H, L) = |[H/[K]]| \mu_G(H, K) = S_G(H\backslash G, K\backslash G)\mu_G(H, K)$$

that the Möbius numbers $\mu_G(H, K) = \widetilde{\chi}(H//\mathcal{S}_G//K) = s_G(H\backslash G, K\backslash G)/S_G(H\backslash G, K\backslash G)$ for all supergroups K of H can be recovered from the G-Stirling tables in degree 1. For instance, from the bottom rows of the Σ_3-Stirling tables in degree 1 from Example 3.18 we read off that $\mu_{\Sigma_3}(1, K) = 3, -1, -1, 1$ for $K = \Sigma_3, A_3, C_2, 1$.

The size of the antichain $\Sigma_G(iS, jT)/\lambda$ in $\Pi^{\mathrm{iso}}(iS)^G$ in general depends on the choice of $\lambda \in \Sigma_G(iS, kU)$. This is the case in the lattice $\Pi(\{1, 2, 3, 4\})$ of classical partitions of a 4-set and also, for most choices of nontrivial $H < G$, in $\Pi(H\backslash G)^G = H/\mathcal{S}_G$. A related fact is that $\mu_\Pi(\widehat{0}, \pi)$ is not a constant function of $\pi \in \Sigma_G(iS, jT)$ as, generally, $S_G(iS, jT)$ does not divide $s_G(iS, jT)$.

5. Equivariant Euler characteristics of posets of partitions

For any group A and natural number $r \ge 1$, let $\varphi_{\mathbf{Z}^r}(A) = |\mathrm{Epi}(\mathbf{Z}^r, A)|$ be the number of epimorphisms of $\mathbf{Z}^r$ onto A. Thus $\varphi_{\mathbf{Z}^r}(A)$ is nonzero if and only A is abelian and generated by r of its elements. Since $\mathrm{Aut}(A)$ acts freely on the set $\mathrm{Epi}(\mathbf{Z}^r, A)$ of epimorphisms of $\mathbf{Z}^r$ onto A, the quotient $d_{\mathbf{Z}^r}(A) = \varphi_{\mathbf{Z}^r}(A)/|\mathrm{Aut}(A)|$ is an integer [10, (1.3)]. Using that the set, $\mathrm{Hom}(\mathbf{Z}^r, G)$, of homomorphisms from $\mathbf{Z}^r$ to G is the disjoint union, $\coprod_{A \le G} \mathrm{Epi}(\mathbf{Z}^r, A)$, over all (abelian) subgroups A of G, we see that the rth reduced equivariant Euler characteristic of the finite G-poset Π (1.1) is the sum

$$\widetilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(\Pi^X) = \frac{1}{|G|} \sum_{A \le G} \widetilde{\chi}(\Pi^A)\varphi_{\mathbf{Z}^r}(A) \tag{5.1}$$

and, by Möbius inversion [19, Proposition 3.7.1], that

$$\widetilde{\chi}(\Pi^H)\varphi_{\mathbf{Z}^r}(H) = \sum_{A \le G} \widetilde{\chi}_r(\Pi, A)|A|\mu_G(A, H)$$

for any subgroup H of G.

We now specialize from general poset to posets of partitions. Let S be a finite G-set. As always, $\Pi(S)$ is the G-poset of partitions of G, and $\Pi^*(S) = \Pi(S) - \{\widehat{0}, \widehat{1}\}$ the G-poset of non-extreme partitions of S (Definition 2.1(5)).

Lemma 5.2. *Suppose that the abelian group A acts on the A-set S such that the action map $A \to \mathrm{Sym}(S)$ is injective (the action is effective). The following conditions are equivalent:*

(1) *A acts isotypically on S.*
(2) *A acts freely on S.*

If A acts isotypically on S then the order of A divides $|S|$.

Proof. If A acts isotypically and A is abelian, the isotropy subgroup at any point of S is the same subgroup, B, of A. The group B acts trivially on S, so B is the trivial subgroup since the action is effective. Thus A acts freely on S. If A acts freely, then $S = m1\backslash A$ as right A-sets and $|S| = m|A|$. $\square$

Lemma 5.3. *Let P be a finite group (not necessarily abelian), m a natural number, and $n = m|P|$.*

(1) *P is isomorphic to a subgroup of Σ_n acting freely on the n-set $\Sigma_{n-1}\backslash\Sigma_n$. The centralizer of this subgroup is the wreath product $P \wr \Sigma_m$ and the normalizer is an extension of the centralizer by the automorphism group $\mathrm{Aut}(P)$ of P.*
(2) *Any two subgroups of Σ_n abstractly isomorphic to P and acting freely on $\Sigma_{n-1}\backslash\Sigma_n$ are conjugate in Σ_n.*

Proof. P acts freely on the free P-set $m1\backslash P$ and the action map $P \to \mathrm{Sym}(m1\backslash P)$ is injective. The centralizer of P in $\mathrm{Sym}(m1\backslash P)$ is the automorphism group $\overline{\mathcal{O}}_P(m1\backslash P, m1\backslash P) = \mathcal{O}_P(1\backslash P, 1\backslash P) \wr \Sigma_m = P \wr \Sigma_m$. The remaining assertions are consequences of the fact that there is, up to isomorphism, just one free P-set on n elements. $\square$

Lemma 5.4. *Let A be any abelian subgroup of Σ_n acting freely on $\Sigma_{n-1}\backslash\Sigma_n$ where $n = m|A|$.*

(1) $\widetilde{\chi}(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)^A) = (-1)^{m+1}\mu_{\Sigma_n}(1, A)|A|^{m-1}(m-1)!$ *when* $n \geq 2$.
(2) $\widetilde{\chi}(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)^A)|\Sigma_n : N_{\Sigma_n}(A)| = (-1)^{n/|A|+1}\frac{\mu_{\Sigma_n}(1,A)}{|\mathrm{Aut}(A)|}(n-1)!\ n \geq 2$.

Proof. (1) As an A-set $\Sigma_{n-1}\backslash\Sigma_n = m1\backslash A$ consists of m free A-orbits. According to Corollary 3.17

$$\begin{aligned}\widetilde{\chi}(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)^A) &= \widetilde{\chi}(\Pi^*(m1\backslash A)^A) = \mu_{\Sigma_n}(1, A)|A|^{m-1}s(m, 1)\\ &= (-1)^{m-1}\mu_{\Sigma_n}(1, A)|A|^{m-1}(m-1)!\end{aligned}$$

This formula also holds when A is the trivial group: The left hand side is $\widetilde{\chi}(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)) = (-1)^{n-1}(n-1)!$ and the right hand side is $(-1)^{n-1}(n-1)!$ as $\mu_{\Sigma_n}(1, 1) = 1$.

(2) This is immediate from (1) as the index of the normalizer of A is known from Lemma 5.3(1) and we remember that $m|A| = n$. $\square$

Proof of Theorem 1.3. According to Lemmas 2.5 and 5.2 we have the expression

$$\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = \frac{1}{n!}\sum_{\substack{[A\leq\Sigma_n]\\ A \text{ free and abelian}}} \widetilde{\chi}(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)^A)\varphi_{\mathbf{Z}^r}(A)|\Sigma_n : N_{\Sigma_n}(A)|$$

for the rth equivariant Euler characteristic. The sum ranges over the set of conjugacy classes of abelian subgroups A of Σ_n acting freely on $\Sigma_{n-1}\backslash\Sigma_n$. By Lemma 5.3 there is a bijective correspondence between this set and the set of isomorphism classes of abelian groups A of order dividing n. Inserting the value from Lemma 5.4(2) for the expression under the summation sign gives

$$\begin{aligned}\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) &= \frac{1}{n}\sum_{\substack{|A|\mid n\\ A \text{abelian}}} (-1)^{n/|A|+1}\mu_{\Sigma_n}(1, A)\frac{\varphi_{\mathbf{Z}^r}(A)}{|\mathrm{Aut}(A)|}\\ &= \frac{1}{n}\sum_{\substack{|A|\mid n\\ A \text{ abelian}}} (-1)^{n/|A|+1}\mu_{\Sigma_n}(1, A)d_{\mathbf{Z}^r}(A)\end{aligned}$$

where the sum ranges over the set of isomorphism classes of abelian groups A of order dividing n. The Möbius function $\mu_{\Sigma_n}(1, A) = \mu_A(1, A)$ is completely known [10, 2.8]. Indeed, write $A = \prod A_p$ as the product of its Sylow p-subgroups A_p. Then $\mu_{\Sigma_n}(1, A) = \prod \mu_{\Sigma_n}(1, A_p)$ and $\mu_{\Sigma_n}(1, A_p) = 0$ unless A_p is an elementary abelian p-group. For an elementary abelian p-group of rank d,

$$\mu_{\Sigma_n}(1, C_p^d) = (-1)^d p^{\binom{d}{2}}.$$

Suppose now that $A = \prod A_p$ where each Sylow p-subgroup $A_p = C_p^{d_p}$ is elementary abelian of rank d_p. By [11, Lemma 2.1], $\mathrm{Aut}(A) = \prod_p \mathrm{Aut}(A_p) = \prod_p \mathrm{GL}_{d_p}(\mathbf{F}_p)$ and clearly $\varphi_{\mathbf{Z}^r}(\prod A_p) = \prod \varphi_{\mathbf{Z}^r}(A_p)$. Since

$$\varphi_{\mathbf{Z}^r}(C_p^d) = \prod_{j=0}^{d-1}(p^r - p^j) = \binom{r}{d}_p |\mathrm{GL}_d(\mathbf{F}_p)|, \qquad d_{\mathbf{Z}^r}(C_p^d) = \frac{\varphi_{\mathbf{Z}^r}(C_p^d)}{|\mathrm{Aut}(C_p^d)|} = \binom{r}{d}_p$$

we have now shown that

$$\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = \frac{1}{n}\sum_{d|n}(-1)^{\frac{n}{d}+1} b_r(d) = \frac{1}{n}(a * b_r)(n)$$

where b_r is the multiplicative function of Eq. (6.1) given by $b_r(p^d) = \mu_{\Sigma_n}(1, C_p^d) d_{\mathbf{Z}^r}(C_p^d)$ and $a(n) = (-1)^{n+1}$. In Corollary 6.8 we shall derive an alternative expression for the Dirichlet convolution $a * b_r$. $\square$

The first equivariant Euler characteristic is the usual Euler characteristic of the quotient Δ-set $\Delta\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)/\Sigma_n$ [16, Proposition 2.13] which is collapsible for $n > 2$ [13]. This explains why $\widetilde{\chi}_1(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = \frac{1}{n}c_1(n) = 0$ for $n > 2$. (The situation in the boolean case briefly mentioned in the Introduction is similar: For $n \geq 2$, $\widetilde{\chi}_1(B^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = 0$ and the quotient Δ-set $\Delta B^*(\Sigma_{n-1}\backslash\Sigma_n)/\Sigma_n$, which is a simplex, is contractible.)

The equivariant reduced Euler characteristics $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = c_r(n)/n$ are multiplicative functions of n for all $r \geq 1$ but the ordinary reduced Euler characteristic $\widetilde{\chi}(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)) = (-1)^{n-1}(n-1)!$ is not. The Liouville function $\lambda(n) = (-1)^{\Omega(n)}$, $\Omega(n) = \sum_p \nu_p(n)$, gives the sign of $c_r(n)$.

6. Some multiplicative arithmetic functions

Let $c_r(n) = (a * b_r)(n)$ denote the Dirichlet convolution [8, Section 1] of the multiplicative [18, VI.3.1, Definition 2] arithmetic functions $a(n) = (-1)^{n+1}$ and $b_r(n)$ where

$$b_r(p^e) = (-1)^e p^{\binom{e}{2}} \binom{r}{e}_p \tag{6.1}$$

for any prime power p^e. As usual, the p-binomial coefficient and the ordinary binomial coefficient

$$\binom{r}{e}_p = \frac{(p^r-1)\cdots(p^r-p^{e-1})}{(p^e-1)\cdots(p^e-p^{e-1})} = \prod_{j=0}^{e-1}\frac{p^{r-j}-1}{p^{e-j}-1}, \qquad \binom{e}{2} = \frac{1}{2}e(e-1) \tag{6.2}$$

count the number of e-dimensional subspaces in an r-dimensional $\mathbf{F}_p$-vector space [19, Proposition 1.3.18] and the number of 2-sets in an e-set, respectively. The sequence $b_0 = \varepsilon = 1, 0, 0, \ldots$ is the unit sequence and $b_1 = \mu_{\Sigma_n}$ is the Möbius function. For any prime p and integers $r \geq 1$ and $e, s \geq 0$

$$\sum_{e=0}^{r} b_r(p^e) p^{s(r-e)} = \prod_{e=0}^{r-1}(p^s - p^e) \tag{6.3}$$

by the 'q-binomial theorem' [19, Equation (62), p 162].

Proposition 6.4. *The multiplicative arithmetic sequences b_r are given by $b_1 = \mu$ and the recurrence relations*

$$b_{r+1}(p^d) = p^d b_r(p^d) - p^{d-1} b_r(p^{d-1})$$

valid for all $r \geq 1$ and all prime powers p^d with $d \geq 0$.

Proof. Use Pascal's identities for ordinary and Gaussian binomial coefficients [19, Equation 17b]

$$\binom{d}{2} = \binom{d-1}{2} + (d-1), \qquad \binom{r+1}{d}_p = p^d \binom{r}{d}_p + \binom{r}{d-1}_p$$

and the definition (Eq. (6.1)) of b_r. $\square$

In the following, 1 is the constant sequence with value $1(n) = 1$ on all $n \geq 1$. Its Dirichlet inverse is the Möbius function $1^{-1} = \mu$. Möbius inversion is the assertion that $\mu * (1 * f) = f$ for any multiplicative sequence f.

Corollary 6.5. $(1 * b_{r+1})(n) = nb_r(n)$ *for all $r, n \geq 1$.*

Proof. The telescopic sum

$$(1 * b_{r+1})(p^d) = \sum_{e=0}^{d} b_{r+1}(p^e) = \sum_{e=0}^{d} (p^e b_r(p^e) - p^{e-1} b_r(p^{e-1}))$$

evaluates to $p^d b_r(p^d)$ at any prime power p^d. $\square$

Proposition 6.6. *The multiplicative arithmetic sequences c_r are given by $c_1 = 1, -2, 0, 0, \dots$ and*

$$c_{r+1}(n) = n(b_r(n) - b_r(n/2)) \quad (\textit{where } b_r(n/2) = 0 \textit{ for odd } n)$$

for all $r, n \geq 1$.

Proof. The two multiplicative sequences $c_1 = a * \mu$ and $1, -2, 0, 0, \dots$ are identical since they agree on all prime powers. For *odd* n, $c_{r+1}(n) = (a * b_{r+1})(n) = (1 * b_{r+1})(n) = nb_r(n)$ by Corollary 6.5. For powers of 2,

$$\begin{aligned} c_{r+1}(2^d) &= (a * b_{r+1})(2^d) = b_{r+1}(2^d) - \sum_{e=0}^{d-1} b_{r+1}(2^e) \\ &= 2^d b_r(2^d) - 2^{d-1} b_r(2^{d-1}) - 2^{d-1} b_r(2^{d-1}) = 2^d (b_r(2^d) - b_r(2^{d-1})) \end{aligned}$$

by the recurrence relation of Proposition 6.4. Thus $c_{r+1}(n) = n(b_r(n) - b_r(n/2))$ for even n by multiplicativity. $\square$

If we introduce $\overline{c}_r(n) = \frac{1}{n} c_r(n)$, Proposition 6.6 states that $\overline{c}_{r+1} = \overline{c}_1 * b_r$ for all $r \geq 0$.

The multiplicative sequences c_r can be defined recursively by $c_1 = 1, -2, 0, 0, 0, \dots$, and, for $r \geq 1$,

$$c_{r+1}(2^d) = \begin{cases} 2c_r(2) & d = 1 \\ 2^d c_r(2^d) + \sum_{j=2}^{d} 2^{d+j-2} c_r(2^{d-j}) & d \geq 2 \end{cases}$$

for powers of 2, while $c_{r+1}(p^d) = p^d c_r(p^d) - p^d c_r(p^{d-1})$ at powers of an *odd* prime p. These relations are consequences of Propositions 6.4 and 6.6. In particular, $c_r(2^d) = 0$ for $r < d$ and $c_r(p^d) = 0$ for $r \leq d$ for an odd prime p.

Corollary 6.7. *The Dirichlet series and their Eulerian expansions of the functions b_r and c_r, $r \geq 1$, are*

$$\sum_{n=1}^{\infty} \frac{b_r(n)}{n^s} = \frac{1}{\zeta(s)\zeta(s-1)\cdots\zeta(s-r+1)} = \prod_p p^{-sr} \prod_{e=0}^{r-1} (p^s - p^e)$$

$$\sum_{n=1}^{\infty} \frac{c_r(n)}{n^s} = \frac{1-2^{1-s}}{\zeta(s-1)\cdots\zeta(s-r+1)} \overset{r\geq 1}{=} (1-2^{1-s}) \prod_p p^{-s(r-1)} \prod_{e=1}^{r-1} (p^s - p^e)$$

where $\zeta(s)$ is the Riemann ζ-function.

Proof. Write $\beta_r(s)$ for the Dirichlet series of $b_r(n)$. According to Corollary 6.5,

$$\zeta(s)\beta_{r+1}(s) = \beta_r(s-1)$$

as $nb_r(n)$, with series $\beta_r(s-1)$, is the Dirichlet convolution of 1, with series $\zeta(s)$, and $b_{r+1}(n)$. (The Dirichlet series of a Dirichlet convolution is the product of the Dirichlet series of the factors [8, Section 1].) The expression for the Dirichlet series of $b_r(n)$ now follows by induction starting with the series, $\zeta(s)^{-1}$, for $b_1 = \mu$. The Dirichlet series of the convolution $c_r = a * b_r$ is the product of this series and the series, $\zeta(s)(1-2^{1-s})$, of $a = 1 * c_1$ (Dirichlet η-function). We evaluate the factors of the Eulerian expansion for the Dirichlet series of b_r [18, VI, Section 3, Lemma 4],

$$1 + b_r(p)p^{-s} + \cdots + b_r(p^r)p^{-rs} = p^{-sr} \sum_{e=0}^{r} (-1)^e \binom{r}{e}_p p^{\binom{e}{2}} p^{s(r-e)} = p^{-sr} \prod_{e=0}^{r-1} (p^s - p^e)$$

with the help of the 'q-binomial theorem' [19, Equation (62), p 162]. □

Let $\lambda_r(n) = |\{\mathbf{Z}^r \geq H \mid |\mathbf{Z}^r : H| = n\}|$ denote the number of subgroups of $\mathbf{Z}^r$ of index n. The function λ_r is multiplicative and completely determined by its values [9],

$$\lambda_r(p^e) = \prod_{j=0}^{e-1} \frac{p^{r+j}-1}{p^{e-j}-1} = \prod_{j=0}^{e-1} \frac{p^{r+e-1-j}-1}{p^{e-j}-1} = \binom{e+r-1}{e}_p$$

on prime powers p^e. Also, let π_k, $k \geq 0$, be the kth power function $\pi_k(n) = n^k$ for all $n \geq 1$ and ι_2 the multiplicative function given by $\iota_2(n) = n$ if $n = 1, 2, 4, \ldots$ is a power of 2 and $\iota_2(n) = 0$ otherwise.

Corollary 6.8. *The Dirichlet inverses of the functions b_r and c_r are*

$$b_r^{-1} = \pi_0 * \pi_1 * \cdots * \pi_{r-1} = \lambda_r, \qquad c_r^{-1} = \iota_2 * \pi_1 * \cdots * \pi_{r-1} = a^{-1} * \lambda_r.$$

Thus $b_r * \lambda_r = \varepsilon$ and $c_r * \lambda_r = a$.

Proof. The Dirichlet series for π_k is $\zeta(s-k)$ and for ι_2 it is $(1-2^{1-s})^{-1}$ reflecting that $a * \iota_2 = \pi_0$. Corollary 6.7 implies that the Dirichlet inverses of the multiplicative sequences b_r and c_r are $\pi_0 * \pi_1 * \cdots * \pi_{r-1}$ and $\iota_2 * \pi_1 * \cdots * \pi_{r-1}$, respectively. We now recognize b_r^{-1} as λ_r by [22, p 206] [20, Note p 113], and then $c_r^{-1} = (a * b_r)^{-1} = a^{-1} * b_r^{-1} = a^{-1} * \lambda_r$. □

We finish with a small observation about the asymptotic behavior of the sequence $c_r(n)$ as r varies.

Lemma 6.9. *For $r \geq 1$ and $n \geq 2$, $c_r(n) = 0 \iff r \leq \max\{\nu_2(n) - 1, \nu_3(n), \nu_5(n), \ldots\}$.*

Proof. The claim is that $c_r(n) = 0$ if and only if n is divisible by 2^{r+1} or by p^r for some odd prime p. Since this is true for $r = 1$ we can assume that $r > 1$. It is enough to let n be a prime power by multiplicativity. For any prime p, $b_r(p^d) = 0 \iff d > r$. If p is odd, $c_r(p^d) = p^d b_{r-1}(p^d)$ so that $c_r(p^d) = 0 \iff d \geq r$. For powers of 2, $c_r(2^d) = 2^d(b_{r-1}(2^d) - b_{r-1}(2^{d-1}))$ so that $c_r(2^d) = 0 \iff d \geq r+1$. □

The lemma shows that the zeros in the sequence $r \to c_r(n)$ for fixed n are concentrated at the beginning.

Corollary 6.10. *For any $n \geq 1$*

$$\frac{c_{r+1}(n)}{c_r(n)} \to n \quad \text{for } r \to \infty.$$

Proof. It is enough to verify this when $n = p^d$ is a prime power and it then follows from the recursion formulas. $\square$

7. The q-primary equivariant Euler characteristics

For any prime number q and any natural number $r \geq 1$, let $Z_q^r = (\mathbf{Z}_q)^{r-1} \times \mathbf{Z}$ where $\mathbf{Z}_q$ is the abelian group of q-adic integers. The rth q-primary equivariant reduced Euler characteristic of the G-poset Π, as defined by Tamanoi [21, (1–5)], is the integer

$$\widetilde{\chi}_r^q(\Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(Z_r^q, G)} \widetilde{\chi}(\Pi^X) = \frac{1}{|G|} \sum_{A \leq G} \widetilde{\chi}(\Pi^A) \varphi_{Z_q^r}(A) \tag{7.1}$$

where $\varphi_{Z_q^r}(A) = |\operatorname{Epi}(Z_q^r, A)|$ denotes the number of epimorphisms of Z_q^r onto the subgroup A of G. (Compared to [21, (1–5)] we work here with reduced Euler characteristics and with a degree shift.) Clearly, $\varphi_{Z_q^r}(A)$ is nonzero if and only if A is generated by r commuting elements of G all of which but one have q-power order. We also let $d_{Z_q^r}(A) = \varphi_{Z_q^r}(A)/|\operatorname{Aut}(A)|$ [10, (1.3)]. As $Z_q^1 = \mathbf{Z}$, the first q-primary equivariant Euler characteristic $\widetilde{\chi}_1^q(\Pi, G)$ is independent of q and coincides with the first equivariant Euler characteristic $\widetilde{\chi}_1(\Pi, G)$.

Remark 7.2 (*Algebraic Topological Interpretation*)**.** The paper [12] offers an alternative perspective on (q-primary) equivariant Euler characteristics. Consider the function $\widetilde{\mu}_\Pi$ defined on all subgroups of G with value 0 on all nonabelian subgroups and satisfying the relations

$$\widetilde{\chi}(\Pi^A) = \sum_{B \leq G} \zeta_G(A, B) \widetilde{\mu}_\Pi(B), \qquad \widetilde{\mu}_\Pi(A) = \sum_{B \leq G} \mu_G(A, B) \widetilde{\chi}(\Pi^B)$$

where the sums are over all subgroups B of G. The first relation is the Möbius inverse [19, Proposition 3.7.2] of the second one which defines $\widetilde{\mu}_\Pi$ recursively [12, p 556]. Also, recall that

$$|\operatorname{Hom}(\mathbf{Z}^r, B)| = \sum_{A \leq G} \varphi_{\mathbf{Z}^r}(A) \zeta_G(A, B), \qquad \varphi_{\mathbf{Z}^r}(B) = \sum_{A \leq G} |\operatorname{Hom}(\mathbf{Z}^r, A)| \mu_G(A, B)$$

where the second relation is the Möbius inverse [19, Proposition 3.7.1] of the first one. Combining two of these equalities we find that

$$\begin{aligned} |G| \widetilde{\chi}_r(\Pi, G) &= \sum_{B \leq G} \widetilde{\chi}(\Pi^B) \varphi_{\mathbf{Z}^r}(B) = \sum_{A, B \leq G} \widetilde{\chi}(\Pi^B) |\operatorname{Hom}(\mathbf{Z}^r, A)| \mu_G(A, B) \\ &= \sum_{A \leq G} |\operatorname{Hom}(\mathbf{Z}^r, A)| \widetilde{\mu}_\Pi(A) = \sum_{A \leq G} |A|^r \widetilde{\mu}_\Pi(A). \end{aligned}$$

Replacing $\mathbf{Z}^r$ by Z_q^r leads to the corresponding expression

$$\widetilde{\chi}_r^q(\Pi, G) = \frac{1}{|G|} \sum_{A \leq G} |A| |A_q|^{r-1} \widetilde{\mu}_\Pi(A) \tag{7.3}$$

for the rth q-primary equivariant reduced Euler characteristic where A_q is the Sylow q-subgroup of A.

In this paper we prefer to work with equivariant *reduced* Euler characteristics. However, if we for a moment consider the equivariant *unreduced* Euler characteristics,

$$\chi_r(\Pi, G) = \widetilde{\chi}_r(\Pi, G) + |\operatorname{Hom}(\mathbf{Z}^r, G)|/|G|, \qquad \chi_r^q(\Pi, G) = \widetilde{\chi}_r^q(\Pi, G) + |\operatorname{Hom}(Z_q^r, G)|/|G|$$

then Eq. (7.3) in degree $r+1$ becomes

$$\chi^q_{r+1}(\Pi, G) = \frac{1}{|G|}\sum_{A\le G} |A||A_q|^r \mu_\Pi(A), \quad r \ge 0$$

where the unreduced function μ_Π is defined in the same way as $\widetilde{\mu}_\Pi$ by using unreduced Euler characteristics [12, p 556]. Let $K(r)$ be the rth Morava K-theory at q [17]. Comparing with [12, Theorem B (Part 2), Theorem 4.12] we see that the $(r+1)$th q-primary equivariant unreduced Euler characteristic $\chi^q_{r+1}(\Pi, G)$ is the $K(r)$-Euler characteristic [12, p 555]

$$\chi^q(K(r), \Pi, G) = \dim_{K(r)^*} K(r)^{\mathrm{even}}(|\Pi|_{hG}) - \dim_{K(r)^*} K(r)^{\mathrm{odd}}(|\Pi|_{hG})$$

of the homotopy orbit space $|\Pi|_{hG} = |\Pi| \times_G EG$ for the G-action on the topological realization $|\Pi|$ of Π. (This was first observed in [21, Propositions 2-3,5-1].)

We now specialize from G-posets in general to the Σ_n-poset $\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)$ of non-extreme partitions of the n-set. It is convenient to declare $\widetilde{\chi}^q_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ to mean 1 for all $r \ge 1$ when $n = 1$ even though the q-primary equivariant reduced Euler characteristics actually equal -1 in these case.

Theorem 7.4. *Let $r \ge 1$ and $n \ge 1$. The rth q-primary equivariant reduced Euler characteristic of the Σ_n-poset $\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)$ is*

$$\widetilde{\chi}^q_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = \begin{cases} b_{r-1}(2^d) - b_{r-1}(2^{d-1}) & q = 2, n = q^d, 0 \le d \le r \\ b_{r-1}(q^d) & q > 2, n = q^d, 0 \le d < r \\ -b_{r-1}(q^d) & q > 2, n = 2q^d, 0 \le d < r \\ 0 & \text{otherwise}. \end{cases}$$

The rth q-primary equivariant reduced Euler characteristic and the rth equivariant reduced Euler characteristic coincide,

$$\widetilde{\chi}^q_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = \widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$$

when $n = q^d$ is a power of q.

Proof. Let $c^q_r = a * b^q_r$ where b^q_r is the multiplicative function with value

$$b^q_r(p^d) = \mu(1, C^d_p) d_{Z^r_q}(C^d_p) = \begin{cases} b_r(p^d) & p = q \\ \mu(p^d) & p \ne q \end{cases}$$

on the prime power p^d. The proof of Theorem 1.3 shows that

$$\widetilde{\chi}^q_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = \frac{1}{n}\sum_{|A||n} (-1)^{n/|A|+1}\mu(1, A) d_{Z^r_q}(A) = \frac{1}{n}(a * b^q_r)(n) = \frac{1}{n} c^q_r(n).$$

Alternatively, $b^q_r(n) = b_r(n_q)\mu(n/n_q)$ where $n_q = q^{\nu_q(n)}$ is the q-part of n. For any prime $p \ne q$ and exponent $d \ge 1$,

$$(1 * b^q_r)(p^d) = \sum_{e=0}^{d} \mu(p^e) = \mu(1) + \mu(p) = 0$$

so the multiplicative convolution $1 * b^q_r$ is nonzero only on powers of q where it agrees with $1 * b_r$. In fact, $1 * b^q_1 = 1 * b_1 = 1 * \mu = \varepsilon$ is nonzero only on 1, and, for $r \ge 2$, $(1 * b^q_r)(q^d) = (1 * b_r)(q^d) = q^d b_{r-1}(q^d)$ (Corollary 6.5) is nonzero only on the r powers q^d for $0 \le d < r$ (Eq. (6.1)). Thus $c^q_r = a * b^q_r = c_1 * 1 * b^q_r$ is nonzero only on natural numbers of the form q^d and $2q^d$ for $0 \le d < r$. At powers of q, $(c_1 * 1 * b_r)(q^d) = (c_1 * 1 * b^q_r)(q^d)$ so that the rth q-primary and the standard equivariant reduced Euler characteristic coincide. In fact, $(c_1 * 1 * b^2_r)(2^d) = (1*b^2_r)(2^d) - 2(1*b^2_r)(2^{d-1}) + 2^d(b_{r-1}(2^d) - b_{r-1}(2^{d-1}))$ and $(c_1*1*b_r)(q^d) = (1*b^q_r)(q^d) = q^d b_{r-1}(q^d)$

for $q > 2$. At $2q^d$, $q > 2$, $(c_1 * 1 * b_r)(q^d) = (1 * b_r)(q^d)$ and $(c_1 * 1 * b_r)(2q^d) = -2(1 * b_r)(q^d)$ which means that the sequence $\frac{1}{n}c_r^q(n) = \frac{1}{n}(a * b_r^q)(n)$ takes opposite values at $n = q^d$ and $n = 2q^d$. $\square$

With fixed r, $c_r^q(n)/n = \widetilde{\chi}_r^2(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ is nonzero only at the $r + 1$ first powers of 2, $n = 2^d$ for $0 \le d \le r$, when $q = 2$ (Fig. 3), and, for an odd prime q, only at the r first powers of q, $n = q^d$ for $0 \le d < r$, and at the double of these numbers (Fig. 4). It is the consequence of a general rule that all row sums of the tables of Figs. 3–4 equal 0:

$$\sum_{\substack{0\le d\le r\\ n=2^d}} \widetilde{\chi}_r^2(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = 0, \qquad \sum_{\substack{0\le d<r\\ n=q^d}} \widetilde{\chi}_r^q(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = 0 \quad (q > 2).$$

For $q = 2$, the sum is telescopic, and for $q > 2$, one uses the q-binomial theorem [19, Equation (62), p 162].

The q-primary equivariant reduced Euler characteristics happen to determine the equivariant reduced Euler characteristics in the sense that

$$\begin{aligned}\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) &= \prod_q \widetilde{\chi}_r^q(\Pi^*(\Sigma_{n_q-1}\backslash\Sigma_{n_q}), \Sigma_{n_q}) \\ &= (b_{r-1}(n_2) - b_{r-1}(n_2/2)) \prod_{q>2} b_{r-1}(n_q)\end{aligned} \tag{7.5}$$

when n has prime factorization $n = \prod_q n_q$ with $n_q = q^{\nu_q(n)}$ a power of the prime q. This rth equivariant Euler characteristicis nonzero if and only if $\nu_2(n) \le r$ and $\nu_q(n) < r$ for all odd primes q (Lemma 6.9).

Proof of Corollary 1.4. Item (1) is the identity $c_r * \lambda_r = a$ from Corollary 6.8, item (2) is a reformulation of (1), and (3) is part of Theorem 7.4. $\square$

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