p-compact groups

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Notation

p prime number
$\mathbf{F}_p$ field with p elements
$\mathbf{Z}_p$ ring of p-adic integers
$\mathbf{Q}_p$ field of p-adic numbers
B_p the $H_*\mathbf{F}_p$-completion of the space B

Finite loop spaces

Definition 1 *A finite loop space is a space BX whose loop space ΩBX is homotopy equivalent to a finite CW-complex.*

Terminology: $\mathbf{R}P^\infty$ is a finite group.

Example 2 *BG where G is a (compact) Lie group.*
Zabrodsky mix of rationally identical Lie groups (Hilton criminal)
$2^\omega \subset G(B\mathrm{SU}(2)) \xrightarrow{\Omega} G(\mathrm{SU}(2)) = \{\mathrm{SU}(2)\}$
(Rector 1971)

Conjecture 3
1. *Any finite loop space with a maximal torus $BT \to BX$ is a compact Lie group*
   TRUE away from 2
2. *Any finite loop space is rationally a compact Lie group*
   FALSE (Andersen, Bauer, Grodal, Pedersen 2004)

Application 4 *(Yau 2002) There are uncountably many λ-ring structures on the power series ring $\mathbf{Z}[[x_1, \ldots, x_n]]$.*

p-compact groups

Definition 5 *A p-compact group is a p-complete space BX such that $H^*(\Omega BX; \mathbf{F}_p)$ is finite (Dwyer–Wilkerson 1994).*

Example 6

- *Rector's uncountable many examples all complete to the same p-compact group $(B\mathrm{SU}(2))_p$*
- *$(BG)_p$ where G is a Lie group and $\pi_0(G)$ a finite p-group*
- *$BT_r = (B\mathrm{U}(1)^r)_p = K(\mathbf{Z}_p^r, 2)$ – p-compact torus of rank r.*
- *$BT_r = (B\check{T}_r)_p$ where $\check{T}_r = (\mathbf{Z}/p^\infty)^r$ is the discrete approximation to T_r.*
- *$BT \to BP \to B\pi$ – p-compact toral group (with discrete approximation $\check{P}$)*
- *Any connected p-complete space with f.g. polynomial $H^*\mathbf{F}_p$-cohomology is a (polynomial, often exotic) p-compact group*

A short history of polynomial p-compact groups

Theorem 7 *(Sullivan) If $n|(p-1)$, then $BS^{2n-1} = B(\mathbf{Z}/p^\infty \rtimes C_n)_p$ is a p-compact group and*

$$H^*(BS^{2n-1};\mathbf{Z}_p) = \mathbf{Z}_p[u]^{C_n} = \mathbf{Z}_p[x_{2n}]$$

and $H^*(\Omega BS^{2n-1};\mathbf{F}_p) = E(y_{2n-1})$ is an exterior algebra over $\mathbf{F}_p$.

Theorem 8 *(Clark–Ewing 1974) Let $W \subset \mathrm{GL}(r,\mathbf{Z}_p) = \mathrm{Aut}(\check{T}_r)$, be a p-adic reflection group of order prime to p. Then $BX(W) = B(\check{T} \rtimes W)_p$ is a p-compact group and*

$$H^*(BX(W);\mathbf{F}_p) = H^*(B\check{T}_r;\mathbf{F}_p)^W$$

is a polynomial and $H^*(\Omega BX(W);\mathbf{F}_p)$ an exterior algebra over $\mathbf{F}_p$.

The next example uses a generalized Clark–Ewing construction and a spectral sequence

$$E_2^{pq} = H^p(K(\mathbf{I}); H^qM) \Longrightarrow \lim{}^{p+q}M$$

for the higher limits of the functor M on the EI-category $\mathbf{I}$.

Example 9 *(Aguadé 1989) The simple reflection group no* $W_{12} \subset \mathrm{GL}(2, \mathbf{Z}_3)$ *maps isomorphically to* $\mathrm{GL}(2, \mathbf{F}_3)$*. Define the* 3*-compact group* $BDI(2)$ *as the homotopy colimit of*

$$\mathbf{Z}/2 \circlearrowright B\mathrm{SU}(3) \longleftarrow BT \circlearrowleft W_{12}$$

Then

$$H^*(BDI(2); \mathbf{F}_3) \cong H^*(BV_2; \mathbf{F}_3)^{\mathrm{GL}(2,\mathbf{F}_3)}$$
$$\cong \mathbf{F}_3[x_{12}, x_{16}], \qquad P^1 x_{12} = x_{16}$$

Example 10 *(Dwyer–Wilkerson 1993) There exists a* 2*-compact group* $BDI(4)$ *such that*

$$H^*(BDI(4); \mathbf{F}_2) = H^*(BV_4; \mathbf{F}_2)^{\mathrm{GL}(4,\mathbf{F}_2)}$$
$$= \mathbf{F}_2[c_8, c_{12}, c_{14}, c_{15}], \quad Sq^4 c_8 = c_{12}, Sq^1 c_{14} = c_{15}$$

$BDI(4)$ *is the homotopy colimit of a diagram of the form*

$$\mathrm{Spin}(7) \supset \mathrm{SU}(2)^3 / \langle(-E, -E, -E)\rangle$$
$$\supset T \rtimes \langle -E \rangle \supset V_4$$

Example 11 *(Quillen's generalized $B\mathsf{U}(n)$, Oliver-Notbohm 1993) Suppose $r|m|p-1$. Let*

$$G(m,r,n) = A(m,r,n) \rtimes \Sigma_n \subset \mathsf{GL}(n, \mathbf{Z}_p)$$

where Σ_n is permutations and $A(m,r,n)$ is diagonal matrices with entries in $C_m \subset C_{p-1} \subset \mathbf{Z}_p^\times$ and determinant in $C_{m/r} \subset C_m$.

Define

$$BC_X \colon \mathbf{A}(G(m,r,n), \mathbf{F}_p^n) \to [\mathbf{pcg}]$$

as the functor that is $BC_{\mathsf{U}(n)}$ plus products of unstable Adams operations ψ^λ, $\lambda \in C_m$.

Then $BXG(m,r,n) = \operatorname{hocolim} BC_X$ is a polynomial center-free p-compact group with

$$\begin{aligned} H^*(BXG(m,r,n); \mathbf{Z}_p) &= \mathbf{Z}_p[x_1, \dots, x_n]^{G(m,r,n)} \\ &= \mathbf{Z}_p[y_1, \dots, y_{n-1}, e] \end{aligned}$$

where y_i is the ith symmetric polynomium in x_i^m and $e = (x_1 \cdots x_n)^{m/r}$.

The *cohomological dimension* of BX is

$$\mathrm{cd}(BX) = \max\{d \geq 0 \mid H^d(\Omega BX; \mathbf{Q}_p) \neq 0\}$$

Example 12 $\mathrm{cd}(BDI(4)) = 45$ *as*

$$H^*(BDI(4); \mathbf{Q}_2) = \mathbf{Q}_2[x_8, x_{12}, x_{28}]$$

$\mathrm{cd}(BG)$ *is* $\dim(G)$, G *a compact Lie group.*

Open Problem 13

- *Is it possible to characterize the class of cohomology algebras*

  $$H^*(BX; \mathbf{F}_p)$$

  for p-compact groups? Can we tell from $H^*(B; \mathbf{F}_p)$ if B is a p-compact group?

- *Do all p-compact groups have discrete approximations?*

- *What is the analogue of the Lie* algebra*? Benson proposes a candidate (unfortunately not containing the Lie algebra of* Spin(7)*!) for the Lie algebra of $BDI(4)$.*

Morphisms and homogeneous spaces

A *morphism* is a pointed map $Bf\colon BX \to BY$. The fibre of Bf is denoted Y/fX or just Y/X.

$$Y/X \longrightarrow BX \xrightarrow{Bf} BY$$

- Bf is a monomorphism if $H^*(Y/X; \mathbf{F}_p)$ is finite iff $H^*(BY; \mathbf{F}_p)$ is a f.g. module over $H^*(BX; \mathbf{F}_p)$.
- Bf is an epimorphism if Y/X is a p-compact group BK. Then

  $$BK \longrightarrow BX \longrightarrow BY$$

  is a short exact sequence of p-compact groups.
- Bf is an isomorphism if the fibre is contractible

Example 14 *$\exists$ monomorphism$B$Spin(7) $\to$ $BDI(4)$ and $\chi(DI(4)/\text{Spin}(7)) = 24$. If $G \to H$ is a monomorphism of Lie groups, then $(BG)_p \to (BH)_p$ is a monomorphism of p-compact groups and $G/H = (G/H)_p$. Similarly for epimorphisms.*

Example 15 $BX\langle 1\rangle \to BX$ *is a* p*-compact group monomorphism. There is a finite covering map*

$$X/X\langle 1\rangle = \pi_0(X) \to BX\langle 1\rangle \to BX$$

so $BX_0 = BX\langle 1\rangle$ *is the* identity component *of* BX*.*
$BX\langle 2\rangle \to BX$ *is a* p*-compact group morphism, so* $BSX = BX\langle 2\rangle$ *is the* universal covering p*-compact group.*

Theorem 16 *Any nontrivial* p*-compact group admits a monomorphism* $B\mathbf{Z}/p \to BX$*. Any* p*-compact group with a nontrivial identity component admits a monomorphism* $B\mathrm{U}(1) \to BX$*.*

Open Problem 17

- *Do all* p*-compact groups admit faithful complex representations?*

- *Investigate homogeneous spaces* Y/X *of* p*-compact groups.*

Ziemanski constructs a faithful complex representation of $DI(4)$ (at $p = 2$) of dimension 70368744177664. This faithful representation of $DI(4)$ is constructed by finding compatible faithful representations of the Lie groups in the diagram.

Theorem 18 *Let* $Bf\colon BX \to BY$ *be a* p*-compact group homomorphism that vanishes on all elements of finite order. Then* Bf *is trivial.*

Centralizers

The centralizer of the morphism Bf is

$$BC_Y(fX) = BC_Y(X) = \mathrm{map}(BX, BY)_{Bf}$$

and Bf is *central* if

$$BC_Y(X) \to BY$$

is a homotopy equivalence.
The p-compact group BA is abelian if the identity map is central: $\mathrm{map}(BA, BA)_{B1} \simeq BA$.

Proposition 19 *If $BX \to BY$ is a central monomorphism then BX is abelian.*

Theorem 20 *If BP is a p-compact toral group then $BC_Y(P)$ is a p-compact group and $BC_Y(P) \to BY$ a monomorphism.*

Theorem 21 (Sullivan conjecture) *The trivial morphism of $BP \to BY$ from a p-compact toral group is central.*

Proof. $\Omega BC_Y(P) = \Omega\,\mathrm{map}(BP, BY)_{B0} = \mathrm{map}(BP, \Omega BY) = \Omega BY$. $\square$

Example 22 *Let G be a Lie group, P be a p-toral Lie group, $f\colon P \to G$ a Lie monomorphism, and $C_G(P)$ the Lie centralizer. Then*

$$\begin{aligned} C_G(P) \times P \to G \rightsquigarrow BC_G(P) \times BP \to BG \\ \rightsquigarrow BC_G(P) \to \mathrm{map}(BP, BG)_{Bf} \end{aligned}$$

gives an isomorphism of p-compact groups.

Proposition 23 *Any morphism $BA \to BX$ from an abelian p-compact group BA factors through its centralizer:*

$$\begin{array}{ccc} & & BC_A(X) \\ & \nearrow & \downarrow \\ BA & \longrightarrow & BX \end{array}$$

Proof.

$$\begin{array}{ccc} \mathrm{map}(BA, BA)_{B1} & \longrightarrow & \mathrm{map}(BA, BX)_{Bf} \\ \simeq \downarrow & & \downarrow \\ BA & \longrightarrow & BX \end{array}$$

$\square$

This factorization gives the fibration sequence

$$C_X(A)/A \to BA \to BC_X(A)$$

The maximal torus

A maximal torus is a monomorphism $BT \to BX$ such that $C_X(T)/T$ homotopically discrete.

The Weyl group $W(X) = \pi_0 \mathcal{W}(X)$ is the component group of Weyl space of self-maps of BT over BX:

$$\begin{array}{ccc} BT & \xrightarrow{\mathcal{W}(X)} & BT \\ & \searrow \quad \swarrow & \\ & BX & \end{array}$$

Theorem 24 1. *Any p-compact group BX has an essentially unique maximal torus*

2. *If BX is connected then*

$$W(X) \hookrightarrow \mathrm{Aut}(\pi_2(BT)) = \mathrm{GL}(\mathbf{Z}_p, r)$$

is a p-adic reflection group

3. $H^*(BX; \mathbf{Z}_p) \otimes \mathbf{Q} = (H^*(BT; \mathbf{Z}_p) \otimes \mathbf{Q})^W$

4. $\chi(X/T) = |W(X)|$

BX is *simple* if its Weyl group representation $W(X) \to \mathrm{GL}(\pi_2(BT) \otimes \mathbf{Q})$ is simple.

Example 25 • $W(BS^{2n-1}) = C_n < \mathrm{GL}(\mathbf{Z}_p, 1)$ *consists of the* n*th roots of unity.*

- $W(BX(W)) = W < \mathrm{GL}(r, \mathbf{Z}_p)$ *(Clark–Ewing)*

- $W(BDI(4)) < \mathrm{GL}(\mathbf{Z}_2, 3)$ *is the simple reflection group no* 24 *of order* 336*, isomorphic to* $\mathbf{Z}/2 \times \mathrm{GL}(3, \mathbf{F}_2)$*.*

- $W(BG) = W(G) \subset GL(r, \mathbf{Z}) \subset \mathrm{GL}(r, \mathbf{Z}_p)$*,* G *Lie group.*

Weyl groups are *never* abstract groups, they are *always* subgroups of $\mathrm{GL}(r, \mathbf{Z}_p)$!

If $H^*(BX; \mathbf{Q}_p) = \mathbf{Q}_p[x_i]$ then W has order $|W| = \prod \frac{1}{2}|x_i|$ and W contains $\sum(\frac{1}{2}|x_i| - 1)$ reflections.

The center of a p-compact group

Theorem 26 *For any p-compact group BX there exists a central monomorphism $BZ(X) \to BX$ where*

- *$BZ(X)$ is abelian, and*
- *any other central monomorphism to BX factors uniquely through $BZ(X) \to BX$.*

In fact, $BZ(X) = BC_X(X) = \mathrm{map}(BX, BX)_{B1}$.

If $BX = BG_p$, G Lie, then $BZ(X) = BZ(G)_p$.

Theorem 27 *For any central monomorphism $BA \to BX$ there is a short exact sequence*

$$BA \longrightarrow BX \longrightarrow B(X/A)$$

of p-compact groups.

$BPX = B(X/Z(X))$, the *adjoint form* of BX, has no center when BX is connected.

Structure of p-compact groups

Proposition 28 *Any abelian p-compact group is the product of a finite abelian p-group and a p-compact torus: $BA = B\pi \times BT$.*

Theorem 29 *For any connected p-compact group BX there is a short exact sequence of p-compact groups*

$$BK \longrightarrow BSX \times BZ(X)_0 \longrightarrow BX$$

where K is finite abelian p-group and $BK \to BSX$ is central.

The corresponding theorem for Lie groups:

$$Z/p \to \mathrm{SU}(p) \times \mathrm{U}(1) \to \mathrm{U}(p)$$
$$(A, z) \to A(zE)$$

Theorem 30 *(Semi-simplicity) BPX and BSX, for any connected p-compact group, are products of simple p-compact groups.*

Any p-compact group is the quotient of $\prod BY_i \times BS$, BY_i simple, BS p-compact torus, by a central finite abelian group.

Decomposing BPX

$\mathbf{A}(X)$ is the category with

objects Monomorphism $BV \xrightarrow{B\nu} BX$, V elementary abelian p-group

morphisms Monomorphisms $\phi\colon V_1 \to V_2$ such that

$$\begin{array}{ccc} BV_1 & \xrightarrow{B\phi} & BV_2 \\ & {\scriptstyle B\nu_1}\searrow \quad \swarrow{\scriptstyle B\nu_2} & \\ & BX & \end{array}$$

commutes up to homotopy.

The functors

$$BC_X\colon \mathbf{A}(X)^{\mathrm{op}} \to \mathsf{Top}$$
$$\pi_i BZC_X\colon \mathbf{A}(X) \to \mathsf{AbGrp}$$

are given by

$$BC_X(B\nu) = \mathrm{map}(BV, BX)_{B\nu}$$
$$\pi_i BZC_X(B\nu) = \pi_i(BZC_X(B\nu))$$

Example 31 $\mathbf{A}_p(B\mathrm{U}(n)) = \mathbf{A}(\Sigma_n, \mathbf{F}_p^n)$ *and* $BC_{\mathrm{U}(n)}(\sum_{\rho\in V^\vee} n_\rho \rho) = \prod B\mathrm{U}(n_\rho)$.

Theorem 32 hocolim $BC_X \xrightarrow{H_*\mathbf{F}_p} BX$

Here, $\mathrm{cd}_{\mathbf{F}_p} BC_X(V) < \mathrm{cd}_{\mathbf{F}_p} BX$ when BX has no center.

The *toral subcategory*

$$\mathbf{A}(X)^{\leq t} = \mathbf{A}(W, t)$$

is generated by all toral objects.

One can often prune the index category. For polynomial p-compact groups, it is enough to take t^P for P a p-subgroup of W.

Example 33 *Lannes theory applied to*

$$\begin{array}{ccc} & & H^*(BT;\mathbf{F}_p) \\ & \swarrow & \uparrow \\ H^*(BV;\mathbf{F}_p) & \longleftarrow & H^*(BX;\mathbf{F}_p) \end{array}$$

shows that all objects are toral when $H^*(BX)$ embeds in $H^*(BT)$ (eg for polynomial p-compact groups for p odd).

Example 34 *If $|W|$ is prime to p, then the polynomial $BX(W)$ is the homotopy colimit of*

$$\overset{W}{\circlearrowright}\atop BT$$

Example 35 *The polynomial 3-compact group BG_2, with Weyl group $W(G_2) \cong \Sigma_3 \times \mathbf{Z}/2$, is the homotopy colimit of the diagram*

$$\overset{\mathbf{Z}/2}{\circlearrowright} \qquad\qquad\qquad \overset{W(G_2)}{\circlearrowright}$$
$$B\mathsf{SU}(3) \longleftarrow BT$$

where $Z(W(G_2)) \cong \mathbf{Z}/2$ acts on $B\mathsf{SU}(3)$ via the unstable Adams operations $\psi^{\pm 1}$

Classification of compact connected Lie groups

Theorem 36 *(Curtis–Wiederhold–Williams, Bourbaki) Let G_1 and G_2 be two compact connected Lie groups. Then*

$$G_1 \text{ and } G_2 \text{ are isomorphic} \iff N(G_1) \text{ and } N(G_2) \text{ are isomorphic}$$

where $N(G_1) \to G_1$, $N(G_2) \to G_2$ are the maximal torus normalizers.

Theorem 37 *(Hämmerli) For any compact connected Lie group G*

$$\mathrm{Out}(G) \cong \frac{\mathrm{Out}(N(G))}{H^1(W(G); T(G))}$$

The Weyl group itself is *not* enough:

$$\mathrm{U}(1) \to N(\mathrm{SU}(2)) \overset{\not\circlearrowleft}{\to} C_2 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$$\mathrm{SO}(2) \to N(\mathrm{SO}(3)) \overset{\circlearrowleft}{\to} C_2 = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

More generally: $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+1)$

Classification of p-compact groups for p odd

The *normalizer of the maximal torus*

$$BN(X) \to BX$$

is the Borel construction for the action of the Weyl space $\mathcal{W}(X)$ on the maximal torus $BT(X)$.

There is a fibration sequence

$$BT(X) \to BN(X) \to BW(X)$$

so that $BN(X)$ is an *extended p-toral group*.

BN is characterized by the data:

1. The Weyl group action $W \to \mathrm{Aut}(BT)$
2. The extension class in $H^3(BW; \pi_2(BT))$

There is a *discrete approximation*

$$\breve{T}(X) \to \breve{N}(X) \to W(X)$$

If $BX = BG$, G Lie, the p-compact group and the Lie (discrete) maximal torus normalizer are (essentially) identical.

Theorem 38 *If p is odd, $\check{N} = \check{T} \rtimes W$ is a semi-direct product for any connected p-compact group. If $p = 2$, $H^3(BW; \pi_2(BT))$ is an elementary abelian 2-group and the extension class may be nonzero.*

Curtis–Wiederhold–Williams tell us for which simple compact Lie groups $N(G)$ splits.

Theorem 39 *The maximal torus and the maximal torus normalizer are characterized by*

1. *$BT \to BX$ is a monomorphism from a p-compact torus and the Euler characteristic $\chi(X/T) \neq 0$*

2. *$BN \to BX$ is a monomorphism from an extended p-toral group and the Euler characteristic $\chi(X/N) = 1$*

Good things about BN

Theorem 40 *Let $Bf\colon BX \to BX$ be an automorphism of BX.*

1. *There exists a lift*

$$\begin{array}{ccc} BN & \xrightarrow{BN(f)} & BN \\ \downarrow & & \downarrow \\ BX & \xrightarrow{Bf} & BX \end{array}$$

which is unique up to homotopy.

Theorem 41 *Let $BV \xrightarrow{B\nu} BX$ be a monomorphism from an elementary abelian p-group BV to BX. There exists a* preferred lift

$$\begin{array}{ccc} & \overset{B\mu}{\nearrow} & BN \\ & & \downarrow \\ BV & \xrightarrow{B\nu} & BX \end{array}$$

such that $BC_N(\mu) \to BC_X(\nu)$ is a maximal torus normalizer. The preferred lift is unique if $B\nu$ is toral.

The maximal torus normalizer informs about the group of components and the center.

Proposition 42 *There is a short exact sequence*

$$W(X_0) \to W(X) \to \pi_0(X)$$

When $p > 2$, $W(X_0)$ is the subgroup generated by

$\{w \in W(X) \mid \pi_2(w) \otimes \mathbf{Q}$
is a reflection in $\pi_2(BT) \otimes \mathbf{Q}$ of order $|w|\}$

Proposition 43 *There is a monomorphism (isomorphism when $p > 2$) $BZ(X) \to BZN(X)$ of centers. The center of a connected BX can be computed from BN.*

Main theorems

Theorem 44 *Let BX_1 and BX_2 be two p-compact groups and BN an extended p-compact torus. Any diagram*

$$\begin{array}{ccc} & BN & \\ {}^{Bj_1}\swarrow & & \searrow^{Bj_2} \\ BX_1 & \overset{\simeq}{\dashrightarrow} & BX_2 \end{array}$$

where the slanted arrows are maximal torus normalizers, can be completed by an isomorphism $BX_1 \to BX_2$ under BN.

Theorem 45 *There is an isomorphism*

$$\mathrm{Out}(BX) \xrightarrow{Bf \to BN(f)} \mathrm{Out}(BN)$$

If BX is connected then

$$\mathrm{Out}(BN) \cong N_{\mathrm{GL}(r,\mathbf{Z}_p)}(W)/W$$

is the Weyl group of the Weyl group.

For *connected* p-compact groups there are bijections

$$\left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{connected } p\text{-compact groups}\end{matrix}\right\} \xrightarrow{(W,\check{T})} \left\{\begin{matrix}\text{Similarity classes of}\\ \mathbf{Z}_p\text{-reflection groups}\end{matrix}\right\}$$

Steenrod's problem:

$$\left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{polynomial } p\text{-compact groups}\end{matrix}\right\} \xrightarrow{(W,\check{T})} \left\{\begin{matrix}\text{Similarity classes of polynomial}\\ \mathbf{Z}_p\text{-reflection groups with } H_1 = 0\end{matrix}\right\}$$

Many p-compact groups are *cohomologically unique*.

Theorem 46 *Let BX be a connected p-compact group. If the Weyl group $W(X) \subset$ GL$(r, \mathbf{Z}_p)$ is determined by its mod p reduction in* GL$(r, \mathbf{F}_p)$, *then BX is a cohomologically unique p-compact group.*

All simple p-compact groups, except possibly the quotients $B(\mathrm{SU}(p^n)/p^r)$, are cohomologically unique p-compact groups.

Computing automorphism groups

There is an exact sequence

$$1 \to C_{\mathrm{GL}(r,\mathbf{Z}_p)}(W)/Z(W) \to N_{\mathrm{GL}(r,\mathbf{Z}_p)}(W)/W \to \mathrm{Out}_{\mathrm{tr}}(W)$$

where $\mathrm{Out}_{\mathrm{tr}}(W)$ consists of trace preserving automorphisms of W. The automorphisms

$$\mathbf{Z}_p^\times \subset C_{\mathrm{GL}(r,\mathbf{Z}_p)}(W)$$

are the unstable Adams operations.

Example 47

$$\mathrm{Out}(B\mathrm{SU}(n)_p) = \begin{cases} \mathbf{Z}_p^\times/\{\pm 1\} & n = 2 \\ \mathbf{Z}_p^\times & n > 2 \end{cases}$$

$$\mathrm{Out}(\underbrace{B\mathrm{SU}(3)_p \times \cdots \times B\mathrm{SU}(3)_p}_{n}) = \mathbf{Z}_p^\times \wr \Sigma_n$$

In general the automorphism group of a product consists of Adams operations on the factors together with permutations of identical factors.

Example 48 *The automorphism group of* BS^{2n-1} *is*

$$\begin{aligned}\mathrm{Out}(BS^{2n-1}) &= N_{\mathrm{GL}(1,\mathbf{Z}_p)}(C_n)/C_n \\ &= \mathbf{Z}_p^\times/C_n\end{aligned}$$

The automorphism group of any Aguadé group is $\mathbf{Z}_p^\times/Z(W)$*.*

Example 49 *At* $p = 3$*,*

$$\begin{aligned}\mathrm{Out}(B\mathrm{F}_4) &= N_{\mathrm{GL}(4,\mathbf{Z}_3)}(W(\mathrm{F}_4)) \\ &= \mathbf{Z}_3^\times/\{\pm 1\} \times \{\alpha\}\end{aligned}$$

where the exceptional isogeny α *has order* 2*.*

Theorem 50 *Let* BX *be a* p*-compact group with identity component* BX_0 *and component group* π*. There is an exact sequence*

$$H^1(\pi; \check{Z}(X_0)) \to \mathrm{Aut}(BX) \to \mathrm{Aut}(B\pi, BX_0)_{BX}$$

where the group to the right is the stabilizer subgroup for the action of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(BX_0)$ *on* $H^2(\pi; \check{Z}(X_0))$*.*

Sketch of proof

BX is totally N-determined if Theorems 44 and 45 hold for BX.

Theorem 51

BX_0 is totally N-determined $\Longrightarrow BX$ is totally N-determined

BPX is totally N-determined $\Longrightarrow BX$ is totally N-determined

BP_1 and BP_2 are totally N-determined $\Longrightarrow BP_1 \times BP_2$ is totally N-determined

Theorem 52 *Let BX be a connected p-compact group with no center. If*

1. *All centralizers $BC_X(V)$, $|V| \leq p^2$, are totally N-determined*
2. $\lim^1 \pi_1 BZC_X = 0 = \lim^2 \pi_2 BZC_X$ *and* $\lim^2 \pi_1 BZC_X = 0 = \lim^3 \pi_2 BZC_X$
3. *The problems with non-uniqueness of preferred lifts can be solved*

Then BX is totally N-determined.

The main problem is the computation of the higher limits.

Fact: Any simple p-compact group is either polynomial or a Lie group.

Proposition 53 *The higher limits are* 0 *on* $\mathbf{A}(X)^{\leq t}$*.*

We only need to compute the higher limits for Lie groups where we may set the functors $= 0$ on all toral objects. Find the *non-toral* elementary abelian p-groups in the simple compact center-free Lie groups PG (not so easy for the E-family!). Use Oliver's cochain complex with

$$\prod_{|V|=p^{r+1}} \mathrm{Hom}_{\mathbf{A}(G)(V)}(\mathrm{St}(V), \pi_i(BZC_G(V)))$$

in degree r. In fact, these Hom-groups are trivial so there is no need to compute the differentials.

Conclusion

Let p be an odd prime.

- A connected p-compact group is completely determined by its Weyl group

- All p-compact groups are known, there do not exist fake copies of BG

- Automorphism groups of p-compact groups are known

- Morphism sets $[BG, BH]$ are not fully understood (admissible homomorphisms may be helpful). Homotopy representation theory is not fully understood.

- The maximal torus conjecture is true at odd primes

- Exactly for which spaces do these methods apply?

Computation of BX^{hG}

A G-action on BX is a fibration

$$BX \longrightarrow BX_{hG} \underset{\longrightarrow}{\overset{BX^{hG}}{\longleftarrow}} BG$$

with fibre BX. The homotopy orbit space BX_{hG} is the total space and the homotopy fixed point space BX^{hG} is the space of sections.

Theorem 54 *Let BX be a connected p-compact group and $G \subset \mathrm{Out}(BX)$ a finite group of automorphisms order prime to p.*

1. *BX^{hG} is a connected p-compact group with*

   $$H^*(BX^{hG}; \mathbf{Q}_p) = S[QH^*(BX; \mathbf{Q}_p)_G]$$

2. *$BX^{hG} \to BX$ is a monomorphism and*

   $$X \simeq X^{hG} \times X/X^{hG}$$

   In particular, X/X^{hG} is an H-space.

3. *BX^{hG} is polynomial if BX is.*

Example 55 *The action of* $\langle\lambda\rangle = C_m \to \mathrm{Out}(BS^{2n-1}) = \mathbf{Z}_p^\times/C_n$ *is trivial if* $m|n$. *Otherwise,* $H^{2n}(\psi^\lambda) = \lambda^n$ *is nontrivial on* $Q\overline{H}^*(BS^{2n-1};\mathbf{Q}_p) = H^{2n}(BS^{2n-1};\mathbf{Q}_p) = \mathbf{Q}_p$. *We conclude that*

$$(BS^{2n-1})^{hC_m} = \begin{cases} BS^{2n-1} & m \mid n \\ * & m \nmid n \end{cases}$$

Example 56 *The Aguadé* p*-compact group*

$$BDI(2) = B\mathsf{F_4}^{h\langle\alpha\rangle} \quad (p=3)$$

is the fixed point 3*-compact group for the action of the exceptional isogeny, and*

$$\mathsf{F_4} \simeq DI(2) \times \mathsf{F_4}/DI(2)$$

so that $\mathsf{F_4}/DI(2)$ *is a* 3*-complete* H*-space.*

Example 57 *The Aguadé* p*-compact group*

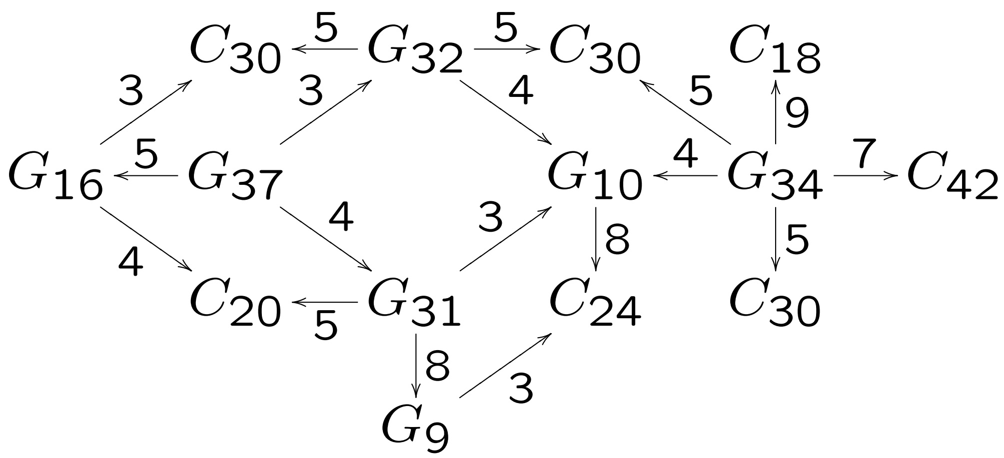
$$BX(W_{31}) = BE_8^{hC_4} \quad (p=5)$$

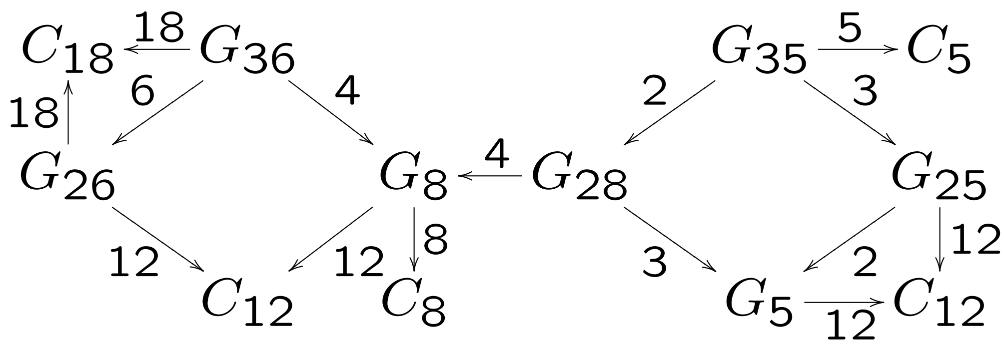
is the fixed point 5*-compact group for the action of* $C_4 = \left\langle\psi^i\right\rangle$ *on* BE_8*, and*

$$E_8 \simeq X(W_{31}) \times E_8/X(W_{31})$$

where $E_8/X(W_{31})$ *is a* 5*-complete* H*-space.*

It is possible to determine the fixed point groups for all simple p-compact groups. Here are the fixed point groups for actions through unstable Adams operations:





(1)

$G_{37} = W(E_8)$, $G_{36} = W(E_7)$, $G_{35} = W(E_6)$

The genus set of BG

Theorem 58 *The genus set $G(BG)$ is uncountably large for any nonabelian compact connected Lie group G.*

Theorem 59 *The set* SNT(BG) *is uncountably large for any nonabelian compact connected Lie group G – except for G in*

$$\{\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{PSU}(2), \mathrm{PSU}(3)\}$$

where $\mathrm{SNT}(BG) = \{BG\}$.

Is it possible to classify $G(BG)$ or SNT(BG)?

Theorem 60 *Two spaces in $G(BG)$ are homotopy equivalent iff they have isomorphic K-theory λ-rings.*

Automorphism groups of spaces in the genus of BG.

Theorem 61 *Let G be a* simple *Lie group and $B \in G(BG)$ a space of the same genus as BG. If there exists an essential map between B and BG, then $B = BG$.*

Homotopy Chevalley groups

Friedlander's homotopy pull-back square is

$$\begin{array}{ccc} B^\tau G(q) & \longrightarrow & BG_p \\ \downarrow & & \downarrow \Delta \\ BG_p & \xrightarrow{(1,\tau\psi^q)} & BG_p \times BG_p \end{array}$$

where G is compact Lie, τ is an auto of G, and q is a prime power prime to p.

Definition 62 *The homotopy Chevalley group is the pull-back*

$$\begin{array}{ccc} B^\tau X(q) & \longrightarrow & BX \\ \downarrow & & \downarrow \Delta \\ BX & \xrightarrow{(1,\tau\psi^q)} & BX \times BX \end{array}$$

where τ is an auto of BX and $q \in \mathbf{Z}_p^\times$.

If $q = 1$ and $\tau = \mathrm{id}$, the pull-back is ΛBX.

There is an exact sequence

$$\cdots \to \pi_i B^\tau X(q) \to \pi_i BX \xrightarrow{1-(\tau\psi^q)_*} \pi_i BX \to \pi_{i-1} B^\tau X(q) \to \cdots$$

For a p-compact torus, $BT(q) = B\mathbf{Z}/p^\nu$ where $\nu = \nu_p(1-q)$.

Proposition 63 *Write* $\tau\psi^q = (\tau\psi^u)\psi^{q_1}$ *where* $q = uq_1$, $u \in C_{p-1}$, $q_1 \equiv 1 \bmod p$. *Suppose that* $G = \langle\tau\psi^u\rangle \subset \mathrm{Out}(BX)$ *has finite order prime to* p *and that* $(\psi^{q_1})^*$ *is the identity on* $H^*(X;\mathbf{F}_p)$. *Then*

$$B^\tau X(q) = BX^{hG}(q_1)$$

where BX^{hG} *is the homotopy fixed point* p*-compact group for the* G*-action.*

The exploration breaks into two steps:

1. What is BX^{hG} when $G \subset \mathrm{Out}(BX)$ has order prime to p?

2. What is $BX(q)$ for $q \equiv 1 \bmod p$?

Since we already know the answer to the first quetsion, we turn to the second question.

p-local finite groups

Let G be a finite group and $S \leq G$ a Sylow p-subgroup.

The fusion system of G is the category $\mathcal{F}_p(G)$ with **objects** the subgroups of S and **morphisms**

$$\mathcal{F}_p(G)(P,Q) = N_G(P,Q)/C_G(P)$$

where $N_G(P,Q) = \{g \in G \mid gPg^{-1} \leq Q\}$.

The centric linking system $\mathcal{L}_p^c(G)$ of G is the category with **objects** the subgroups of S that are p-centric in G and **morphisms**

$$\mathcal{L}_p^c(G)(P,Q) = N_G(P,Q)/O^p(C_G(P))$$

$O^p(H)$ is the minimal normal subgroup of H of p-power index.

A p-group $P \leq G$ is p-centric if

$$C_G(P) = Z(P) \times O^p(C_G(P))$$

and $O^p(C_G(P))$ has order prime to p.

There is a functor $\mathcal{L}_p^c(G) \to \mathcal{F}_p(G)$.

Theorem 64 *$BG_p \simeq |\mathcal{L}_p^c(G)|_p$ for any finite group G.*

Definition 65 *A p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ consisting of a p-group S, an abstract (saturated) fusion system $\mathcal{F}$ over the group S, and an abstract centric linking system $\mathcal{L}$ associated to $\mathcal{F}$ via a functor $\mathcal{L} \to \mathcal{F}$.*

The classifying space is the space $|\mathcal{L}|_p$. The whole p-local finite group triple is recoverable from $|\mathcal{L}|_p$.

A p-local finite group mimicks the conjugacy relations that hold bewteen the subgroups of the Sylow subgroup of a finite group.

Are there any p-local finite groups that are not finite groups?

What is $BX(q)$? $q \equiv 1 \bmod p$, $q \neq 1$

Proposition 66 *Let $B\nu\colon BV \to BX$ be a toral monomorphism with connected centralizer* $\mathrm{map}(BV, BX)_{B\nu} = BC_X(\nu)$.

1. *There is a unique lift of $B\nu$ to $BX(q)$.*

2. $\mathrm{map}(BV, BX(q))_{B\nu(q)} = BC_X(V)(q)$.

We obtain a map

$$\mathrm{hocolim}_{\mathbf{A}(X)^{\leq t}} BC_X(V)(q) \to BX(q)$$

that for polynomial p-compact groups often is an $H_*\mathbf{F}_p$-equivalence. We are trying to move a homotopy inverse limit around a homotopy direct limit.

Example 67 *If W has order prime to p,*

$$BX(W)(q) = (B(\mathbf{Z}/p^{\nu(1-q)} \rtimes W))_p$$

In particular,

$$BS^{2n-1}(q) = (B(\mathbf{Z}/p^{\nu(1-q)} \rtimes C_n))_p$$

Example 68 *$BDI(2)(q)$ is $H_*\mathbf{F}_3$ equivalent to the homotopy colimit of*

$$BSU(3,q) \overset{\mathbf{Z}/2}{\circlearrowright} \longleftarrow B(\mathbf{Z}/p^{\nu(1-q)})^2 \overset{W_{12}}{\circlearrowright}$$

$BDI(2)$ is the limit of the spaces

$$\begin{aligned} BDI(2)(2^{2n+1}) &= B\mathrm{F}_4{}^{h\langle\alpha\rangle}(2^{2n+1}) \\ &= (B\,{}^2\mathrm{F}_4(2^{2n+1}))_3 \end{aligned}$$

so the inifinte Ree group $\bigcup_n {}^2\mathrm{F}_4(2^{2n+1})$ is a discrete approximation to $BDI(2)$.

Theorem 69 *Let BX be a simply connected p-compact group, τ an automorphism of BX of finite order, and q a prime power. Assume that the order of τ and q are prime to p. Then $B^\tau X(q)$ is the classifying space of a p-local finite group.*

These p-local finite groups are exotic when BX is an Aguadé p-compact group or one of Quillen's $BXG(m,r,n)$ with $r > 2$. At $p = 2$, we have $\mathrm{Sol}(q) = DI(4)(q)$.

Open Problem 70 *$BX(q)$, $q \equiv 1 \bmod p$, only depends on $\nu(1-q)$.*

2-compact groups

Problematic things about 2-compact groups

- $H^1(W;\check{T}) \neq 0$, $H^2(W;\check{T}) \neq 0$
- $BZ(X) \to BZN(X)$ is not surjective
- $BN(X)$ does not even determine $\pi_0(X)$
- $\mathrm{Out}(BX) \to \mathrm{Out}(BN(X))$ is not surjective

Good things about 2-compact groups

- The simple $\mathbf{Z}_2$-reflection groups are $W(G)$ and $W(DI(4))$
- The *only* extensions $\check{T} \to \check{N} \to W$ that are realizable by connected 2-compact groups come from $BG \times BDI(4)^m$
- $BZ(G) \neq BZN(G)$ only when G contains direct $\mathrm{SO}(2n+1)$-factors

The maximal torus normalizer pair: $BX \rightsquigarrow (BN, BN_0)$

The maximal torus normalizer pair informs about $\pi_0(X)$.

Two 2-compact groups with the same maximal torus normalizer pair are isomorphic?

Example 71 Out(BPU(4)) $= \mathbf{Z}_2^\times$ *and* $\mathbf{Z}/2 = H^1(W;\check{T}) \subset$ Out(BN). *The diagram*

$$\begin{array}{ccc} BN & \xrightarrow{B\alpha} & BN \\ {\scriptstyle Bj}\downarrow & & \downarrow{\scriptstyle Bj} \\ B\mathrm{PU}(4) & \dashrightarrow & B\mathrm{PU}(4) \end{array}$$

has no solution.

Definition 72 BX *is N-determined if there exists a solution,* $B\alpha \in H^1(W;\check{T})$*, to*

$$\begin{array}{ccc} BN & \overset{B\alpha}{\dashrightarrow} & BN \\ {\scriptstyle Bj}\downarrow & & \downarrow{\scriptstyle Bj'} \\ BX & \underset{\cong}{\overset{Bf}{\dashrightarrow}} & BX' \end{array} \tag{2}$$

for any other BX' *with the same maximal torus normalizer pair.*

Definition 73 BX *has N-determined automorphisms if*

$$\mathrm{Out}(BX) \to W_0 \backslash \mathrm{Aut}(BN)$$

is injective.

Totally N-determined = N-determined + N-determined automorphisms

Uniquely N-determined = N-determined with unique solution to diagram (2) + N-determined automorphisms

Conjecture 74 *All (connected) 2-compact groups are (uniquely) totally N-determined.*

Lemma 75 *If BX is connected and uniquely N-determined, then*

$$\mathrm{Aut}(BX) \cong \frac{\mathrm{Out}(BN)}{H^1(W; \check{T})}$$

Already known: If $BX = BG$, G connected Lie, or $BX = BDI(4)$, then BX has N-determined automorphisms and diagram (2) has at most one solution.

Theorem 76 *1. Let BX be a 2-compact group. If BX_0 has N-determined automorphisms and*

$$H^1(\pi; \check{Z}(X_0)) \to H^1(\pi; \check{Z}(N_0))$$

is injective, then BX has N-determined automorphisms

2. If BX is connected and BPX has N-determined automorphisms, then BX has N-determined automorphisms

3. If BX_1 and BX_2 are connected and have N-detetermined automorphisms, then $BX_1 \times BX_2$ has N-determined automorphisms

Theorem 77 *Suppose that BX is connected and has no center. If*

1. *the centralizer $BC_X(\nu)$ of any monomorphism $\nu\colon B\mathbf{Z}/p \to BX$ has N-determined automorphisms, and*
2. $\lim^1 \pi_1 BZC_X = 0 = \lim^2 \pi_2 BZC_X$

then BX has N-determined automorphisms.

Theorem 78 *1. Let BX be a LHS 2-compact group. If BX_0 is uniquely N-determined and*

$$H^2(\pi; \check{Z}(X_0)) \to H^2(\pi; \check{Z}(N_0))$$

is injective, then BX is N-determined

2. *If BX is connected and BPX is N-determined, then BX is N-determined*

3. *If BX_1 and BX_2 are N-determined, then $BX_1 \times BX_2$ is N-determined*

Conjecture 79 *All 2-compact groups are LHS*

Theorem 80 *Suppose that BX is connected and has no center. If*

1. *All centralizers $BC_X(V)$, $|V| \leq p^2$, are totally N-determined*

2. *The problems with non-uniqueness of preferred lifts and non-uniqueness of $B\alpha$ in Diagram 2 can be solved*

3. $\lim^2 \pi_1 BZC_X = 0 = \lim^3 \pi_2 BZC_X$

then BX is N-determined.

The plan is to verify that the simple 2-compact groups are uniquely N-determined in the cases

- Classical matrix groups: $\mathrm{PGL}(n+1,\mathbf{C})$, $\mathrm{PSL}(2n,\mathbf{R})$, $\mathrm{SL}(2n+1,\mathbf{R})$, $\mathrm{PGL}(n,\mathbf{H})$
- G_2, F_4, $DI(4)$
- E_6, E_7, E_8

Theorem 81 *The above simple 2-compact groups outside the E-family are uniquely N-determined with automorphism groups*

$$\mathrm{Aut}(\mathrm{PGL}(n+1,\mathbf{C})) = \begin{cases} \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times & n=1 \\ \mathbf{Z}_2^\times & n>1 \end{cases}$$

$$\mathrm{Aut}(\mathrm{PSL}(2n,\mathbf{R})) = \begin{cases} \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times \times \Sigma_3 & n=4 \\ \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times \times \langle c_1\rangle & n>4 \text{ even} \\ \mathbf{Z}_2^\times & n>4 \text{ odd} \end{cases}$$

$$\mathrm{Aut}(\mathrm{SL}(2n+1,\mathbf{R})) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times, \quad n\geq 2$$

$$\mathrm{Aut}(\mathrm{PGL}(n,\mathbf{H})) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times, \quad n\geq 3$$

$$\mathrm{Aut}(\mathrm{G}_2) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times \times C_2, \quad \mathrm{Aut}(\mathrm{F}_4) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$$

$$\mathrm{Aut}(\mathrm{DI}(4)) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$$

where $\langle c_1\rangle$ is a group of order two.

PGL$(n+1,\mathbf{C})$: Computation of higher limits

Proposition 82 *The higher limits vanish over the toral subcategory* $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$.

For $V \subset \mathrm{PGL}(n+1,\mathbf{C})$ let $[\ ,\ ]\colon V \times V \to \mathbf{F}_2$ be the symplectic form $[u\mathbf{C}^\times, v\mathbf{C}^\times] = [u,v] = \pm E$.

Lemma 83 $V \subset \mathrm{PGL}(n+1,\mathbf{C})$ *is toral iff* $[V,V]=0$.

Lemma 84 *If* $n+1$ *is odd,* $\mathrm{PGL}(n+1,\mathbf{C})$ *contains no nontoral objects. For each* $d \geq 1$ *there is a unique elementary abelian* $H^d \subset \mathrm{PGL}(2^d m,\mathbf{C})$ *with non-degenerate symplectic form, and*

$$C_{\mathrm{PGL}(2^d m,\mathbf{C})}(H^d) = H^d \times \mathrm{PGL}(m,\mathbf{C})$$
$$\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))(H^d) = \mathrm{Sp}(2d)$$

Any other nontoral has the form $H^d \times E$ *where* $E \subset \mathrm{PGL}(m,\mathbf{C})$ *is toral, and*

$$\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))(H^d \times E) = \begin{pmatrix} \mathrm{Sp}(2d) & * \\ 0 & \mathbf{A}(E) \end{pmatrix}$$

If we let

$$[E] = \mathsf{Hom}_{\mathbf{A}(E)}(\mathsf{St}(E), \pi_1(BZC_{\mathsf{PGL}(n+1,\mathbf{C})}(E)))$$

then Oliver's cochain complex has the form

$$0 \to [H] \xrightarrow{d^1} \prod_{1 \le i \le [m/2]} [H\#L[m-i,i]] \xrightarrow{d^2}$$

$$[H\#P[1,1,m-2] \times \prod_{2<i<[m/2]} [H\#P[1,i-1,m-i]]$$

where we only list some of the nontoral rank four objects. We need to show that d^1 is injective and that $\ker d^2 = \operatorname{im} d^1$. The computer program *magma* says that

$$[H] = \mathsf{Hom}_{\mathsf{Sp}(2)}(\mathsf{St}(H), H) \cong \mathbf{F}_2$$
$$[H\#L[m-i,i]] = \mathsf{Hom}_{\mathsf{Sp}(3,1)}(\mathsf{St}(V), V) \cong \mathbf{F}_2,$$
$$[H\#P[1,1,m-2]] \cong \mathbf{F}_2$$
$$[H\#P[1,i-1,m-i]] \cong \mathbf{F}_2 \times \mathbf{F}_2$$

and further computations show exactness.