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NILPOTENT SPACES OF SECTIONS

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ABSTRACT. The space of sections of a fibration is nilpotent provided the base is finite CW-complex and the fiber is nilpotent. Moreover, localization commutes with the formation of section spaces.

1. Introduction. In their famous book on localization theory and later in the proceedings of the Vancouver 1977 conference, Hilton et al. showed that (any component of) the space of maps of a finite CW-complex into a nilpotent space is itself nilpotent [5, II. 2.6; 6, Theorem A]. The main purpose of this paper is to state and prove a twisted and relative version of their result.

To be more precise, let now, and throughout the paper, $A \xrightarrow{i} X$ be a cofibration, $p\colon Y \to B$ a fibration with fiber F, $u\colon X \to Y$ a continuous map, and let

$$F_u(X, A; Y, B) := \{v\colon X \to Y \mid vi = ui, pv = pu\}$$

be the space of all lifts of $pu\colon X \to B$ which agree with u on A. In this set-up, $F_u(X, A; Y, B)$ is a nilpotent space provided F is nilpotent and (X, A) is a finite relative CW-complex (Theorem 4.1).

The proof of this assertion proceeds essentially as in the case considered by Hilton et al. of ordinary mapping spaces. A key move, though, is to establish a usable generalization of the principally refined Postnikov systems of nilpotent spaces. This is done in §3 and could be of some independent interest.

The content of the remaining sections is as follows. §2 contains the basic definitions, a few preliminary lemmas, and nilpotency of the function space is proved in a particularly easy case corresponding to the easy construction of localization functors in the simply connected category. Nilpotency in the general case is proved in §4 by means of the refined Postnikov towers of §3. This corresponds to the standard way of constructing localization functors in the nilpotent category. Finally, in §5, we show that fiberwise localization induces localization of the function space.

The following conventions are in force throughout the paper: X, the base B, and the fiber F are 0-connected spaces; a space is said to be nilpotent if all its

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components are nilpotent; G denotes an arbitrary group and M a G-module; the function space $F_u(X, A; Y, B)$ is equipped with the compactly generated topology associated to the compact-open topology.

2. The case of a simple fiber. In this section we recall some basic facts, introduce the notation, and prove the main result in a particularly easy case.

The G-action on M can be realized geometrically as a based, cellular action on the Eilenberg-Mac Lane space $K(M, n+1)$, $n > 0$. Let EG be a free, contractible G-space and put

$$L(M, n+1) := EG \times_G K(M, n+1).$$

The obvious maps

$$L(M, n+1) \underset{\check{k}}{\overset{\hat{k}}{\rightleftarrows}} K(G,1)$$

make $L(M, n+1)$ into an object in the category $\mathscr{T}K(G,1)$ of spaces over and under $K(G,1)$ [**10**].

Let now N be another G-module and $\mathrm{Hom}_G(M, N)$ the set of G-module homomorphisms of M into N. Also, let $\langle L(M, n+1), L(N, n+1)\rangle_{K(G,1)}$ denote the set of $\mathscr{T}K(G,1)$-homotopy classes of maps of $L(M, n+1)$ into $L(N, n+1)$.

LEMMA 2.1. *There is a bijective correspondence*

$$\mathrm{Hom}_G(M, N) \leftrightarrow \langle L(M, n+1), L(N, n+1)\rangle_{K(G,1)}.$$

PROOF. By the universal nature of $\hat{k}\colon L(N, n+1) \to K(G,1)$,

$$\langle L(M, n+1), L(N, n+1)\rangle_{K(G,1)} = H^{n+1}(L(M, n+1), K(G,1); N).$$

The Serre spectral sequence for $\hat{k}\colon L(M, n+1) \to K(G,1)$ with local coefficients N shows [**10**, (5) p. 4] that $H^{n+1}(L(M, n+1), K(G,1); N) = \mathrm{Hom}_G(M, N)$. $\square$

Recall that we have a path fibration

$$\overline{P}L(M, n+1) \to L(M, n+1)$$

in $\mathscr{T}K(G,1)$. The path space is defined [**10**] as

$$\overline{P}L(M, n+1) := \{\sigma\colon I \to L(M, n+1) \mid \hat{k}\sigma(I) = \hat{k}\sigma(0), \sigma(0) = \check{k}\hat{k}\sigma(0)\}$$

and the projection onto $L(M, n+1)$ is evaluation at $1 \in I = [0,1]$ as usual.

Let now $v\colon X \to \overline{P}L(M, n+1)$ be any map into this path space. Using the fact that $\hat{k}\colon \overline{P}L(M, n+1) \to K(G,1)$ is a homotopy equivalence, one may prove

LEMMA 2.2. $F_v(X, A; \overline{P}L(M, n+1), K(G,1))$ *is contractible.*

Let $Z \xrightarrow{f} K$ be a space over some other space K and let $L \rightleftarrows K$ be an object of $\mathscr{T}K$.

DEFINITION 2.3. A K-principal fibration over Z is any fibration obtained as the pullback of the path fibration $\overline{P}L \to L$ in $\mathscr{T}K$ along a map $k\colon Z \to L$ over K.

Suppose for instance that the base space B is a space over K, $B \xrightarrow{f} K$, so that any space over B also becomes a space over K. Suppose furthermore that we have a commutative diagram of the form

$$\begin{array}{ccc} Y & \xrightarrow{\bar{k}} & \bar{P}L \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ Y_2 & \xrightarrow{k} & L \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ B & \xrightarrow{f} & K \end{array}$$

where $p = p_1 p_2$ and p_2: $Y \to Y_2$ is the K-principal fibration defined by k. Put $u_2 = p_2 u$ and $u_1 = p_1 u_2$. In this situation we have the simple but important

LEMMA 2.4. *If $F_{u_2}(X, A; Y_2, B)$ is nilpotent, so is $F_u(X, A; Y, B)$.*

PROOF. The pullback diagram of the K-principal fibration p_2 induces another pullback diagram

$$\begin{array}{ccc} F_u(X, A; Y, B) & \to & F_{\bar{k}u}(X, A; \bar{P}L, K) \\ \downarrow & & \downarrow \\ F_u(X, A; Y_2, B) & \to & F_{ku}(X, A; L, K) \end{array}$$

of function spaces. Here, $F_{\bar{k}u}(X, A; \bar{P}L, K)$ is contractible by Lemma 2.2, so the assertion fôllows from [**5**, II.2.2]. The requirement of [**5**, II.2.2] that all spaces be connected can be met, as in the proof of [**6**, Theorem A], by introducing suitable covering spaces of $F_{ku}(X, A; L, K)$. □

The material introduced at this point suffices to verify the nilpotency statement in a special case. Recall that F is simple if $\pi_1(F)$ acts trivially on $\pi_*(F)$, that F is finitely anticonnected if $\pi_j(F) = 0$ for j sufficiently large, and that (X, A) is finitely coconnected if $H^j(X, A; M) = 0$ for any local coefficient system M for j sufficiently large.

PROPOSITION 2.5. *Suppose that the fiber F is simple and that either (X, A) is finitely coconnected or F is finitely anticonnected. Then $F_u(X, A; Y, B)$ is nilpotent.*

PROOF. Make B into a space over $K(G, 1)$, where $G := \pi_1(B)$, by choosing a map $B \to K(G, 1)$ which is the identity on π_1.

Since F is simple, G acts on $\pi_*(F)$ and we may erect Postnikov towers [**10**]

$$\begin{array}{ccccccccccc} & & & & & L(\pi_i F, i+1) & & & & & \\ & & & & & \uparrow & & & & & \\ Y \to Y_{r+1} \to & \cdots & \to & Y_{i+1} & \to & Y_i & \to & \cdots & \to & Y_2 & \to Y_1 = B \\ & & & \uparrow & & & & & & \uparrow & \\ & & & K(\pi_i F, i) & & & & & & K(\pi_1 F, 1) & \end{array}$$

where $Y \to Y_{r+1}$ is $(r+1)$-connected and each stage $Y_{i+1} \to Y_i$, $1 \leqslant i \leqslant r$, is a $K(G, 1)$-principal fibration. $F_u(X, A; Y_2, B)$ is nilpotent, even simple, by [**12**], so repeated applications of Lemma 2.4 show that $F_u(X, A; Y_{r+1}, B)$ is nilpotent. But the stated finiteness assumptions assure that this space converges to $F_u(X, A; Y, B)$ as r goes to infinity. Hence the latter space is nilpotent. □

It is not true in general that $F_u(X, A; Y, B)$ is simple even though F is simple: The fundamental group of the space of degree 0 maps $S^1 \times S^1 \to S^2$ is not abelian [**2**, **8**].

3. Refinements of Postnikov towers. Nilpotent spaces have the important property that their Postnikov systems admit principal refinements. In this section we shall relativize this concept.

First, we consider fibrations with acyclic fibers.

Let N be a nilpotent group. The lower central series of N,

$$\Gamma^1(N) \rhd \Gamma^2(N) \rhd \cdots \rhd \Gamma^i(N) \rhd \Gamma^{i+1}(N) \rhd \cdots,$$

defined inductively by $\Gamma^1(N) = N$, $\Gamma^{i+1}(N) = [N, \Gamma^i(N)]$, $i \geqslant 1$, is finite, in the sense that $\Gamma^{c+1}(N) = \{1\}$ for some c, by the very definition of nilpotency. Any automorphism of N restricts to an automorphism of $\Gamma^i(N)$ for all $i \geqslant 1$. In other words, $\mathrm{Aut}(N)$ acts on $\Gamma^i(N)$ and hence also on $\Gamma^i(N)/\Gamma^{i+1}(N)$ for all $i \geqslant 1$. Moreover, inner automorphisms act trivially on $\Gamma^i(N)/\Gamma^{i+1}(N)$ and hence $\mathrm{Out}(N) := \mathrm{Aut}(N)/N$ acts on these subquotients. Similarly, $\mathrm{Aut}(N)/\Gamma^i(N)$ acts on $\Gamma^i(N)/\Gamma^{i+1}(N)$ and even on $H_1(\Gamma^i(N)) = \Gamma^i(N)/[\Gamma^i(N), \Gamma^i(N)]$.

Now assume that $K(N, 1) \to Y \to B$ is a fibration with an acyclic, nilpotent fiber. The associated semiaction [**4**, p. 142] of $\pi_1(B)$ on N is a homomorphism ψ: $\pi_1(B) \to \mathrm{Out}(N)$. B, and hence also Y, becomes a space over $K(\mathrm{Out}(N), 1)$ if we realize ψ geometrically as a map of B into $K(\mathrm{Out}(N), 1)$.

LEMMA 3.1. *Suppose that the fibration p: $Y \to B$ has an acyclic, nilpotent space $K(N, 1)$ as fiber. Then there is a factorization*

$$\begin{tikzcd} & & & & L(\Gamma^i/\Gamma^{i+1}, 2) & & \\ Y = Y_{c+1} \arrow[r] & Y_c \arrow[r] & \cdots \arrow[r] & Y_{i+1} \arrow[r] & Y_i \arrow[u] \arrow[r] & \cdots \arrow[r] & Y_1 = B \\ & & & K(\Gamma^i/\Gamma^{i+1}, 1) \arrow[u] & & & \end{tikzcd}$$

of p: $Y \to B$ into a finite string of $K(\mathrm{Out}(N), 1)$-principal fibrations.

PROOF. W.l.o.g., p is the universal example [**4**, Theorem 2.1] of such a fibration. Then $Y = K(\mathrm{Aut}(N), 1)$ and $\pi_1(B) = \mathrm{Out}(N)$. In the Serre spectral sequence for p with local coefficients $\mathrm{Out}(N) \to \mathrm{Aut}(\Gamma^1(N)/\Gamma^2(N))$ one has, as in the proof of [**10**, Theorem 4.1], that

$$\begin{aligned} E_2^{01} &= \mathrm{Hom}_{\mathrm{Aut}(N)/\Gamma^1(N)}\big(H_1\Gamma^1(N), \Gamma^1(N)/\Gamma^2(N)\big) \\ &= \mathrm{Hom}_{\mathrm{Aut}(N)}\big(\Gamma^1(N), \Gamma^1(N)/\Gamma^2(N)\big). \end{aligned}$$

Hence the projection map $\Gamma^1(N) \to \Gamma^1(N)/\Gamma^2(N)$, which is $\mathrm{Aut}(N)$-equivariant, is geometrically realizable [**10**, p. 5 and Theorem 3.1] by a map $Y_1 \to L(\Gamma^1(N)/\Gamma^2(N), 2)$ over $K(\mathrm{Out}(N), 1)$. By pullback of the path fibration we get a factorization

$$\begin{array}{ccccc} Y & \rightarrow & Y_2 & \rightarrow & Y_1 = B \\ \uparrow & & \uparrow & & \\ K(\Gamma^1,1) & & K(\Gamma^1/\Gamma^2,1) & & \end{array}$$

of $Y \to B$ such that $Y_2 \to Y_1$ is a $K(\mathrm{Out}(N),1)$-principal fibration and the restriction of $Y \to Y_2$ to the fibre induces the projection $\Gamma^1(N) \to \Gamma^1(N)/\Gamma^2(N)$ on π_1. This implies that the homotopy fiber of $Y \to Y_2$ is $K(\Gamma^2(N),1)$ and $\pi_1(Y_2) = \mathrm{Aut}(N)/\Gamma^2(N)$.

Repeat the argument with Y_1 replaced by Y_2, etc. The process eventually stops, because the homotopy fiber, $K(\Gamma^{c+1}(N),1)$, of $Y \to Y_{c+1}$ is contractible if $\Gamma^{c+1}(N) = 1$. □

Next, we consider the abelian case. Suppose that the group G fits into an exact sequence of groups of the form

$$N \xrightarrow{\kappa} G \xrightarrow{\lambda} Q \to 1.$$

Then N acts on M through κ. Define the lower central N-series of M,

$$\Gamma_N^1(M) \supset \Gamma_N^2(M) \supset \cdots \supset \Gamma_N^i(M) \supset \Gamma_N^{i+1}(M) \supset \cdots,$$

by setting $\Gamma_N^1(M) = M$ and letting $\Gamma_N^{i+1}(M)$ be the subgroup generated by $\{nm - m \mid n \in N, m \in \Gamma_N^i(M)\}$. N acts nilpotently on M if $\Gamma_N^{c+1}(M) = 0$ for some c.

LEMMA 3.2. *$\Gamma_N^i(M)$ is a G-submodule for all $i \geqslant 1$.*

PROOF. For all $n \in N$, $m \in \Gamma_N^i(M)$, and $g \in G$,

$$g(nm - m) = (gng^{-1})gm - gm \in \Gamma_N^{i+1}(M),$$

for $gng^{-1} \in \kappa(N)$ and, arguing by induction, we may assume that $gm \in \Gamma_N^i(M)$. □

Thus all the subquotients $\Gamma_N^i(M)/\Gamma_N^{i+1}(M)$ are G-modules and even Q-modules since N acts trivially here.

Assume now that $K(M,n) \to Y \to B$ is a $K(G,1)$-principal fibration, $n \geqslant 1$.

LEMMA 3.3. *Suppose that N acts nilpotently on M. Then there is a factorization*

$$\begin{array}{ccccccccc} & & & & & & L(\Gamma^i/\Gamma^{i+1}, n+1) & & \\ & & & & & & \uparrow & & \\ Y = Y_{c+1} \to Y_c \to & \cdots & \to & Y_{i+1} & \to & & Y_i & \to \cdots & \to Y_1 = B \\ & & & \uparrow & & & & & \\ & & & K(\Gamma^i/\Gamma^{i+1}, n) & & & & & \end{array}$$

of $p\colon Y \to B$ into a finite string of $K(Q,1)$-principal fibrations.

PROOF. W.l.o.g., $B = L(M, n+1) = EG \times_G K(M, n+1)$, $Y = \overline{P}L(M, n+1)$, and $p\colon Y \to B$ is the path fibration in $\mathcal{T}K(G,1)$.

Since the Milnor construction [7] of universal numerable, principal bundles is functorial, there exists a map $E\lambda\colon EG \to EQ$ over $\lambda\colon K(G,1) \to K(Q,1)$ such that $E\lambda(eg) = E\lambda(e)\lambda(g)$, $e \in EG$, $g \in G$. By Lemma 2.1, there also exists a G-equivariant, based map

$$\rho\colon K(\Gamma_N^i(M), n+1) \to K(\Gamma_N^i(M)/\Gamma_N^{i+1}(M), n+1)$$

corresponding to the projection $\Gamma_N^i(M) \to \Gamma_N^i(M)/\Gamma_N^{i+1}(M)$ of G-modules. Considering the target as a Q-space, this means that $\rho(gx) = \lambda(g)\rho(x)$ for all $g \in G$ and $x \in K(\Gamma_N^i(M), n+1)$. $E\lambda \times \rho$ induces, by passing to orbit spaces, a map

$$\begin{array}{ccc} L\left(\Gamma_N^i(M), n+1\right) & \to & L\left(\Gamma_N^i(M)/\Gamma_N^{i+1}(M), n+1\right) \\ \downarrow\uparrow & & \downarrow\uparrow \\ K(G,1) & \xrightarrow{\lambda} & K(Q,1) \end{array}$$

over and under λ. Since this map lifts to the path spaces, we obtain, by forming pullback, a factorization

$$\begin{array}{ccccc} \bar{P}L\left(\Gamma_N^i(M), n+1\right) & \to & Y_{i+1} & \to & L\left(\Gamma_N^i(M), n+1\right) \\ \uparrow & & \uparrow & & \\ K(\Gamma^i, n) & & K\left(\Gamma^i/\Gamma^{i+1}, n\right) & & \end{array}$$

of the path fibration over $L = L(\Gamma_N^i(M), n+1)$. Here, $Y_{i+1} \to L$ is a $K(Q,1)$-principal fibration and the homotopy fiber of $\bar{P}L \to Y_{i+1}$ is $K(\Gamma_N^{i+1}(M), n)$. This implies that $Y_{i+1} = L(\Gamma_N^{i+1}(M), n+1)$ and $\bar{P}Y_{i+1} = \bar{P}L$.

If N acts nilpotently on M, then this process eventually stops, for the homotopy fiber $K(\Gamma_N^{c+1}(M), n)$ of $\bar{P}L(\Gamma_N^c(M), n+1) \to Y_{c+1}$ is contractible if $\Gamma_N^{c+1}(M) = 0$. $\square$

A combination of Lemma 3.1, Lemma 3.3, [**4**], and [**10**] yields

THEOREM 3.4. *Let p: $Y \to B$ be a fibration with nilpotent fiber F. For $r \geqslant 1$ there exist factorizations*

$$Y \to Y_{s+1} \to \cdots \to Y_{i+1} \to Y_i \to \cdots \to Y_2 \to Y_1 = B$$

of p: $Y \to B$ such that $Y \to Y_{s+1}$ is $(r+1)$-connected and each stage $Y_{i+1} \to Y_i$, $1 \leqslant i \leqslant s$, is a K-principal fibration where either $K = K(\operatorname{Out}\pi_1(F), 1)$ or $K = K(\pi_1(B), 1)$.

PROOF. There exist Postnikov decompositions of the form [**4, 10**]:

$$\begin{array}{ccccccccc} Y \to Y_{r+1} & \to \cdots \to & Y_{i+1} & \to & Y_i & \to \cdots \to & Y_2 & \to & Y_1 = B \\ & & \uparrow & & & & \uparrow & & \\ & & K(\pi_i(F), i) & & & & K(\pi_1(F), 1) & & \end{array}$$

The first stage, $Y_2 \to Y_1$, is a fibration of the type discussed in Lemma 3.1. The next stages, $Y_{i+1} \to Y_i$, $i \geqslant 2$, are $K(\pi_1(Y), 1)$-principal fibrations corresponding to the action of $\pi_1(Y)$ on $\pi_i(F) = \pi_{i+1}(B, Y)$. But $\pi_1(Y)$ fits into the exact sequence

$$\pi_1(F) \to \pi_1(Y) \to \pi_1(B) \to 1$$

and $\pi_1(F)$ acts nilpotently on $\pi_i(F)$. Thus $Y_{i+1} \to Y_i$ is a fibration of the type discussed in Lemma 3.3. $\square$

The converse of Theorem 3.4 is also true, for the restriction to the fiber of factorizations as in the theorem are principal refinements of the Postnikov tower of F.

4. The case of a nilpotent fiber. In this section we shall prove the main result in its full generality.

In fact, all preparations have been made already so we proceed straightway to the formulation of

THEOREM 4.1. *Suppose that the fiber F of $p\colon Y \to B$ is nilpotent and that either (X, A) is finitely coconnected or F is finitely anticonnected. Then $F_u(X, A; Y, B)$ is nilpotent.*

In order to prove this statement, replace the "crude" Postnikov tower by the refined one of Theorem 3.4 and proceed as in Proposition 2.5.

For an application of Theorem 4.1, suppose that F is a G-space and $\overline{X}$ a free (principal) G-space such that $(\overline{X}/G, \overline{A}/G) = (X, A)$ for some G-invariant subspace $\overline{A} \subset \overline{X}$. Denote by $F_f(\overline{X}, \overline{A}; F)^G$ the space of all G-equivariant extensions of $f \mid \overline{A}$, where $f\colon \overline{X} \to F$ is some G-map.

COROLLARY 4.2. *$F_f(\overline{X}, \overline{A}; F)^G$ is nilpotent under the assumptions of Theorem* 4.1.

PROOF. Apply the theorem to the space of sections of the associated fiber bundle $\overline{X} \times_G F \to X$. □

In particular, section spaces and equivariant function spaces admit localizations. That these localizations behave as expected will be shown in the next, final section.

5. Localizations of section spaces. Throughout this section we make the additional assumption that (X, A) is a relative CW-complex with finite skeleta.

For any family P of primes, the P-localized module M_P is canonically a G-module and the localization map $e\colon M \to M_P$ a G-homomorphism.

Given a homomorphism $\pi_1(X) \to G$, M and M_P become local coefficient systems in X.

LEMMA 5.1. *The coefficient group homomorphism*

$$e_*\colon H^*(X, A; M) \to H^*(X, A; M_P)$$

is a P-localization.

PROOF. Let $\Gamma^*(X, A; M)$, $\Gamma^*(X, A; M_P)$ denote the cellular cochain complexes of (X, A) with local coefficients M, M_P. Since localization commutes with direct sum, the commutative diagram

$$\begin{array}{ccc} \bigoplus_\alpha M(h_\alpha(E_0)) & \xrightarrow{\cong} & \Gamma^q(X, A; M) \\ \downarrow e & & \downarrow e_* \\ \bigoplus_\alpha M(h_\alpha(E_0))_P & \xrightarrow{\cong} & \Gamma^q(X, A; M_P) \end{array}$$

where $\{h_\alpha\colon \Delta^q \to X\}$ are the q-cells of (X, A) and $E_0 \in \Delta^q$ the base point, shows that e_* P-localizes on the cochain level. To complete the proof, observe that e_* is a cochain map and localization an exact functor. □

Let also $e\colon L \to L_{(P)}$ denote the $\mathscr{T}K(G,1)$-morphism of $L := L(M, n+1)$ into $L_{(P)} := L(M_P, n+1)$, $n > 0$, corresponding to $e\colon M \to M_P$.

In the following,

$$F_u^0(-,-;-,-)$$

denotes the path-component of $F_u(-,-;-,-)$ containing u; $\underline{m}$, where m is a fiber map, denotes the map on function spaces defined as post composition with m.

LEMMA 5.2. *For any map $v\colon X \to \overline{P}L$,*

$$\underline{\overline{P}e} : F_v^0(X, A; \overline{P}L, L) \to F_{\overline{P}ev}^0\big(X, A; \overline{P}L_{(P)}; L_{(P)}\big)$$

is a P-localization.

PROOF. This follows from Lemma 5.1 and the decomposition [**12**] of the function spaces in question as products of Eilenberg-Mac Lane spaces. □

Consider the fiberwise P-localization $e\colon Y \to Y_{(P)}$ constructed in [**1, 9**] for any fibration with nilpotent fiber.

THEOREM 5.3. *Suppose that F is nilpotent and that either (X, A) is finite or F is finitely anticonnected. Then*

$$\mathbf{e}\colon F_u^0(X, A; Y, B) \to F_{eu}^0(X, A; Y_{(P)}, B)$$

is a P-localization.

The proof is again by induction in the refined Postnikov tower of p and proceeds as that of [**6**, Theorem B] once [**6**, Theorem 2.1] has been replaced by Lemma 5.2.

In the situation of Corollary 4.2 and under the same assumptions as in Theorem 5.3 we have

COROLLARY 5.4. $F_f^0(\overline{X}, \overline{A}; F)_P^G = F_{ef}^0(\overline{X}, \overline{A}; F_P)^G$.

PROOF. $(\overline{X} \times_G F)_{(P)} = \overline{X} \times_G F_P$. □

We finish this paper by considering an application of Corollary 5.4.

EXAMPLE 5.5. The antipodal map defines a $\mathbf{Z}/2$-action on S^n. Suppose that n is even (and positive) and let 1 denote the identity map. Then any component of $F_1(S^n, S^m; S^n)^{\mathbf{Z}/2}$ is rationally homotopy equivalent to S^{n-m-1} if m is even, $0 \leqslant m < n$, and to S^{2n-m-2} if m is odd, $-1 \leqslant m < n$ (where $S^{-1} = \varnothing$).

The assertions in Example 5.5 follow from the decomposition [**12**] of the section space for $S^n \times_{\mathbf{Z}/2} S^n_{(0)} \to \mathbf{R}P^n$ induced from the Postnikov tower; cf. [**3, 13**]. Note that the results of [**3** or **15**] cannot be used as neither the fibration nor the base is nilpotent.

Added in proof. Only $K(\pi_1(B), 1)$-principal fibrations are needed in Theorem 3.4 since in Lemma 3.1, $\mathrm{Out}(N)$ can easily be replaced by $\pi_1(B)$.

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