Eksamen juni 2004

Matematik 3GT

Books, notes, calculators, and computers are allowed at this three hour written exam. You may write your answers in pencil.

Problem 1

Let $f\colon X \to Y$ be a map between two topological spaces. Show that the following two statements are equivalent:

(1) f is continuous and closed
(2) $\overline{f(A)} = f(\overline{A})$ for all $A \subset X$.

Problem 2

Let $f\colon K \to X$ be a continuous map of a compact space K into a locally path-connected Hausdorff space X. Show that the path-components of $X - f(K)$ are open.

Problem 3

Let X be a normal space and βX its Stone–Čech compactification. We consider X as a subset of βX. Let y be a point in the remainder $\beta X - X$. Assume that y is the limit of a sequence of points from X. We will show that this assumption leads to a contradiction.

(1) Show that y is the limit of a sequence of points $x_n \in X$, $n \in \mathbf{Z}_+$, all of whose points are distinct, ie such that $x_n \neq x_m$ for $n \neq m$.
[Hint: If (x_n) converges to y, where x_n is in X and y in $\beta X - X$, consider the subsequence (x_{n_k}) where

$$n_k = \begin{cases} 1 & k = 1 \\ \min\{n > n_{k-1} \mid x_n \notin \{x_{n_1}, \dots, x_{n_{k-1}}\}\} & k > 1 \end{cases}$$

is defined recursively.]
(2) Let $A = \{x_1, x_3, x_5, \dots\}$ be the set of odd and $B = \{x_2, x_4, x_6, \dots\}$ the set of even elements of the sequence. Show that $\overline{A} = A \cup \{y\}$ and $\overline{B} = B \cup \{y\}$. What is $\overline{A} \cap \overline{B}$? (Closures are taken in βX.)
(3) Show that A and B are are disjoint closed subsets of X and that there exists a continuous function $f\colon X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.
(4) Show that there exists a continuous function $\overline{f}\colon \beta X \to [0,1]$ such that $\overline{f}(A) = 0$ and $\overline{f}(B) = 1$. What does that tell you about $\overline{A} \cap \overline{B}$?
(5) Explain why we now have a contradiction and conclude that no point of $\beta X - X$ is the limit of a sequence of points from X.

Assume now that X is normal and noncompact.

(6) Explain why $\beta X - X$ is nonempty.
(7) Explain why βX does not satisfy the conclusion of the Sequence lemma. Conclude that βX is not first countable and not metrizable.

(THE END)