Eksamen juni 2003

Matematik 3GT

Books, notes, calculators, and computers are allowed at this three hour written exam. You may write your answers in pencil.

Problem 1 (20 points)

Let S_Ω denote the smallest uncountable well-ordered set.

(1) Show that S_Ω contains a subset with the order type of the positive integers $\mathbf{Z}_+$. [Hint: S_Ω does not have a largest element.]

(2) Find an element of S_Ω that does not have an immediate predecessor. [Hint: Use $\mathbf{Z}_+ \subset S_\Omega$.]

Problem 2 (10 points)

If Z is a topological space and $C \subset Z$ a subset, we define the *boundary* of C by the equation

$$\partial C = \overline{C} \cap \overline{(Z - C)}$$

Let X and Y be topological spaces, A a subset of X and B a subset of Y. Then $A \times B$ is a subset of $X \times Y$. Show that

$$\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$$

[Hint: $(X \times Y) - (A \times B) = (X - A) \times Y \cup X \times (Y - B)$]

Problem 3 (20 points)

Let $f\colon S^1 \to \mathbf{R}$ be a continuous map of the circle to the real line.

(1) Show that there is a point x on the circle so that $f(x) = f(-x)$. [Hint: The odd map $g(x) = f(x) - f(-x)$ must take the value 0 at some point.]

(2) Is it possible to imbed the circle S^1 in the real line $\mathbf{R}$?

Problem 4 (40 points)

Let $X = \mathbf{Z}_+$ be the set of positive integers with the discrete topology and $\beta(X)$ its Stone–Čech compactification. We consider X as a subset of $\beta(X)$. Let A be any subset of $X \subset \beta(X)$. Let U be an open subset of $\beta(X)$.

(1) Show that there is a continuous function $F\colon \beta(X) \to \{0, 1\}$ defined on the compactification such that $F(A) = 0$ and $F(X - A) = 1$. Deduce that $\overline{A}$ and $\overline{X - A}$ are disjoint where closures are taken in $\beta(X)$.

(2) Show that $\beta(X) - \overline{A} = \overline{X - A}$ and that $\overline{A}$ is open and closed in $\beta(X)$. [Hint: You may use without proof the general fact that $\overline{C} - \overline{D} \subset \overline{C - D}$.]

(3) Show that $\overline{U} = \overline{U \cap X}$ and that $\overline{U}$ is open and closed in $\beta(X)$.

(4) Show that the connected components of $\beta(X)$ are one-point sets.

(THE END)