Eksamen januar 2005
JMM

Matematik 3GT

Books, notes, calculators, and computers are allowed at this three hour written exam. You may write your answers in pencil. An unjustified answer counts as no answer.

Problem 1

Let $f\colon X \to Y$ be a map between two topological spaces. Show that the following three statements are equivalent:

(1) f is continuous and open.
(2) $f^{-1}(\mathrm{Int}(B)) = \mathrm{Int}(f^{-1}(B))$ for all $B \subset Y$.
(3) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ for all $B \subset Y$.

Problem 2

Let X be a topological space and $A \subset X$ a subset. The *exterior* and the *boundary* of A are the subsets

$$\mathrm{Ext}(A) = X - \overline{A}, \qquad \mathrm{Bd}(A) = \overline{A} \cap \overline{X - A}$$

of X.

(1) Explain why the exterior of A is open and the boundary of A is closed.
(2) Show that $X = \mathrm{Int}(A) \cup \mathrm{Bd}(A) \cup \mathrm{Ext}(A)$ and that these three sets are pairwise disjoint.
(3) Let $u\colon [0,1] \to X$ be a continuous path in X from a point $u(0) \in \mathrm{Int}(A)$ in the interior of A to a point $u(1) \in \mathrm{Ext}(A)$ in the exterior of A. Show that there is a $t \in [0,1]$ such that $u(t)$ is in the boundary, $\mathrm{Bd}(A)$, of A.

Problem 3

Consider the spaces

$$\{0\} \cup \{1/n \mid n \in \mathbf{Z}_+\} = A \subset I = [0,1]$$

equipped with their standard topologies (as subspaces of $\mathbf{R}$ with the standard topology). Show that there does not exist a continuous map $R\colon I \times I \to I \times \{0\} \cup A \times I$ such that $R(x \times t) = x \times t$ for all $x \times t$ in $I \times \{0\} \cup A \times I$.

You may proceed as follows: Assume that the map R exists.

(1) Show that $R(x_n \times 1) = \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right) \times 0$ for some point $x_n \in [\frac{1}{n+1}, \frac{1}{n}]$.
(2) Obtain a contradiction with the continuity of R.

Problem 4

Let X be a Hausdorff space equipped with an ascending sequence of closed subspaces $X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n \subset \cdots \subset X$. Assume that the topology on X is coherent with this filtration in the sense that $X = \bigcup_{n=0}^{\infty} X_n$ and

$$A \text{ is closed} \iff A \cap X_n \text{ is closed for each } n$$

holds for all subsets A of X.

Let C be a compact subset of X. Choose a point $t_n \in C \cap (X_n - X_{n-1})$ for all $n \in \mathbf{Z}_+$ for which this intersection is nonempty and let T be the set of all the points t_n.

(1) Show that T is closed and that any subspace of T is closed. What can you say about the subspace topology on T?
(2) Show that T is finite.
(3) Conclude that C is contained in X_N for some $N \in \mathbf{Z}_+$.

(THE END)