Eksamen januar 2004

Matematik 3GT

Books, notes, calculators, and computers are allowed at this three hour written exam. You may write your answers in pencil.

Problem 1

Let $\mathbf{R}$ be the real line (with standard topology) and let $I = [0,1]$ be the unit interval (considered as a set).

For each natural number $k \in \mathbf{Z}_+$, let D_k be the set of all finite sequences

$$(I_1, \dots, I_k, x_1, \dots, x_k)$$

where $I_1, \dots, I_k \subset I$ are disjoint closed subintervals of I with rational endpoints and $x_1, \dots, x_k \in \mathbf{Q}$ are rational numbers.

(1) Show that the set $D = \bigcup_{k=1}^{\infty} D_k$ is countable.

For each element $(I_1, \dots, I_k, x_1, \dots, x_k) \in D_k$, let $x(I_1, \dots, I_k, x_1, \dots, x_k) \in \mathbf{R}^I$ be the element given by

$$\pi_t x(I_1, \dots, I_k, x_1, \dots, x_k) = \begin{cases} x_j & t \in I_j \text{ for some } j \in \{1, \dots, k\} \\ 0 & t \notin I_1 \cup \dots \cup I_k \end{cases}$$

where $\pi_t \colon \mathbf{R}^I \to \mathbf{R}$, $t \in I$, is the projection map.

(2) Show that $x(D) = \bigcup_{k=1}^{\infty} x(D_k)$ is dense in $\mathbf{R}^I$ (with the product topology).

(3) Show that $\mathbf{R}^J$ (with the product topology) does not contain any dense countable subsets when the cardinality of J is strictly bigger than that of the power set $\mathcal{P}(\mathbf{Z}_+)$.

Hint for (3): Let $D \subset \mathbf{R}^J$ be a dense subset. Show that the map

$$J \to \mathcal{P}(D) \colon j \mapsto D \cap \pi_j^{-1}((2003, 2004))$$

is injective. You may use without proof that $\overline{D \cap U} = \overline{U}$ for any open set $U \subset \mathbf{R}^J$.

Problem 2

Suppose that X is a locally compact Hausdorff space and A a nonempty closed subset of X. Let $\omega(X - A) = (X - A) \cup \{\omega\}$ denote the one-point (Alexandroff) compactification of $X - A$ and X/A the quotient space of X obtained by identifying all points of A to one point and making no further identifications. Are these two spaces homeomorphic?

(1) Explain why $X - A$ is locally compact Hausdorff.

(2) Let $f \colon X \to \omega(X - A)$ be the surjective map that is the identity on $X - A$ and sends A to ω. Show that f is continuous.

(3) Show that f induces a continuous bijective map $\overline{f} \colon X/A \to \omega(X - A)$.

(4) Show that $\overline{f}$ is a homeomorphism if and only if X/A is compact.

Let now $X = \mathbf{R}$ and $A = \bigcup_{n \in \mathbf{Z}} [2n, 2n+1]$.

(5) Is X/A compact? Is $\overline{f} \colon X/A \to \omega(X - A)$ a homeomorphism?

(6) Show that X/A is not first countable at the point corresponding to A.

(7) Describe a subspace of $\mathbf{R}^2$ that is homeomorphic to $\omega(X - A)$. (A proof is not required.) Can you find a subspace of $\mathbf{R}^2$ that is homeomorphic to X/A?

(THE END)