Eksamen i Matematik 3GT
Januar 2003

# Matematik 3GT

Books, notes, calculators, and computers are allowed at this 3 hour written exam. You may write your answers in pencil.

**Problem 1** (10 points)

Let $X$ be a strictly simply ordered set. Assume that $X$ is compact in the order topology. Show that $X$ has a smallest and a largest element.

**Problem 2** (40 points)

Let $X$ be a compact metrizable space.

(1) Show that $X$ contains a dense countable subset $A$. (Hint: Cover $X$ by finitely many $1/n$-balls.)

(2) Show that the topology on $X$ is induced by a metric $d\colon X \times X \to [0,1]$ with values in the interval $[0,1]$.

(3) Let $A \subset X$ be as in (1) and the metric $d$ as in (2). Show that the map

$$f\colon X \to [0,1]^A \quad \text{given by} \quad f(x) = \big(d(x,a)\big)_{a\in A}$$

is an imbedding and that the image $f(X) \subset [0,1]^A$ is closed.

(4) Show that any closed subspace of $[0,1]^\omega$ is compact and metrizable.

**Problem 3** (20 points)

For a locally compact Hausdorff space $X$, let $X^\bullet = X \cup \{\infty\}$ denote its one-point-compactification. For a continuous function $f\colon X \to Y$ between two such spaces, let $f^\bullet\colon X^\bullet \to Y^\bullet$ be the extension of $f$ given by $f(\infty) = \infty$.

(1) Show that $f^\bullet\colon X^\bullet \to Y^\bullet$ is continuous if and only if $f^{-1}(K)$ is compact for any compact subspace $K \subset Y$.

(2) Is $g^\bullet\colon \mathbf{R}^\bullet \to (\mathbf{R}^2)^\bullet$ continuous when $g\colon \mathbf{R} \to \mathbf{R}^2$ is the continuous map $g(t) = (\cos t, \sin t)$?

**Problem 4** (50 points)

Let $P^2 = S^2/\sim$ be the quotient space of the unit 2-sphere

$$S^2 = \{(x_1,x_2,x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbf{R}^3$$

by the equivalence relation $x \sim y \Leftrightarrow x = \pm y$. The quotient map, sending a point to its equivalence class, is denoted $p\colon S^2 \to P^2$.

(1) Show that the map $p$ is open and closed.

(2) Show that $P^2$ has a countable basis.

(3) Show that $P^2$ is Hausdorff.

(4) Let $U = \{(x_1,x_2,x_3) \in S^2 \mid x_3 > 0\} \subset S^2$ be the upper open hemisphere. Show that $p|U\colon U \to P^2$ is an imbedding.

(5) The complement $P^2 - p(U)$ is a familiar topological space. Which one? (Just give the answer, no justification is necessary.)

(THE END)