Eksamen august 2005
JMM

# Matematik 3GT

Books, notes, calculators, and computers are allowed at this three hour written exam. You may write your answers in pencil. An unjustified answer counts as no answer.

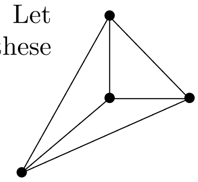
**Problem 1** (30%)

In $\mathbf{R}^2$ with the standard topology, let $B = \{x \times y \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ be the open unit ball around $0 \times 0$. Consider a continuous path $u\colon [0,1] \to \mathbf{R}^2$ starting and ending at the point $u(0) = 2 \times 0 = u(1)$ outside $B$.

(1) Show that $u^{-1}(B)$ is the union of at most countably many disjoint open intervals $(a_i, b_i)$.
(2) Show that $u^{-1}(0 \times 0)$ is contained in the union of finitely many of the open intervals $(a_i, b_i)$.
(3) Show that $u$ maps the end-points, $a_i$ and $b_i$, of the intervals $(a_i, b_i)$ to the unit circle $\overline{B} - B$.

**Problem 2** (20%)

Let $\Delta \subset \mathbf{R}^3$ be the subspace of $\mathbf{R}^3$ (with the standard topology) consisting of the four points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and the straight line segments between these points. Let $f\colon \Delta \to \Delta$ be a homeomorphism. How many possibilities are there for $f(0,0,0)$? Can all these possibilities actually be realized by some homeomorphism?



**Problem 3** (50%)

The (standard) circle $S^1$ is the subspace of $\mathbf{R}^2$ (with standard topology) given by

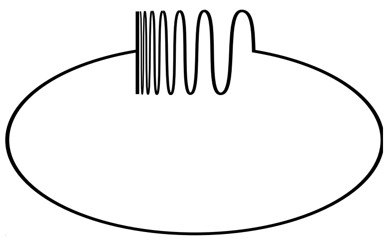
$$S^1 = \{x \times y \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

Let $e\colon \mathbf{R} \to S^1$ be the continuous map given by $e(t) = \cos(2\pi t) \times \sin(2\pi t)$, $t \in \mathbf{R}$.

The Warsaw circle is the subspace $W$ of $\mathbf{R}^2$ (with standard topology) obtained as the union of the topologist's sine curve

$$\overline{S} = (\{0\} \times [-1,1]) \cup \{x \times \sin(1/x) \mid 0 < x \leq 1/\pi\}$$

with the image of an injective continuous curve $c\colon [0,1] \to \mathbf{R}^2$ that is disjoint from $\overline{S}$ except that $c(0) = 0 \times 0$ and $c(1) = 1/\pi \times 0$.



$$\begin{tikzcd}
& \mathbf{R} \arrow[dr, "e"] & \\
W \arrow[ur, dotted, "s_1"] \arrow[rr, "q"'] & & S^1 \arrow[ul, dotted, "s_2"'] \\
& \mathbf{R} \arrow[ul, "c_1"] \arrow[ur, "c_2"'] &
\end{tikzcd}$$

FIGURE 1. The Warsaw circle

(1) Explain that $S^1$ and $W$ are compact Hausdorff spaces.
(2) Explain that there is a compactification $c_1 : \mathbf{R} \to W$ with remainder $W - c_1(\mathbf{R})$ homeomorphic to the interval $[-1,1]$ and a compactification $c_2 : \mathbf{R} \to S^1$ with remainder $S^1 - c_2(\mathbf{R})$ equal to a point.
(3) Find a quotient map $q\colon W \to S^1$ such that $q \circ c_1 = c_2$ inducing a homeomorphism between $W/(\{0\} \times [-1,1])$ and $S^1$.
(4) Show that there does not exist a continuous map $s_2 : S^1 \to \mathbf{R}$ such that $e \circ s_2$ is the identity map of $S^1$. (You may use without proof that $S^1$ does not embed into $\mathbf{R}$.)
(5) Does there exist a continuous map $s_1 : W \to \mathbf{R}$ such that $e \circ s_1 = q$?

(THE END)