Solutions to the June 2004 exam

Problem 1

(1) \Longrightarrow (2): Since f is continuous, $f(A) \subset f(\overline{A}) \subset \overline{f(A)}$. Since f is closed, $f(\overline{A})$ is closed and therefore these two inclusions imply $\overline{f(A)} = f(\overline{A})$.

(2) \Longrightarrow (1): Since $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$, f is continuous. If A is closed, then $A = \overline{A}$ so that $f(A) = f(\overline{A}) = \overline{f(A)}$ is also closed. This means that f is closed. **Problem 2**

Any continuous map of a compact space to a Hausdorff space is closed. Open subsets of locally path-connected spaces have open path-components.

Problem 3 [2, Ex 38.9]

(1) Suppose that $x_n \in X$ converges to $y \in \beta X - X$. We recursively define a subsequence by x_{n_k} by

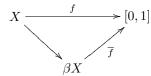
$$n_k = \begin{cases} 1 & k = 1\\ \min\{n > n_{k-1} \mid x_n \notin \{x_{n_1}, \dots, x_{n_{k-1}}\}\} & k > 1 \end{cases}$$

This definition makes sense since the set we are taking the minimal element of a nonempty set. Since x_n converges to y, the subsequence x_{n_k} also converges to y. Clearly, no two points of the subsequence x_{n_k} are identical. We call this subsequence x_n again.

(2) Note that y is in the closure of A as any neighborhood of y contains a point from A. Since therefore $A \subset A \cup \{y\} \subset \overline{A}$, it suffices to show that $A \cup \{y\}$ is closed, ie that the complement of $A \cup \{y\}$ is open: Let z be a point in the complement. Since z is not the limit of the sequence (x_{2n+1}) (there is just one limit point, namely y, in the Hausdorff space βX) there exists a neighborhood, U, of z containing only finitely many points from the sequence. Then $U - (U \cap (A \cup \{y\}))$ is an open neighborhood of z disjoint from $A \cup \{y\}$.

This shows that $\overline{A} = A \cup \{y\}$. Similarly, $\overline{B} = B \cup \{y\}$. Therefore the intersection $\overline{A} \cap \overline{B} = \{y\} \neq \emptyset$.

- (3) A and B are disjoint since no two points of the sequence x_n are identical. The set A is closed in X since $\operatorname{Cl}_X A = X \cap \overline{A} = X \cap (A \cup \{y\}) = A$. Similarly for B, of course. By Urysohn's characterization of normal spaces, there exists a continuous function $f: X \to [0, 1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.
- (4) The universal property of the Stone–Čech compactification $[1, \S 27]$ says that there exists a unique continuous map \overline{f} into the compact Hausdorff space [0, 1] such that the diagram



commutes. Since $\overline{A} \subset \overline{f}^{-1}(0)$ and $\overline{B} \subset \overline{f}^{-1}(1)$, \overline{A} and \overline{B} are disjoint.

- $(5) \hspace{0.1 cm} \emptyset \stackrel{(2)}{\neq} \overline{A} \cap \overline{B} \stackrel{(4)}{=} \emptyset \not \sharp$
- (6) X is a proper subspace of βX since βX is compact which X is not.
- (7) First countable spaces satisfy the sequence lemma. If $\beta X X = \overline{X} X$ is nonempty then βX does not satisfy the sequence lemma as we have just seen. Thus βX is not first countable, in particular not metrizable.

Here are two applications:

- $\beta \mathbf{Z}_+$ is compact but not sequentially compact: The sequence \mathbf{Z}_+ in $\beta \mathbf{Z}_+$ has no convergent subsequence (neither in \mathbf{Z}_+ nor in $\beta \mathbf{Z}_+ \mathbf{Z}_+$).
- $\beta \mathbf{R}$ is connected but not path-connected: $\beta \mathbf{R}$ is connected because it has a dense connected subspace, \mathbf{R} . It is not path-connected since there can be no path between a point in \mathbf{R} and a point outside \mathbf{R} as any such path would also give a a sequence of points in \mathbf{R} converging to a point outside \mathbf{R} (the first point of the path that is not in \mathbf{R}). To see this note that as \mathbf{R} is locally compact, \mathbf{R} is open in the compact Hausdorff space $\beta \mathbf{R}$ [1, 30.8]. The remainder $\beta \mathbf{R} - \mathbf{R}$ is therefore closed. Let now $p: [0, 1] \rightarrow \beta \mathbf{R}$ be a path from $p(0) \in \mathbf{R}$ to $p(1) \in \beta \mathbf{R} - \mathbf{R}$. Then $p^{-1}(\beta \mathbf{R} - \mathbf{R})$ is a compact subset of [0, 1] so it contains its greatest lower bound $t_0 = \inf p^{-1}(\beta \mathbf{R} - \mathbf{R})$. This lower bound is positive for 0 belongs to $p^{-1}(\mathbf{R})$. Hence $t_n = t_0 - \frac{1}{n}$ is a sequence in $p^{-1}(\mathbf{R})$ (for $n \gg 0$) that converges to $t_0 \notin p^{-1}(\mathbf{R})$. The image sequence $p(t_n)$ is then a sequence in \mathbf{R} that converges to a point $p(t_0)$ outside \mathbf{R} . But that is impossible!

References

- [1] Jesper M. Møller, General topology, http://www.math.ku.dk/ moller/e03/3gt/notes/gtnotes.dvi.
- James R. Munkres, Topology. Second edition, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063