Solutions to the June 2003 exam

Problem 1

(1) Let 0 be the smallest element of S_Ω and n, the nth iterated immediate successor of 0. It is possible to construct such a sequence since all elements but the largest in a well-ordered set has an immediate successor [Ex 10.2.(a)]. (The uncountable ordered set S_Ω has no largest element since any section of it is countable while S_ω itself is uncountable.) Then $0 < 1 < 2 < \cdots < n-1 < n < \cdots$ so the well-ordered subset $\{n \mid n \in \mathbf{Z}_+\} \subset S_\Omega$ has the order type of $\mathbf{Z}_+$. (This means that $h\colon \mathbf{Z}_+ \to S_\Omega$ is recursively defined by $h(n) = \min(S_\Omega - \{h(1), \dots, h(n-1)\})$ [Thm 8.4].)

(2) Put $\omega = \sup \mathbf{Z}_+$, the least upper bound of $\mathbf{Z}_+$ in the well-ordered set S_Ω [Ex 10.1]. None of the elements of $\mathbf{Z}_+$ are immediate predecessors of ω, so an immediate predecessor of ω would be an upper bound for $\mathbf{Z}_+$, smaller than the least upper bound; cf. [Ex 24.12.(c)]. (Also the smallest element 0 of S_Ω has no immediate predecessor for it has no predecessors at all!)

Problem 2

$$\partial(A \times B) \overset{\text{def}}{=} \overline{A \times B} \cap \overline{(X \times Y) - (A \times B)} \overset{\text{Hint}}{=} \overline{A \times B} \cap \overline{(X-A) \times Y \cup X \times (Y-B)}$$

$$\overset{\text{[Ex 19.9]}}{=} (\overline{A} \times \overline{B}) \cap (\overline{X-A} \times Y \cup X \times \overline{Y-B}) \overset{\text{[Ex 1.2]}}{=} \big((\overline{A} \cap \overline{X-A}) \times (\overline{B} \cap Y)\big) \cup \big((\overline{A} \cap X) \times (\overline{B} \cap \overline{Y-B})\big)$$

$$\overset{\text{def}}{=} (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$$

Problem 3

(1) [Ex 24.2] We must show that $g(x) = 0$ for some x. If $g(x) = 0$ for all x, then there is nothing to prove. If not, g assumes both positive and negative values because $g(-x) = -g(x)$. Since S^1 is connected [Thm 23.5], g also takes the value 0 at some point [Thm 24.3].

(2) It is not possible to imbed S^1 in $\mathbf{R}$ for there do not exist injective continuous maps $S^1 \hookrightarrow \mathbf{R}$.

Comment: The Borsuk–Ulam theorem says that for any continuous map $f\colon S^n \to \mathbf{R}^n$, $n \geq 1$, there is a point $x \in S^n$ such that $f(x) = f(-x)$.

Problem 4 [Ex 38.7] [2, 1]

(1) Let $F\colon \beta(X) \to \{0,1\}$ be the extension [Thm 38.4] of the continuous function $f\colon X \to \{0,1\}$ given by $f(A) = 0$ and $f(X - A) = 1$. Then $\overline{A} \subset F^{-1}(0)$ and $\overline{X-A} \subset F^{-1}(1)$ so these two subsets are disjoint; in other words $\overline{X-A} \subset \beta(X) - \overline{A}$.

(2) The inclusions

$$\beta(X) - \overline{A} \overset{\text{def}}{=} \overline{X} - \overline{A} \overset{[Ex 17.8]}{\subset} \overline{X-A} \overset{(1)}{\subset} \beta(X) - \overline{A}$$

tell us that $\beta(X) - \overline{A} = \overline{X-A}$. In particular, $\overline{A}$ is open (and closed).

(3) Since $U \cap X$ is a subset of U, it is clear that $\overline{U \cap X} \subset \overline{U}$ [Ex 17.6.(a)]. Conversely, let x be a point in $\overline{U}$ and V any neighborhood of x. Then $V \cap U \neq \emptyset$ is nonempty for x lies in the closure of U, and hence $(V \cap U) \cap X = V \cap (U \cap X) \neq \emptyset$ is also nonempty as X is dense. Thus every neighborhood V of x intersects $U \cap X$ nontrivially. This means that $x \in \overline{U \cap X}$. We conclude that $\overline{U \cap X} = \overline{U}$. From (2) (with $A = U \cap X$) we see that $\overline{U}$ is open (and closed).

(4) Let Y be any subset of $\beta(X)$ contaning at least two distinct points, x and y. We shall show that Y is not connected. Let $U \subset \beta(X)$ be an open set such that $x \in U$ and $y \notin \overline{U}$; such an open set U exists because $\beta(X)$ is Hausdorff [Definition, p. 237]. Then $Y = (Y \cap \overline{U}) \cup (Y - \overline{U})$ is a separation of Y, so Y is not connected.

References

[1] Russell C. Walker, *The Stone-Čech compactification*, Springer-Verlag, New York, 1974, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 83. MR 52 #1595

[2] Nancy M. Warren, *Properties of Stone-Čech compactifications of discrete spaces*, Proc. Amer. Math. Soc. **33** (1972), 599–606. MR 45 #1123