Eksamen januar 2005
JMM

Solutions to the January 2005 exam

Problem 1

(1) $\Longrightarrow$ (2): Since f is continuous $f^{-1}(B^\circ) \subset f^{-1}(B)^\circ$ [2, 18.1] [1, 13.3]. The other inclusion, $f^{-1}(B)^\circ \subset f^{-1}(B^\circ)$, equivalent to $f(f^{-1}(B)^\circ) \subset B^\circ$, follows because $f(f^{-1}(B)^\circ)$ is open and contained in $f(f^{-1}(B))) \subset B$.

(2) $\Longrightarrow$ (3): $X - f^{-1}(\overline{B}) = f^{-1}(Y - \overline{B}) = f^{-1}((Y-B)^\circ) \overset{(2)}{=} f^{-1}(Y-B)^\circ = (X - f^{-1}(B))^\circ = X - \overline{f^{-1}(B)}$ so that $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ for all $B \subset Y$.

(3) $\Longrightarrow$ (2): $X - f^{-1}(B)^\circ = \overline{X - f^{-1}(B)} = \overline{f^{-1}(Y-B)} \overset{(3)}{=} f^{-1}(\overline{Y-B}) = f^{-1}(Y - B^\circ) = X - f^{-1}(B^\circ)$ so that $f^{-1}(B^\circ) = f^{-1}(B)^\circ$ for all $B \subset Y$.

(2) $\Longrightarrow$ (1): The inclusion $f^{-1}(B^\circ) \subset f^{-1}(B)^\circ$, valid for all $B \subset Y$, tells us that f is continuous. Since $A^\circ \subset f^{-1}f(A)^\circ \overset{(2)}{=} f^{-1}(f(A)^\circ)$, we have $f(A^\circ) \subset f(A)^\circ$ for all $A \subset X$. In particular, $f(A) = f(A^\circ) \subset f(A)^\circ \subset f(A)$, ie $f(A) = f(A)^\circ$, when A is open. Thus f is open.

Problem 2

(1) $\mathrm{Ext}(A)$ is open because it is the complement of the closed set $\overline{A}$. $\mathrm{Bd}\,A$ is closed because it is the intersection of two closed sets.

(2) $X - (\mathrm{Int}(A) \cup \mathrm{Ext}(A)) = (X - \mathrm{Int}(A)) \cap \overline{A} = \overline{X - A} \cap \overline{A} = \mathrm{Bd}(A)$ where $\mathrm{Int}(A) \cap \mathrm{Ext}(A) \subset A \cap (X - \overline{A}) \subset A \cap (X - A) = \emptyset$.

(3) Assume that $u(I) \cap \mathrm{Bd}(A) = \emptyset$. Then $[0,1] = u^{-1}(\mathrm{Int}\,A) \cup u^{-1}(\mathrm{Ext}\,A)$ is a union of two disjoint open nonempty sets, contradicting connectedness [2, §23] [1, 16.2] of the unit interval [2, 24.2] [1, 17.3].

Problem 3

(1) Consider the straight line $L_n \subset I \times I$ between the points $\frac{1}{n+1} \times 1$ and $\frac{1}{n} \times 1$ of $I \times \{0\} \cup A \times I$. The image $R(L_n)$ is a connected subspace [1, 16.3] of $I \times \{0\} \cup A \times I$ containing the two end-points of L_n. Since $I \times \{0\} \cup A \times I$ with the point $P_n = \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right) \times 0$ removed is not connected, $R(L_n)$ contains P_n.

(2) The sequence $t_n \times 1$ converges to 0×1 and the image sequence $R(t_n \times 1) = P_n$ converges to $0 \times 0 \neq 0 \times 1 = R(0 \times 1)$ contradicting continuity of R [1, 15.12].

Alternative solution: Assume that the map R exists. Then R is a quotient map because it is a map from a compact space onto a Hausdorff space [1, 18.8]. But $I \times \{0\} \cup A \times I$ is not locally connected [1, 17.17] so it can not be the quotient space of the locally connected space $I \times I$.

Problem 4

(1) $T \cap X_n$ is closed for all n because it is a finite set in a Hausdorff space. Since the topology on X is coherent with the filtration, T is closed. The same argument applies to any subspace of T. Since any subspace of T is closed, T has the discrete topology.

(2) Closed, discrete subspaces of compact spaces are finite [1, 18.13].

(3) Since T is finite there is an N such that $C \cap (X_{n+1} - X_n) = \emptyset$ for all $n \geq N$. This means that $C \subset X_N$.

References

[1] Jesper M. Møller, *General topology*, `http://www.math.ku.dk/ moller/e03/3gt/notes/gtnotes.dvi`.

[2] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063