Solutions to the January 2004 exam

Problem 1 [Ex 30.16]

(1) The set D_k is countable for there is an injective map

$$D_k \hookrightarrow \overbrace{(\mathbf{Q}\times\mathbf{Q})\times\cdots\times(\mathbf{Q}\times\mathbf{Q})}^{k}\times\overbrace{\mathbf{Q}\times\cdots\times\mathbf{Q}}^{k} = \mathbf{Q}^{3k}$$

of D_k into a countable set [Cor 7.3]. As a countable union of countable sets, $D = \bigcup_{k\in\mathbf{Z}_+} D_k$ is countable [Thm 7.5].

(2) The basis open sets in $\mathbf{R}^I$ are finite intersections $\bigcap_{j=1}^k \pi_{i_j}^{-1}(U_{i_j})$ where $i_1,\dots,i_k$ are k distinct points in I and $U_{i_1},\dots,U_{i_k}$ are k open subsets of $\mathbf{R}$. Choose disjoint closed subintervals I_j such that $i_j \in I_j$ and choose $x_j \in U_{i_j}\cap\mathbf{Q}$, $j=1,\dots,k$. Then $x(I_1,\dots,I_k,x_1,\dots,x_k) \in \bigcap_{j=1}^k \pi_{i_j}^{-1}(U_{i_j})$ for $\pi_{i_j}x(I_1,\dots,I_k,x_1,\dots,x_k) = x_j \in U_{i_j}$ for all $j=1,\dots,k$. This shows that any (basis) open set contains an element of $x(D)$.

(3) Let D be a dense subset of $\mathbf{R}^J$ for some set J. Let $f\colon J\to\mathcal{P}(D)$ be the map from the index set J to the power set of D given by $f(j) = D\cap\pi_j^{-1}(2003,2004)$. For j and k two points of J we have

$$\begin{aligned} f(j)=f(k) &\iff D\cap\pi_j^{-1}(2003,2004) = D\cap\pi_k^{-1}(2003,2004)\\ &\implies \overline{D\cap\pi_j^{-1}(2003,2004)} = \overline{D\cap\pi_k^{-1}(2003,2004)}\\ &\iff \overline{\pi_j^{-1}(2003,2004)} = \overline{\pi_k^{-1}(2003,2004)}\\ &\overset{[\text{Thm 19.5}]}{\iff} \pi_j^{-1}[2003,2004] = \pi_k^{-1}[2003,2004] \iff j=k \end{aligned}$$

and therefore f is injective. Thus $\mathrm{card}J \le \mathrm{card}\mathcal{P}(D)$.

Here is a better proof (due to one of the students) that f is injective: Let j and k be two distinct points in J. Then $f(j)\neq f(k)$ for

$$\begin{aligned} f(j)-f(k) &= \big(\pi_j^{-1}(2003,2004) - \pi_k^{-1}(2003,2004)\big)\cap D\\ &\supset \big(\pi_j^{-1}(2003,2004)\cap\pi_k^{-1}(2002,2003)\big)\cap D \neq \emptyset \end{aligned}$$

since D is dense.

Problem 2

(1) Any open subset of a locally compact Hausdorff space is a locally compact Hausdorff space [Thm 29.3].

(2) The open sets in $\omega(X-A) = (X-A)\cup\{\omega\}$ are of the form U, where U is open in $X-A$, or $(X-A)-C\cup\{\omega\}$, where $C\subset X-A$ is compact. In the first case, $f^{-1}(U)=U$ is open in X since U is open in $X-A$ which is open in X. In the second case $f^{-1}((X-A)-C\cup\{\omega\}) = (X-A)-C\cup A = X-C$ is open since C is closed because it is a compact subset of a Hausdorff space.

(3) The universal property of quotients spaces implies that the map f factors through X/A since it sends A to one point [Thm 22.2]. The induced continuous map $\overline{f}\colon X/A\to\omega(X-A)$ is clearly bijective.

(4) If X is compact, the quotient space X/A is also compact [Thm 26.5]. The map $\overline{f}$ is now a continuous bijective map of a compact space onto a Hausdorff space, hence a homeomorphism [Thm 26.6]. In fact $\overline{f}$ is a homeomorphism if and only if X/A is compact.

(5) Let $p\colon X\to X/A$ denote the quotient map. The quotient space X/A contains the infinite discrete closed subspace $\{p(2n-\frac{1}{2}) \mid n\in\mathbf{Z}\}$ so it is not compact [Thm 28.1]. [The notes [1] contain a similar argument showing that $X/A = \mathbf{R}/\mathbf{Z}$ is not even locally compact.] On the other hand, The Alexandroff compactfication $\omega(X-A)$ is compact. These two spaces are therefore not homeomorphic.

(6) Let $\{U_n\}$ be a countable collection of neighborhoods of A. Let U be the neighborhood of A that in the interval $(2n-1, 2n)$ equals $U_n \cap (2n-1, 2n)$ with one point removed. Then U does not contain any of the U_n. Thus X/A is not first countable at the point $p(A)$ (as shown in [1]).

(7) The Alexandroff compactification $\omega(X - A)$ is homeomorphic to the Hawaiian Earring [Example 1, p 436] $\bigcup_{n \in \mathbf{Z}_+} C_{1/n}$ (as shown in [1]). Since the quotient space X/A is not first countable it can not embed into a first countable space [Thm 30.2] (as observed in [1]).

References

[1] Jesper M. Møller, *General topology*, `http://www.math.ku.dk/ moller/e03/3gt/notes/gtnotes.dvi`.