Mat 3GT Exam January 2003

Solutions

Problem 1

Solution 1: Apply the Extreme value theorem [Thm 27.4] to the identity map $X \to X$.
Solution 2: Copy the last two paragraphs on p. 174.

Problem 2

(1) Solution 1: [Ex 30.4] For each n there is by compactness a finite set $A_n \subset X$ such that all points in X are within distance $1/n$ from A_n. Then $A = \bigcup A_n$ is a countable dense subset for $d(x, A) \leq d(x, A_n) \leq 1/n$ for all $x \in X$ and all $n \geq 1$.
Solution 2: A careful inspection of pp. 191–192 and [Ex 30.5] reveals (no, that isn't very well written) the following theorem:

Theorem 0.1. *Let X be a metrizable space. Then*

$$X \text{ is 2nd countable} \Leftrightarrow X \text{ has a countable dense subset} \Leftrightarrow X \text{ is Lindelöf}$$

In our case, X is metrizable and compact, so it is metrizable and Lindelöf, so it has a countable dense subset.
(2) For any metric, there is a bounded metric giving the same topology [Thm 20.1].
(3) If $f(x) = f(y)$, the continuous [Ex 20.3] functions $z \to d(x, z)$ and $z \to d(y, z)$ agree on the dense subspace A, so they agree on X [Ex 18.13, Ex 31.5]. In particular they have the same value at the point x, so $d(x, y) = d(x, x) = 0$, so $x = y$. We have shown that f is injective.
The map f is continuous because each coordinate, $x \to d(x, a)$, is continuous [Thm 19.6, Ex 20.3].
The continuous bijection $f \colon X \to f(X)$ is a homeomorphism because X is compact and $[0, 1]^A$ is Hausdorff [Thm 26.6, Thm 19.4, Thm 31.2].
The image $f(X)$ is closed because it is a compact subspace of a Hausdorff space [Thm 26.3, Thm 26.5].
(4) A countable product of metrizable spaces is metrizable [Thm 20.5], any product of compact spaces is compact [Thm 37.3]; in particular, $[0, 1]^\omega$ is compact and metrizable. Any subspace of a metrizable space is metrizable [Ex 21.1], any closed subspace of a compact space is compact [Thm 26.2]; in particular, any closed subspace of $[0, 1]^\omega$ is metrizable and compact.

Problem 3

(1) The extension $f^\bullet$ is continuous at all points of X (where it agrees with the continuous map f). The only problem is continuity at $\infty \in X$. This explains the first step in the following argument:

$$\begin{aligned}
f^\bullet \text{ is continuous } &\Leftrightarrow f^\bullet \text{ is continuous at } \infty \in X \\
&\Leftrightarrow \text{For any neighborhood } V \subset Y \text{ of } \infty \text{ there is a neighborhood } U \subset X \text{ of } \infty \text{ such that } U \subset f^{-1}(V) \\
&\Leftrightarrow \text{For any compact } K \subset Y \text{ there is a compact } L \subset X \text{ such that } X - L \subset f^{-1}(Y - K) \\
&\Leftrightarrow \text{For any compact } K \subset Y \text{ there is a compact } L \subset X \text{ such that } f^{-1}(K) \subset L \\
&\Leftrightarrow \text{For any compact } K \subset Y,\ f^{-1}(K) \text{ is compact}
\end{aligned}$$

For the final step, note that if $f^{-1}(K) \subset L$ then $f^{-1}(K)$ is compact [Thm 26.2] as a closed subspace [Thm 26.3] of a compact space.
(2) The map $g^\bullet$ is not continuous for $g^{-1}(S^1) = \mathbf{R}$ is not compact [Example 1,p. 164].

Problem 4

(1) The saturation $p^{-1}p(A) = A \cup (-A)$ of any open (resp. closed) subspace $A \subset S^2$ is open (resp. closed) because $x \to -x$ is a homeomorphism of S^2. (To see that p is closed one may also note that any continuous map of a compact space into a Hausdorff space is closed [Ex 26.6].)
(2) Solution 1: The map $p\colon S^2 \to P^2$ is perfect, S^2 is 2nd countable (in fact a manifold), and perfect maps preserve 2nd countability [Ex 31.7(d)].
Solution 2: Let $\{B_n\}$ be a countable basis for S^2. Since p is open, $p(B_n)$ is open for all n; indeed $\{p(B_n)\}$ is a countable basis for P^2.
(3) Solution 1: The map $p\colon S^2 \to P^2$ is perfect, S^2 is Hausdorff (in fact a manifold), and perfect maps preserve the Hausdorff property [Ex 31.7(a)].
Solution 2: The map $p\colon S^2 \to P^2$ is a closed quotient map, S^2 is normal (in fact a manifold), and closed quotient maps preserve normality [Ex 31.6].
Solution 3: It is also possible to give a simple ad hoc argument for this particular map.
(4) Solution 1: The restriction $q = p|U\colon U \to p(U)$ is bijective because $U \cap -U = \emptyset$; it is continuous because it is the restriction of a continuous map [Thm 18.2]; it is open because it is the restriction of an open map to an open subspace [Ex 22.5]; so it is a homeomorphism.
Solution 2: The restriction $q = p|U\colon U \to p(U)$ is bijective because $U \cap -U = \emptyset$; it is also a quotient map since U is open [Thm 22.1]; a bijective quotient map is a homeomorphism (either directly from the definition or because there exists a continuous map $f\colon p(U) \to U$ such that $f \circ q$ is the identity on U [Thm 22.2] so that $q^{-1} = f$ is continuous).
(5) $P^2 - p(U) = S^1$.
Conclusion: P^2 is a compact manifold [Thm 60.3], $P^2 = S^1 \cup B((0,0),1)$ where S^1 is the circle and $B((0,0),1)$ the open unit ball in $\mathbf{R}^2$.