Eksamen august 2005
JMM

Solutions to the August 2005 exam

Problem 1

(1) $u^{-1}(B) \subset (0,1) \subset [0,1]$ is open in $[0,1]$ (by continuity), $(0,1)$, and $\mathbf{R}$ (since $(0,1)$ is open in $\mathbf{R}$). Any open and bounded subset of $\mathbf{R}$ is the union of at most countably many disjoint open intervals. (Open subsets of the locally path-connected space $\mathbf{R}$ have open path-components. The open, path-connected, bounded subsets of $\mathbf{R}$ are open intervals. Each of these disjoint open intervals contains a rational number so there are at most countably many of them.)

(2) $u^{-1}(0,0)$ is a closed subset of the compact space $[0,1]$ so it is itself compact. It is covered by the open intervals (a_i, b_i). By compactness, it is covered by finitely many of them.

(3) By continuity of u, $u(a_i) \in u([a_i,b_i]) = u(\overline{(a_i,b_i)}) \subset \overline{u(a_i,b_i)} \subset \overline{B}$. As the intervals are disjoint, $a_i \notin u^{-1}(B)$, so $u(a_i) \notin B$. Thus $u(a_i)$ is in $\overline{B} - B$. Similarly for the other end-point, b_i.

Problem 2

The point $O = (0,0,0)$ has the property that any neighborhood contains a connected neighborhood U such that $U - \{O\}$ has three components. Only the four listed points have this property. Thus any self-homeomorphism of Δ must permute the four points. Any such permutation can be realized by an affine automorphism of $\mathbf{R}^3$.

Problem 3

(1) The Heine-Borel theorem says that S^1 and W are compact Hausdorff spaces since they are closed and bounded subspaces of $\mathbf{R}^2$.

(2) The curve $S^1 - \{0 \times 1\}$ has a parameterization of the form $c_2\colon (0,1) \to \mathbf{R}^2$ where c_2 is a homeomorphism. Similarly, the curve $W - [-1,1]$ has a parameterization of the form $c_1\colon (0,1) \to \mathbf{R}^2$ where c_1 is a homeomorphism. The maps c_1 and c_2 can be more or less explicit. The images of c_1 and c_2 are clearly dense in W and S^1, respectively.

(3) Define $q\colon W \to S^1$ by $qc_1(t) = c_2(t)$ for all $t \in \mathbf{R}$ and $q(W - c_1(\mathbf{R})) = S^1 - c_2(\mathbf{R})$. The map q is continuous as we showed in (General Toplogy, Thm 21.3) or since the pre-image of the intersection of S^1 with any open ball in $\mathbf{R}^2$ is open in W. (Consider separately the open subsets of S^1 containing the point at infinity and those not containing the point at infinity.) The map q is a closed quotient map by the Closed map Lemma (General Toplogy, Lemma 18.8). Since q sends $[-1,1]$ to a point in S^1 there is a factorization $\overline{q}\colon W/[-1,1] \to S^1$ which is again a quotient map (General Topology, Thm 14.15) and, in fact, a homeomorphism as it is bijective (General Topology, Cor 14.9).

(4) Suppose that s_2 exists. Then s_2 is injective and, by the Closed map Lemma, an embedding of the compact space S^1 into the Hausdorff space $\mathbf{R}$. This contradicts the non-existence of such embeddings.

(5) Suppose that s_1 exists. Then $s_1([-1,1]) \subset e^{-1}(1 \times 0) = \mathbf{Z}$. By conectedness, $s_1([-1,1])$ is a point. Thus s_1 factors through $W/[-1,1] = S^1$ to give a map s_2 as in the previous question. Since the map s_2 does not exits, nor does s_1.