2nd April 2005

Munkres §38

Ex. 38.4. Let $X \to \beta X$ be the Stone–Čech compactification and $X \to cX$ an arbitrary compactification of the completely regular space X. By the universal property of the Stone–Čech compactification, the map $X \to cX$ extends uniquely

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & cX \\ & \beta X \arrow[ur] & \end{tikzcd}$$

to a continuous map $\beta X \to cX$. Any continuous map of a compact space to a Hausdorff space is closed. In particular, $\beta X \to cX$ is closed. It is also surjective for it has a dense image since $X \to cX$ has a dense image. Thus $\beta X \to cX$ is a closed quotient map.

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Ex. 38.5.

(a). For any $\varepsilon > 0$ there exists an $\alpha \in S_\Omega = [0, \Omega)$ such that $|f(\beta) - f(\alpha)| < \varepsilon$ for all $\beta > \alpha$. For if no such element existed we could find an increasing sequence of elements $\gamma_n \in (0, \Omega)$ such that $|f(\gamma_n) - f(\gamma_{n-1})| \geq \varepsilon$ for all n. But any increasing sequence in $(0, \Omega)$ converges to its least upper bound whereas the image sequence $f(\gamma_n) \in \mathbf{R}$ does not converge; this contradicts continuity of the function $f\colon (0, \Omega) \to \mathbf{R}$. So in particular, there exist elements α_n such that $|f(\beta) - f(\alpha_n)| < 1/n$ for all $\beta > \alpha_n$. Let α be an upper bound for these elements. Then f is constant on (α, Ω).

(b). Since any real function on $(0, \Omega)$ is eventually constant, any real function, in particular any bounded real function, on $(0, \Omega)$ extends to the one-point-compactification $(0, \Omega]$. But the Stone-Čech compactification is characterized by this property [Thm 38.5] so $(0, \Omega] = \beta(0, \Omega)$.

(c). Use that any compactification of $(0, \Omega)$ is a quotient of $(0, \Omega]$ [Ex 38.4].

Ex. 38.6. ([1, Thm 6.1.14]) Let X be a completely regular space and $\beta(X)$ its Stone-Čech compactification. Then

X is nonconnected $\Leftrightarrow$ There exists a continuous surjective function $X \to \{0, 1\}$

$\overset{[\text{Thm 38.4}]}{\Rightarrow}$ There exists a continuous surjective function $\beta(X) \to \{0, 1\} \Leftrightarrow \beta(X)$ is nonconnected

If X is connected then also βX is connected since it has a connected dense subset [3, Thm 23.4]

Ex. 38.7. ([Exam June 03, Problem 4] [5, 6, 4]) Let X be a discrete space; A a subset of $X \subset \beta(X)$ and U an open subset of $\beta(X)$.

(1) Let $F\colon \beta(X) \to \{0, 1\}$ be the extension [Thm 38.4] of the continuous function $f\colon X \to \{0, 1\}$ given by $f(A) = 0$ and $f(X - A) = 1$. Then $\overline{A} \subset F^{-1}(0)$ and $\overline{X - A} \subset F^{-1}(1)$ so these two subsets are disjoint; in other words $\overline{X - A} \subset \beta(X) - \overline{A}$. The inclusions

$$\beta(X) - \overline{A} \overset{\text{def}}{=} \overline{X} - \overline{A} \overset{[Ex17.8]}{\subset} \overline{X - A} \subset \beta(X) - \overline{A}$$

tell us that $\beta(X) - \overline{A} = \overline{X - A}$. In particular, $\overline{A}$ is open (and closed).

(2) Since $U \cap X$ is a subset of U, it is clear that $\overline{U \cap X} \subset \overline{U}$ [Ex 17.6.(a)]. Conversely, let x be a point in $\overline{U}$ and V any neighborhood of x. Then $V \cap U \neq \emptyset$ is nonempty for x lies in the closure of U, and hence $(V \cap U) \cap X = V \cap (U \cap X) \neq \emptyset$ is also nonempty as X is dense. Thus every neighborhood V of x intersects $U \cap X$ nontrivially. This means that $x \in \overline{U \cap X}$. We conclude that $\overline{U \cap X} = \overline{U}$. From (1) (with $A = U \cap X$) we see that $\overline{U}$ is open (and closed).

(3) Let Y be any subset of $\beta(X)$ contaning at least two distinct points, x and y. We shall show that Y is not connected. Let $U \subset \beta(X)$ be an open set such that $x \in U$ and $y \notin \overline{U}$; such an open set U exists because $\beta(X)$ is Hausdorff [Definition, p. 237]. Then $Y = (Y \cap \overline{U}) \cup (Y - \overline{U})$ is a separation of Y, so Y is not connected.

A Hausdorff space is said to be extremally disconnected if the closure of every open set is open. A space is totally disconnected if the connected components are one-point sets. Any extremally disconnected space is totally disconnected. We have shown that $\beta(X)$ is extremally disconnected.

Ex. 38.8. The compact Hausdorff space I^I is a compactification of $\mathbf{Z}_+$ since [3, Ex 30.16] it has a countable dense subset (and is not finite). Any compactification of $\mathbf{Z}_+$ is a quotient of the Stone–Čech compactification $\beta\mathbf{Z}_+$ [3, Ex 38.4]. In particular, I^I is a quotient of $\beta\mathbf{Z}_+$ so $\mathrm{card}\beta\mathbf{Z}_+ \geq \mathrm{card} I^I$.

Ex. 38.9. ([Exam June 04, Problem 3])

(a). Suppose that $x_n \in X$ converges to $y \in \beta X - X$. We will show that then y is actually the limit point of two sequences with no points in common. The first step is to find a subsequence where no two points are identical. We recursively define a subsequence x_{n_k} by

$$n_k = \begin{cases} 1 & k = 1 \\ \min\{n > n_{k-1} \mid x_n \notin \{x_{n_1}, \ldots, x_{n_{k-1}}\}\} & k > 1 \end{cases}$$

This definition makes sense since the set we are taking the minimal element of a nonempty set. Since x_n converges to y, the subsequence x_{n_k} also converges to y. Clearly, no two points of the subsequence x_{n_k} are identical. We call this subsequence x_n again.

Let now $A = \{x_1, x_3, \ldots\}$ be the set of odd points and $B = \{x_2, x_4, \ldots\}$ the set of even points in this sequence. We claim that $\overline{A} = A \cup \{y\}$ and $\overline{B} = B \cup \{y\}$.

Any neighborhood of y contains a point from A, so y is in the closure of A. Since $A \subset A \cup \{y\} \subset \overline{A}$, it suffices to show that $A \subset A \cup \{y\}$ is closed, ie that the complement of $A \cup \{y\}$ is open: Let z be a point in the complement. Since z is not the limit of the sequence (x_{2n+1}) (there is just one limit point, namely y, in the Hausdorff space βX) there exists a neighborhood of z, even one that doesn't contain y, containing only finitely many elements from this sequence. Since z is not in A we can remove these finitely many points from the neighborhood to get a neighborhood of z that is disjoint from $A \cup \{y\}$.

This shows that $\overline{A} = A \cup \{y\}$. Similarly, $\overline{B} = B \cup \{y\}$. Therefore the intersection $\overline{A} \cap \overline{B} = \{y\} \neq \emptyset$.

On the other hand, the sets A and B are disjoint since no two points of the sequence x_n are identical. They are closed subsets of X for $\mathrm{Cl}_X A = X \cap \overline{A} = X \cap (A \cup \{y\}) = A$ and similarly for B, of course. By Urysohn's characterization of normal spaces, there exists a continuous function $f\colon X \to [0,1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$. The universal property of the Stone–Čech compactification [2, §27] says that there exists a unique continuous map $\overline{f}$ into the compact Hausdorff space $[0,1]$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & {[0,1]} \\ & \beta X \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes. Since $\overline{A} \subset \overline{f}^{-1}(0)$ and $\overline{B} \subset \overline{f}^{-1}(1)$, $\overline{A}$ and $\overline{B}$ are disjoint.

We have now shown that $\overline{A} \cap \overline{B}$ is both empty an nonempty. This contradiction means that no point in $\beta X - X$ can be the limit of a sequence of points in X.

(b). Assume that X is *normal* and noncompact. X is a proper subspace of βX since βX is compact which X is not. No point in $\beta X - X = \overline{X} - X$ is the limit of a sequence of points in X. Thus βX does not satisfy the Sequence lemma so βX is not first countable, in particular not metrizable.

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