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Munkres §32

Ex. 32.1. Let Y be a closed subspace of the normal space X. Then Y is Hausdorff [Thm 17.11]. Let A and B be disjoint closed subspaces of Y. Since A and B are closed also in X, they can be separated in X by disjoint open sets U and V. Then $Y \cap U$ and $V \cap Y$ are open sets in Y separating A and B.

Ex. 32.3. Look at [Thm 29.2] and [Lemma 31.1]. By [Ex 33.7], locally compact Hausdorff spaces are even completely regular.

Ex. 32.4. Let A and B be disjoint closed subsets of a regular Lindelöf space. We proceed as in the proof of [Thm 32.1]. Each point $a \in A$ has an open neighborhood U_a with closure $\overline{U}_a$ disjoint from B. Applying the Lindelöf property to the open covering $\{U_a\}_{a\in A} \cup \{X - A\}$ we get a countable open covering $\{U_i\}_{i\in\mathbf{Z}_+}$ of A such that the closure of each U_i is disjoint from B. Similarly, there is a countable open covering $\{V_i\}_{i\in\mathbf{Z}_+}$ of B such that the closure of each V_i is disjoint from A. Now the open set $\bigcup U_i$ contains A and $\bigcup V_i$ contains B but these two sets are not necessarily disjoint. If we put $U'_1 = U_1 - \overline{V_1}$, $U'_2 = U_2 - \overline{V_1} - \overline{V_2}, \dots,$ $U'_i = U_i - \overline{V_1} - \cdots - \overline{V_i}, \dots$ we subtract no points from A so that the open sets $\{U'_i\}$ still form an open covering of A. Similarly, the open sets $\{V'_i\}$, where $V'_i = V_i - \overline{U_1} - \cdots - \overline{U_i}$, cover B. Moreover, the open sets $\bigcup U'_i$ and $\bigcup V'_i$ are disjoint for U'_i is disjoint from $V_1 \cup \cdots \cup V_i$ and V'_i is disjoint from $U_1 \cup \cdots \cup U_i$.

Ex. 32.5. $\mathbf{R}^\omega$ (in product topology) is metrizable [Thm 20.5], in particular normal [Thm 32.2]. $\mathbf{R}^\omega$ in the uniform topology is, by its very definition [Definition p. 124], metrizable, hence normal.

Ex. 32.6. Let X be completely normal and let A and B be separated subspaces of X; this means that $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Note that A and B are contained in the open subspace $X - (\overline{A} \cap \overline{B}) = (X - \overline{A}) \cup (X - \overline{B})$ where their closures are disjoint. (The closure of A in $X - (\overline{A} \cap \overline{B})$ is $\overline{A} - \overline{B}$ [Thm 17.4].) The subspace $X - (\overline{A} \cap \overline{B})$ is normal so it contains disjoint open subsets $U \supset A$ and $V \supset B$. Since U and V are open in an open subspace, they are open [Lemma 16.2].

Conversely, suppose that X satisfies the condition (and is a T_1-space). Let Y be any subspace of X and A and B two disjoint closed subspaces of Y. Since $\overline{A} \cap Y$ and $\overline{B} \cap Y$ are disjoint [Thm 17.4], $\overline{A} \cap B = \overline{A} \cap (Y \cap B) = (\overline{A} \cap Y) \cap (B \cap Y) = \emptyset$, and, similarly, $A \cap \overline{B} = \emptyset$. By assumption, A and B can then be separated by disjoint open sets. If we also assume that X is T_1 then it follows that Y is normal.

References